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On the theory of generalized *M*-polysymmetric hypergroups

Christos G. Massouros[1], Jean D. Mittas[2]

[1] 54, Klious st., GR15561 Cholargos, Athens, Greece
Email: masouros@hol.gr
[2] 5, Edmond Abbot, GR54643 Thessaloniki, Greece

Abstract. This paper deals with the subhypergroups of the generalized M-polysymmetric hypergroups ($GM - PH$) and it proves that they are invertible, ultra-closed, complete parts and that the quotient of the GM-PH with its subhypergroups gives always abelian groups. It also includes the definition and the study of the monogenic subhypergroups of the GM-PH as well as the homomorhpisms of the $GM - PH$.

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1 Introduction

J. Mittas motivated from the theory of the algebraic closed fields introduced in [4] a special type of completely regular hypergroup, the *polysymmetric hypergroup* and he studied some of its fundamental properties as well. C. N. Yatras in his dissertation [6], written under the direction of J. Mittas, studied this structure in depth under the name *(Mittas) M-polysymmetric hypergroup* (see also [7, 8]). Next, J. Mittas generalized this hypergroup and introduced the *Generalized M-polysymmetric hypergroup* ($GM - PH$), which is a set H equipped with a hypercomposition $x + y$ that satisfies the axioms:

GM1 $(x + y) + z = x + (y + z)$ for all $x, y, z \in H$.

GM2 $x + y = y + x$ for all $x, y \in H$.

GM3 There exists at least one neutral element $e \in H$ (i.e. $x \in e + x$, for all $x \in H$). The set of neutral elements is denoted by U.

GM4a For each $x \in H$, there exists at least one $x' \in H$, opposite or symmetric of x with respect to every element of U. That is

$$(\forall x \in H)(\exists x' \in H)(\forall e \in U)[e \in x + x'].$$

The set of the symmetric elements of x, will be denoted by S(x).

GM4b $x + x' = U$ for all $x \in H, x' \in S(x)$.

GM4c If $(x + y) \cap U \neq \emptyset$, then $x + y = U$ for all $x, y \in H$.

GM5 For each $x, y, z \in H$ with $z \in x + y$ and for each $(x', y', z') \in S(x) \times S(y) \times S(z)$ it holds $z' \in x' + y'$.

A first study of this structure was done in [5] where one can find properties and interesting examples of this hypergroup. For the self-sufficiency of this paper it is mentioned that in [5] it is proved that if x, y, z, w are elements of a GM-PH H, then the implications

(i) $(x + y) \cap (z + w) \neq \emptyset \Rightarrow x + y = z + w$

and

(ii) $x + y = z + w \Rightarrow y + w' = z + x'$

are valid. Also it is proved that for every $x \in H$, the sets $C(x) = U + x = e + x$ (where e is any element of U) form a partition in H, that the equalities $x + y = e + x + y = (e + x) + (e + y)$ are valid and that for every $x, y \in H, x + y$ is a class of this partition. The quotient set of the above mentioned partition becomes an abelian group under the setwise composition. This group is called the reduction group of H.

Lemma 1.1. $(x + y) + U = x + y$, *for every* $x, y \in H$.

Proof. If $z \in x + y$ then, $(x + y) \cap (z + e) \neq \emptyset$, $e \in U$. Therefore $x + y = z + e$. Thus $(x + y) + U = (z + e) + U = z + (e + U) = z + U = x + y$. □

The example which is presented below indicates the relation between the M-polysymmetric hypergroups and the generalized M-polysymmetric hypergroups.

Example 1.1. Let K be the set of the points of a double circular conical surface of revolution around the axis Oz of the $Oxyz$ system. K becomes a M-polysymmetric hypergroup, with identity the vertex of the conical surface, by defining

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = \{(x, y, z) \in K, z = z_1 + z_2\} \quad (1)$$

that is, the hypersum of any two elements (x_1, y_1, z_1), (x_2, y_2, z_2) of K are all the points of the circle of the conical surface with center $z_1 + z_2$, while the opposite of an arbitrary element (x, y, z) of K are all the elements of the symmetric circle in which (x, y, z) belongs, i.e. the circle with center $(0, 0, -z)$.

