Operators and Hyperoperators Acting on Hypergroups

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Abstract. External operations and hyperoperations on hypergroups leads to structures that are called hypermodules, supermodules, hypermoduloids and supermoduloids. These structures give applications in the theory of graphs and in geometries.

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INTRODUCTION

In a non empty set H, a **hypercomposition** is a function from H × H to the powerset P(H) of H. This notion was introduced in mathematics together with the notion of the **hypergroup** by F. Marty [3]. The axioms that endow the pair (H, ·) of a nonempty set H and a hypercomposition "·" with the hypergroup structure are:

i. a(bc) = (ab)c for all a, b, c ∈ H (associativity)
ii. aH = Ha = H for all a ∈ H (reproductivity)

Also F. Marty defined the two **induced hypercompositions** (the left and the right division) that derive from the hypercomposition of the hypergroup, i.e.

$$a/b = \{x \in H \mid a \in xb\} \quad \text{and} \quad b\backslash a = \{y \in H \mid a \in by\}$$

When "·" is commutative, a/b=b\a. Consequences of the above definitions are:

i. $ab \neq \varnothing$, for all a,b ∈ H.
ii. $a/b \neq \varnothing$ and $a\backslash b \neq \varnothing$, for all a,b ∈ H.
iii. the nonempty result of the induced hypercompositions is equivalent to the reproductive axiom.
iv. $(a/b)/c = a/(c\cdot b)$, $c\backslash(b\backslash a) = (b\cdot c)\backslash a$, $(b\backslash a)/c = b\backslash(a/c)$, for all a,b,c ∈ H (mixed associativity)

A **transposition hypergroup** [1] is a hypergroup (H,·) that satisfies a postulated property of transposition i.e. $(b\backslash a) \cap (c/d) \neq \varnothing \Rightarrow (ad) \cap (bc) \neq \varnothing$. A **join space**, also **join hypergroup**, is a commutative transposition hypergroup [11]. A **quasicanonical hypergroup** is a transposition hypergroup containing a scalar identity, that is, there exists e ∈ H such that ea = ae = a, for each a ∈ H. For each a ∈ H, one has that e/a = a\e is a singleton, which is denoted by a^{-1} [6, 1]. A commutative quasicanonical hypergroup is called **canonical hypergroup** [10].

A non void set Y endowed with a composition "·" and a hypercomposition "+" is called a **hyperringoid** [9] if:

i. (Y, +) is a hypergroup
ii. (Y, ·) is a semigroup
iii. the composition is bilaterally distributive to the hypercomposition.

If (Y,+) is a transposition hypergroup, then the hyperringoid is called transposition hyperringoid, while if (Y,+) is a canonical hypergroup, then the structure (Y,+,·) was named hyperring by M. Krasner [2].

The notions of the set of operators and hyperoperators from a hyperringoid Y, over an arbitrary non void set M, were introduced in [7], in order to describe the action of the state transition function in the theory of Automata. Y is a set of **operators** over M, if there exists an external operation from M×Y to M, such that $(s\kappa)\lambda = s(\kappa\lambda)$, for all s∈M and κ,λ∈Y and moreover s1=s for all s∈M, when Y is a unitary hyperringoid. If there exists an external hyperoperation from M×Y to P(M) which satisfies the above axiom with the variation that s∈s1, when Y is a unitary hyperringoid, then Y is a set of **hyperoperators** over M.

If M is a hypergroup and Y a hyperringoid of operators over M such that, for each κ,λ∈Y and s,t∈M, the axioms: (i) $(s+t)\lambda = s\lambda + t\lambda$, (ii) $s(\kappa+\lambda) \subseteq s\kappa + s\lambda$ hold, then M is called **right hypermoduloid** over Y. If Y is

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a set of hyperoperators, then M is called **right supermoduloid**. If the second of the above axioms holds as an equality, then the hypermoduloid is called **strongly distributive**. There is a similar definition of the **left hypermoduloid** and the **left supermoduloid** over Y in which the elements of Y operate from the left side. When M is both right and left hypermoduloid (resp. supermoduloid) over Y it is called **Y-hypermoduloid** (resp. **Y-supermoduloid**) [8]. If M is a canonical hypergroup, the set of operators Y is a hyperring, and s1=s, s0=0 for all $s \in M$, then M is named **right hypermodule**, while it is named **right supermodule** if Y is a set of hyperoperators [5].