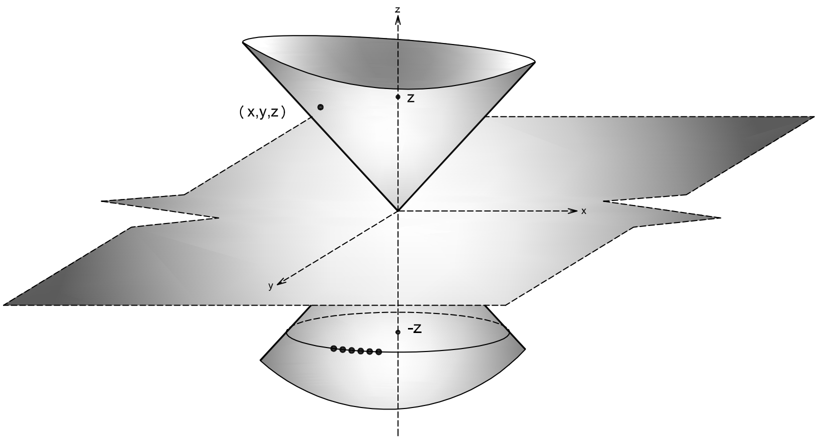

Fig. 1.

Now if we consider the union of the set K of the points of the above double circular conical surface with the set U of the points of the plane xOy, then $\bar{K} = K \cup U$ endowed with the hypercomposition (1) becomes a GM-PH. In this hypergroup the set of neutral elements is the set of the points of the plane xOy. A remarkable family of subhypergroups of $\bar{K}$ is formed by the sets $\overline{k}_p = \{(x, y, \lambda p), \lambda \in Z, p \text{ prime}\} \cup U$.

The next two Propositions show the way of constructing a M-PH from a GM-PH and vice versa. Their proof is straightforward through the verification of the axioms. In what follows $A..B$ denotes the set containing exactly those elements in A that are not in B.

Proposition 1.1. *Let $(H, +)$ be a GM-PH and let U be the set of its neutral elements. If e is an element, different from the elements of H, then the set $H' = [H \cup \{e\}]..U$ becomes a M-PH by defining a hypercomposition "$\dot{+}$" as follows:*

$$x \dot{+} y = x + y \text{ if } x, y \in H..U \text{ and } y \notin S(x),$$
$$x \dot{+} x' = [(x + x') \cup \{e\}]..U \text{ if } x' \in S(x),$$
$$x \dot{+} e = x + U \text{ if } x \in H..U,$$
$$e \dot{+} e = e,$$

and the mapping $f \colon H \to H'$ with

$$f(x) = \begin{cases} x & \text{if } x \in H..U \\ e & \text{if } x \in U \end{cases}$$

is a normal homomorphism.

Proposition 1.2. *Let $(H, +)$ be a M-PH and let $(U, +)$ be a total hypergroup. In the set $H' = [H..\{0\}] \cup U$ a hypercomposition "$\dot{+}$" is defined as follows:*

$x \dotplus y = x + y$ *if* $x, y \in H..\{0\}$ *and* $y \notin S(x)$

$x \dotplus x' = [(x + x') \cup U]..\{0\}$ *if* $x' \in S(x)$

$U \dotplus x = x \dotplus U = x + 0$ *if* $x \in H..\{0\}$

$e_1 \dotplus e_2 = e_1 + e_2$ *if* $e_1, e_2 \in U$

Then $(H', \dotplus)$ *becomes a GM-PH and the mapping* $f \colon H' \to H$ *with*

$$f(x) = \begin{cases} x & \text{if } x \in H..U \\ 0 & \text{if } x \in U \end{cases}$$

is a normal homomorphism.

2 Subhypergroups of GM-PH

A subset h of a hypergroup H is a subhypergroup of $(H, \cdot)$ if and only if $xh = hx = h$ for all $x \in h$. A consequence of the axioms of the GM-PH is that for every neutral element $e \in U$, the equality $e + U = U$ holds, thus:

Proposition 2.1. *The set of neutral elements U is the minimum in the sense of inclusion subhypergroup of a GM-PH.*

A subhypergroup h of a hypergroup H is a subhypergroup of operationally equivalent elements if $xy = hy$ and $yx = yh$ whenever $x \in h$ and $y \notin h$ [1]. In [5] it has been proved that $e + x = U + x$, for all $e \in U$. Thus:

Proposition 2.2. *U is a subhypergroup of operationally equivalent elements.*

A subhypergroup h of a hypergroup H is a subhypergroup of inseparable elements if $xy \cap h \neq \emptyset$ implies $h \subseteq xy$ whenever $x \notin h$ or $y \notin h$ [1]. Thus according to axiom GM4c it holds:

Proposition 2.3. *U is a subhypergroup of inseparable elements.*

A subhypergroup h of a hypergroup H is a subhypergroup of essentially indistinguishable elements if h is a subhypergroup of operationally equivalent and inseparable elements [1], thus, because of the Propositions 2.2 and 2.3, it holds:

Proposition 2.4. *U is a subhypergroup of essentially indistinguishable elements.*

In a GM-PH H the sets $C(x) = U + x = e + x$ (where e is an element of U) form a partition of H, and for every $x, y \in H$, $x + y$ is a class of this partition (see [5], Theorem 2.4). If $G(H) = \{C(x), x \in H\}$, then

Proposition 2.5. *Every subhypergroup of a GM-PH is a union of classes from G(H).*

A consequence of the axioms of the GM-PH is that for every two neutral elements e_1, e_2 it holds that $e_1, e_2 \in e_1 + e_2$ and that $e_1 + e_2 \subseteq U$. Hence $S(e_1) = S(e_2) = U$. Thus:

Proposition 2.6. *The set of neutral elements U of a generalized M-polysymmetric hypergroup (GM-PH) H is a generalized M-polysymmetric subhypergroup (GM-PSH) of H.*

Next let h be a subhypergroup of H, different from U. Then $x + h = h$, for all $x \in h$. Thus for $x \in h$, there exists $y \in h$ such that $x \in x + y$. Let $x' \in S(x)$. Then $x' + x \subseteq x' + x + y \Rightarrow U \subseteq U + y \Rightarrow U = C(y) \Rightarrow y \in U$. Therefore $h \cap U \neq \emptyset$. Suppose that $e_i \in h \cap U$. Since $e_i + e_i = U$ and $e_i + e_i \quad \subseteq h$, it follows that $U \subseteq h$, from where it derives that the entire class $C(x) = x + U$ is contained in h, as well. Now, since $x + h = h$, for all $x \in h$ and $U \subseteq h$, it derives that for all $x \in h$, there exists $x' \in h$ such that $e \in x + x'$, which means that $S(x) \cap h \neq \emptyset$. But $S(x) = C(x')$, and as it was proved above, if an element belongs to h the whole class of this element belongs to h. Thus:

Proposition 2.7. *Every subhypergroup of a generalized M-polysymmetric hypergroup (GM-PH) H is a generalized M-polysymmetric subhypergroup (GM-PSH) of H with the same set of neutral elements.*

Next, since the intersection of any two subhypergroups of a GM-PH is non void, because it contains U, it derives:

Proposition 2.8. *The set of the subhypergroups of a GM-PH is a complete lattice.*

A subhypergroup h of a hypergroup H is an invertible subhypergroup (from the right) if $ah \neq bh$ implies $ah \cap bh = \emptyset$.

Proposition 2.9. *The subhypergroups of a GM-PH are invertible.*

Proof. Let h be a subhypergroup of a GM-PH $(H, +)$. According to Lemma 2.1. [5], if $(a + h) \cap (b + h) \neq \emptyset$, then $a + h = b + h$ and so the Proposition 2.9. $\square$

For the elements a, b of a hypergroup H the induced hypercompositions a/b and $b\backslash a$ are defined as follows $a/b = \{t \in H \mid a \in tb\}$ and $b\backslash a = \{t \in H \mid a \in bt\}$. A subhypergroup h of a hypergroup H is closed if $a, b \in h$ impies $a/b \subseteq h$ and $b\backslash a \subseteq h$. The GM-PH are commutative so the two induced hypercompositions coincide and $a/b = b\backslash a$, will be denoted by $a \div b$.

Since the invertible subhypergroups are closed it holds:

Corollary 2.1. *The subhypergroups of a GM-PH are closed.*

Proposition 2.10. *If h is a subhypergroup of a GM-PH H, then the relation $y \equiv x mod(h) \Leftrightarrow y \in C_h\ (x)$ is an normal equivalence relation.*

Proof. Since the subhypergroups of a GM-PH are invertible, it derives that every subhypergroup h defines in H a partition. The classes of this partition are the sets $C_h(x) = x + h$, $x \in H$. If $C_h(x)$ and $C_h(y)$ are two classes of this partition, then:

$$C_h(x) + C_h(y) = (x + h) + (y + h) = (x + y) + h = \bigcup_{z \in x+y} (z + h)$$
$$= \{C_h(z) \mid z \in x + y\}. \qquad \square$$

Hence the Proposition.

Proposition 2.11. *For every subhypergroup h of a GM-PH H, the quotient set H/h is an abelian group under the setwise composition.*

Proof. Since the relation mod(h) is a normal equivalence relation the set of the classes H/h becomes a hypergroup under the hypercomposition:

$$C_h(x) + C_h(y) = \{C_h(z) \mid z \in x + y\}.$$

But the set $\{C_h(z) \mid z \in x + y\}$ is a singleton. Indeed $z \in x + y$ implies that $x+y = e+z$ and therefore $C_h(x)+C_h(y) = x+y+h = z+e+h = z+h = C_h(z)$. $\square$

A subhypergroup h of a hypergroup H is ultra-closed (from the right) if for every $x \in H$, it holds $xh \cap x(H..h) = \emptyset$.

Proposition 2.12. *The subhypergroups of a GM-PH are ultra-closed.*

Proof. Let h be a subhypergroup of a GM-PH $(H, +)$. Suppose that $z \in h$, $x \in H, y \in H..h$ and $(x + z) \cap (x + y) \neq \emptyset$, then $x + z = x + y$. According to Corollary 2.6 [5] this last equality implies that $x + x' = y + z'$ for all $x' \in S(x)$, $z' \in S(z)$. But $x + x' = U$, thus $y \in S(z')$, so $y \in h$, which is absurd and therefore the proposition. $\square$

A non empty subset A of a hypergroup H is a complete part of H, if the following implication holds:

$$\forall n \in N, \forall (x_1, x_2, \ldots, x_n) \in H^n, \prod_{i=1}^{n} x_i \cap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A$$

As it is mentioned above in a GM-PH, $x + y$ is a class of the partition $G(H)$, thus, for every $n \in N$ the sum $x_1 + x_2 + \ldots + x_n$ is a class of this partition and if A is a subhypergroup of H, then, according to Proposition 2.5, A is a union of classes of this partition. Thus $x_1 + x_2 + \ldots + x_n \subseteq A$, and therefore

Proposition 2.13. *The subhypergroups of a GM-PH are complete parts.*

Since the heart of a hypergroup is the intersection of all subhypergroups which are complete parts, it follows that:

Proposition 2.14. *The heart of a GM-PH is the total subhypergroup U of its neutral elements.*

Proposition 2.15. *A non empty subset h of a GM-PH H is a subhypergroup of H if and only if $x + S(y) \subseteq h$, for all $x, y \in h$.*

Proof. The above condition is obvious when h is a subhypergroup of H. Conversely now, let $x + S(y) \subseteq h$, for all $x, y \in h$, then $x + x' \subseteq h, x' \in S(x)$, thus $U \subseteq h$. Suppose next that $e \in U$, then $e + S(x) \subseteq h$, for all $x \in h$, so $S(x) \subseteq h$, for all $x \in h$, from which it derives that $S(S(h)) = h$. Now if x is an element of h, then $x + h = x + S(S(h)) \subseteq h$. Next let $t \in h$, then $t + S(x) \subseteq h$ or $t + S(x) + x \subseteq x + h$ or $t + U \subseteq x + h$ from where it derives that $t \in x + h$, that is $h \subseteq x + h$. So $x + h = h$. □

Corollary 2.2. *A non empty subset h of a GM-PH H is a subhypergroup of H if and only if it is stable under the hypercomposition and if for every element $x \in h$, its symmetric set $S(x)$ is a subset of h.*

Next suppose that $t \in x \div y$, then $x \in t + y$ or $t \in x + y'$ for all $y' \in S(y)$, hence $x \div y \subseteq x + S(y)$. Now if $t \in x + S(y)$, then $t \in x + y', y' \in S(y)$ and so $x \in t + y$. Thus $t \in x \div y$ and therefore $x + S(x) \subseteq x \div y$. Hence $x \div y = x + S(x)$ an so the Proposition:

Proposition 2.16. *A non empty subset h of a GM-PH H is a subhypergroup of H if and only if $x \div y \subseteq h$, for all $x, y \in h$.*

Let X be a subset of a GM-PH H. Since the set of the subhypergroups of a GM-PH is a complete lattice, the smallest in the sense of inclusion, subhypergroup $h(X)$ of H, which contains X can be corresponded to X. If $X = \emptyset$, then $h(X) = U$. If $X \neq \emptyset$, then X and all the symmetric elements of the elements of X belongs to $h(X)$, i.e. $X \cup S(X) \subseteq h(X)$. Also $h(X)$ contains all the finite sums $\sum_{i=1}^{k} x_i, x_i \in X \cup S(X)$. Let $\overline{X}$ be the set of all elements $x \in H$ which belong to sums of the type $\sum_{i=1}^{k} x_i, x_i \in X \cup S(X)$. Suppose that $x, y \in \overline{X}$. Then $x \in \sum_{i=1}^{m} x_i$ and $y \in \sum_{i=1}^{n} y_i$, $x_i, y_i \in X \cup S(X)$. If $y' \in S(y)$, then $y' \in \sum_{i=1}^{n} y'_i$ and therefore $x + y' \subseteq \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y'_i = \sum_{i=1}^{m+n} z_i$, where $z_i \in X \cup S(X)$. Thus $x + S(y) \subseteq \overline{X}$ and according to Proposition 2.15, $\overline{X}$ is a subhypergroup of H. Since $\overline{X}$ contains all the elements of the union $X \cup S(X)$, it derives that $h(X) \subseteq \overline{X}$. But every subhypergroup of H that contains X, contains every sum of finitely many elements from the union $X \cup S(X)$ as well, which means that $\overline{X} \subseteq h(X)$. Hence:

Proposition 2.17. *The subhypergroup of a GM-PH which is generated from a non empty set X consists of the unions of all finite sums of the elements that are contained in the set $X \cup S(X)$, where $S(X) = \bigcup_{x \in X} S(x)$.*

3 Monogenic Subhypergroups of GM-PH

This paragraph contains the study of the monogenic subhypergroups of a GM-PH, i.e. the subhypergroups, which is generated by a single element (see also [3]). So let $(H, +)$ be a GM-PH, let x be an arbitrary element of H and let $h(x)$ be the subhypergroup which is generated by this element. Then, it holds

$$n \cdot x = \begin{cases} x + x + \cdots + x & (n \text{ times}) & \text{if } n > 0, \\ U & & \text{if } n = 0, \\ x' + x' + \cdots + x' & (-n \text{ times}) & \text{if } n < 0. \end{cases} \tag{1}$$

Next,

$$m \cdot x + n \cdot x = \begin{cases} (m+n) \cdot x & \text{if } mn > 0, \\ (m+n) \cdot x + U & \text{if } mn < 0. \end{cases} \tag{2}$$

From the above it derives that

$$(m+n) \cdot x \subseteq m \cdot x + n \cdot x. \tag{3}$$

Proposition 3.1. *For every $x \in H$ it holds:*

$$h(x) = \Big[\bigcup_{k \in Z} k \cdot x\Big] \cup [x + U].$$

Proof. According to Proposition 2.17, the subhypergroup of a GM-PH which is generated from a non empty set X consists of the unions of all the finite sums of the elements that are contained in the set $X \cup S(X)$, thus, according to (1), for the singleton $\{x\}$, it holds:

$$h(x) = \bigcup_{(m,n) \in N^2} m \cdot x + n \cdot x'$$

and according to (2), it is $m \cdot x + n \cdot x' = (m+n) \cdot x + U$. So the Proposition. □

Let's define now a symbol $\omega(x)$ which is a natural number, but it can even be the $+\infty$. $\omega(x)$ will be named the *order of x* and simultaneously the *order of the monogenic subhypergroup $h(x)$*. Two cases can appear such that one revokes the other:

I. $U \cap k \cdot x = \emptyset$ is valid, for all $k \in Z..\{0\}$. Then the order of x and of $h(x)$ is defined to be the infinity and it is written $\omega(x) = +\infty$.

Proposition 3.2. *$\omega(x) = +\infty$, if and only if it holds $m_1 \cdot x \cap m_2 \cdot x = \emptyset$, for every $m_1, m_2 \in Z$, with $m_1 \neq m_2$.*

Proof. Supposing that $U \cap m \cdot x = \emptyset$, for all $m \in Z, m \neq 0$ and assuming that $m = m_1 + m_2$, it holds:

$$\begin{aligned} U \cap m \cdot x = \emptyset &\Rightarrow U \cap (m_1 \cdot x + m_2 \cdot x) = \emptyset \\ &\Rightarrow m_1 \cdot x' \cap m_2 \cdot x = \emptyset \Rightarrow -m_1 \cdot x \cap m_2 \cdot x = \emptyset \end{aligned}$$

Conversely now, if the intersection $m_1 \cdot x \cap m \cdot x$ is void for every $m_1, m_2 \in Z$ with $m_1 \neq m_2$, then $U \cap (m_1 \cdot x + m_2 \cdot x') = \emptyset$ and therefore $U \cap (m_1 - m_2) \cdot x = \emptyset$. Thus $U \cap m \cdot x = \emptyset$. □

II. There exist $k \in Z$, with $k \neq 0$ such that $U \cap k \cdot x \neq \emptyset$. Then, because of the axiom GM4c, the equality $kx = U$ holds. Let p be the minimum positive integer, such that $p \cdot x = U$. Then the order of x and of $h(x)$ is defined to be the positive integer p and it is written $\omega(x) = p$.

Proposition 3.3. *$\omega(x) = p, p \in N$ if and only if there exist $m_1, m_2 \in Z$, with $m_1 \neq m_2$, such that $m_1 \cdot x \cap m_2 \cdot x \neq \emptyset$.*

Now suppose that $\omega(x) = p$ and assume that there exists $m \in Z..\{0\}$ such that $mx = U$. Then $m = kp + r$, $0 \leqslant r < p$. Thus $(kp + r)x = U$ or $kpx + rx = U$ or $U + rx = U$, hence $rx = U$. But $r < p$ and r is the minimum non zero positive integer which has this property, therefore $r = 0$, and so the Proposition:

Proposition 3.4. *If $\omega(x) = p, p \in N$, then $mx = U, m \in Z..\{0\}$ if and only if $m = kp$.*

As it is mentioned above, in [5] it is proved that the sets $x + U, x \in H$ form a partition in H and $x + y$ is a class of this partition for all $x, y \in H$. Thus the sets $kx, k \in Z..\{1\}$ and the set $x + U$ are the classes of this partition (note that according to Lemma 1.1, $kx + U = kx$, for all $k \in Z..\{1\}$). When $\omega(x) = +\infty$, then the sets $kx, U + x$ are disjoint for all $k \in Z..\{1\}$, while when $\omega(x) = p$, these sets coincides with the ones of the family $\{U, U + x, 2x, \dots, (p-1)x\}$.

Proposition 3.5. *If $\omega(x) = +\infty$, then the reduction group $h(x)/U$ of $h(x)$ is isomorphic to the additive group Z of integers, while when $\omega(x) = p, p \in N$, then the reduction group is isomorphic to the additive group Z_p of the integers mod(p).*

4 Homomorphisms of GM-PH

This study mainly refers to the normal (or good) homomorhisms. According to the terminology that M. Krasner introduced [2], if H and H' are two hypergroups, then a homomorphism from H to H' is a mapping $\phi \colon H \to P(H')$ such that $\phi(x + y) \subseteq \phi(x) + \phi(y)$, for every $x, y \in H$. ϕ is named strong if the above relation holds as an equality. A homomorphism is named strict if ϕ is a mapping from H to H' such that $\phi(x + y) \subseteq \phi(x) + \phi(y)$, for every $x, y \in H$. A strict homomorphism is called normal if $\phi(x + y) = \phi(x) + \phi(y)$, for every $x, y \in H$.

Let's suppose that H and H' are two GM-PH, with sets of neutral elements U and U' respectively and let ϕ be a normal homomorphism from H to H'. As usual [2], the kernel of ϕ, which is denoted by kerϕ, is defined to be the subset $\phi^{-1}(\phi(U))$ of H and the homomorphic image $\phi(H)$ of H, is denoted by Imϕ.

Proposition 4.1. *If ϕ is a normal homomorphism from H to H', then*

i) $\phi(U) = U'$,
ii) $\phi(S(x)) = S(\phi(x))$ *for all* $x \in H$,
iii) *Imϕ is a subhypergroup of H',*
iv) *kerϕ is a subhypergroup of H.*

Proof.

(i) Let x be an element of H. Then $\phi(x) \in \phi(x + U) = \phi(x) + \phi(U)$. Since in [5] it is proved that the implication $x \in x + y \Rightarrow y \in U$ holds, it derives that $\phi(U) = U'$.

(ii) It has been proved that $S(x)$ is a class of the partition which is defined by U, i.e. that $S(x) = x' + U$ [5]. Also since $\phi(U) = U'$ it derives that $\phi(x + x') = U'$ or $\phi(x) + \phi(x') = U'$, i.e. $\phi(x') \in S(\phi(x))$. Thus $\phi(S(x)) = \phi(x' + U) = \phi(x') + \phi(U) = \phi(x') + U' = S(\phi(x)) + U' = S(\phi(x))$.

(iii) Let y be an arbitrary element of $\phi(H)$. Then $y = \phi(x)$ for some $x \in H$. Thus $y + \phi(H) = \phi(x) + \phi(H) = \phi(x + H) = \phi(H)$.

(iv) Since $\phi(U) = U'$ it derives that $\phi(x+x') = U', x' \in S(x)$ or equivalently that $\phi(x) + \phi(x') = U'$. If it is assumed that x belongs to kerϕ, then $U' + \phi(x') = U'$, i.e. $\phi(x') = U'$. So $S(x) \subseteq$ kerϕ, for all $x \in$ kerϕ. Therefore if $x, y \in H$, then $\phi[y + S(x)] = \phi(x) + \phi(S(x)) = U' + U' = U'$. Hence $y + S(x) \subseteq$ kerϕ and because of Proposition 2.15 kerϕ is a subhypergroup of H. □

A direct consequence of the above (iii) is the following Proposition:

Proposition 4.2. *If ϕ is a normal homomorphism from H to H' then the homomorphic image of every subhypergroup of H is a subhypergroup of H'.*

Now let h be a subhypergroup of $\phi(H)$. If x, y are elements of $\phi^{-1}(h)$ then $\phi(x) \in h$ and $S(\phi(y)) = \phi(S(y)) \subseteq h$. Thus $\phi(x) + \phi(S(y)) \subseteq h$, or $\phi(x + S(y)) \subseteq h$, or $x + S(x) \subseteq \phi^{-1}(h)$. Hence because of Proposition 2.15 it holds:

Proposition 4.3. *If ϕ is a normal homomorphism from H to H' then the inverse image of every subhypergroup of $\phi(H)$ is a subhypergroup of H.*

Although in normal homomorphism the equality $\phi(x+y) = \phi(x)+\phi(y)$ holds, the respective equality is not valid for the induced hypercomposition. Generally for the induced hypercomposition the inclusion $\phi(x \div y) \subseteq \phi(x) \div \phi(y)$ is valid. Indeed if $z \in \phi(x \div y)$, then there exists $w \in x \div y$ such that $\phi(w) = z$. Since $w \in x \div y$, it derives that $x \in w + y$. Therefore $\phi(x) \in \phi(w + y)$, or $\phi(x) \in \phi(w) + \phi(y)$, or $\phi(w) \in \phi(x) \div \phi(y)$. Thus $z \in \phi(x) \div \phi(y)$ and so the inclusion

$\phi(x \div y) \subseteq \phi(x) \div \phi(y)$ holds. But since in the GM-PH the equality $x \colon y = x + S(y)$ is valid, it derives that $\phi(x \div y) = \phi(x + S(y)) = \phi(x) + \phi(S(y)) = \phi(x) + S(\phi(y)) = \phi(x) \div \phi(y)$. Hence the Proposition:

Proposition 4.4. *If ϕ is a normal homomorphism between two GM-PH, then the equality $\phi(x \div y) = \phi(x) \div \phi(y)$ is valid.*

Since kerϕ is a subhypergroup of H, it derives, because of Proposition 2.11, that $H/\ker\phi$ is an abelian group. Also $\phi(H)/U'$ is the reduction group of $\phi(H)$. For these two groups it holds:

Proposition 4.5. *The abelian groups $H/ker\phi$ and $\phi(H)/U'$ are isomorphic.*

Proof. Consider the mapping $\psi \colon H/\ker\phi \to \phi(H)/U'$ with $\psi(x + \ker\phi) = \phi(x) + U'$. Obviously ψ is a surjection for which it holds: $\psi[(x + \ker\phi) + (y + \ker\phi)] = \psi(x + y + \ker\phi) = \{\psi(z + \ker\phi) \mid z \in x + y\} = \{\phi(z) + U' \mid z \in x + y\} = \phi(x + y) + U' = [\phi(x) + U'] + [\phi(y) + U'] = \psi(x + \ker\phi) + \psi(y + \ker\phi)$. Thus ψ is an epimorphism. Also ψ is a monomorphism, since if it is supposed that from $x + \ker\phi \neq y + \ker\phi$ derives the equality $\phi(x) + U' = \phi(y) + U'$, then the following implications lead to a contradiction: Indeed $\phi(x) + U' = \phi(y) + U' \Rightarrow \phi(x') + \phi(x) + U' = \phi(x') + \phi(y) + U' \Rightarrow \phi(x' + x) + U' = \phi(x' + y) + U' \Rightarrow \phi(U) + U' = \phi(x' + y) + U' \Rightarrow U' = \phi(x' + y) + U' \Rightarrow \phi(x' + y) = U' \Rightarrow x' + y \in \ker\phi \Rightarrow x + \ker\phi = y + \ker\phi$. So the Proposition. □

Corollary 4.1. *If ϕ is a normal epimorphism, then the abelian group $H/ker\phi$ is isomorphic to the reduction group of H'.*

Corollary 4.2. *If $ker\phi = U$, then the reduction group of H is isomorphic to the reduction group of $\phi(H)$.*

Every normal homomorphism ϕ from H to H' defines in a natural way a mapping $\overline{\phi}$ from H to the reduction group of H' as follows: $\overline{\phi}(x) = \phi(x) + U$. One can easily verify that $\overline{\phi}$ is a normal homomorphism. $\overline{\phi}$ is called reduction homomorphism.

Proposition 4.6. *If ϕ is a normal homomorphism from H to H', then for the reduction homomorphism $\overline{\phi}$ the equality $\overline{\phi} = \psi\sigma$ is valid, where σ is the function that maps each element $x \in H$ to the element $x + ker\phi$ of $H/ker\phi$ and ψ is the isomorphism from $H/ker\phi$ to the reduction group of $\phi(H)$.*

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