HYPERMODULOIDS

The set of the operators over a non empty set M, can define in M a hypercomposition and when the set of the operators is a unitary hyperringoid, M enriched with this hypercomposition, becomes a hypergroup.

Definition 2.1. An element s_2 of M is called **connected** with an element s_1 of M, if there exists an element λ of Y such that $s_2=s_1\lambda$, when Y is a set of operators over M, or $s_2 \in s_1\lambda$, when Y is a set of hyperoperators over M.

It must be mentioned that s_2 being connected to s_1, does not necessarily imply that s_1 is connected to s_2.

With the use of the notion of the connected elements, a hypercomposition can be defined in M, as follows:

$$(2.1) \qquad s_1+s_2 = \begin{cases} \{s \in M \mid s=s_1\kappa \text{ and } s_2=s\lambda, \text{ with } \kappa,\lambda \in Y\}, & \text{if } s_2 \text{ is connected to } s_1 \\ \{s_1, s_2\}, & \text{if } s_2 \text{ is not connected to } s_1 \end{cases}$$

Proposition 2.1. *If the set of the operators Y over a non void set M is a unitary hyperringoid, then M endowed with the hypercomposition (2.1) becomes a hypergroup.*

Corollary 2.1. *The set of vertices of a directed graph, is endowed with the structure of the hypergroup, if the result of the hypercomposition of two vertices v_i and v_j is the set of the vertices which appear in all the possible paths that connect v_i to v_j, or the biset $\{v_i, v_j\}$, if there do not exist any connecting paths from vertex v_i to vertex v_j.*

Proposition 2.2. *If M_1, M_2 are two right Y-hypermoduloids, then $M=M_1 \times M_2$ becomes a right Y-hypermoduloid, if M is endowed with the hypercomposition:*

$$(s_1,t_1) + (s_2,t_2) = \{ (s,t) \;/\; s \in s_1+s_2,\ t \in t_1+t_2 \}$$

and the external operation from $M \times Y$ to M:

$$(s,t)\lambda = (s\lambda,t\lambda)$$

M is not strongly distributive, even when M_1 and M_2 are strongly distributive.

Let H and H´ be two hypergroups and let $R \subseteq H \times H'$ be a binary relation from H to H´.

Definition 2.2. R is called **homomorphic relation**, if, for all $(a_1,b_1), (a_2,b_2) \in R$ it holds:

$$(\forall x \in a_1+a_2)(\exists y \in b_1+b_2)\,[(x,y) \in R] \ \text{ and } \ (\forall y' \in b_1+b_2)(\exists x' \in a_1+a_2)\,[(x',y') \in R] \quad (D_1)$$

or equivalently for all $x \in a_1+a_2$ and for all $y \in b_1+b_2$ it holds:

$$[\, \{ x \} \times (b_1+b_2) \,] \cap R \neq \varnothing \quad \text{and} \quad [\, (a_1+a_2) \times \{ y \} \,] \cap R \neq \varnothing \qquad (D_1')$$

Let Y and Y´ be two hyperringoids and let $R \subseteq Y \times Y'$ be a binary relation from Y to Y´.

Definition 2.3. R will be called **homomorphic relation**, if it satisfies the axioms of the Definition 2.2. and, moreover, if for every $(a_1,b_1) \in R$ and $(a_2,b_2) \in R$ it holds:

$$(a_1a_2,\ b_1b_2) \in R \qquad (D_2)$$

A homomorphic relation which is also an equivalence relation is named **congruence relation**.

Proposition 2.3. *If M is a strongly distributive hypermoduloid over a hyperringoid Y, then the relation*

$$T = \{ (k,k') \in Y \times Y \;/\; (\forall s \in M)\ sk = sk' \}$$

is a congruence relation.

It is easy to verify that if an equivalence relation R in a hyperringoid Y satisfies the property:

$$xRy \text{ and } w \in E \ \Rightarrow\ xwRyw \text{ and } wxRwy \qquad [D_2']$$

then it satisfies the axiom $[D_2]$ of the Definition 2.3. An equivalence relation whish satisfies $[D_2']$, is called **compatible** to the composition. It is possible though that an equivalence relation satisfies only one of the conditions of the second part of $[D_2']$. Such a relation is called **right** or, resp. **left compatible** to the composition.

Lemma 2.1. *Every congruence relation R in a hypergroup H is a normal equivalence relation and therefore the set H/R becomes a hypergroup under the hypercomposition*

$$(2.2) \qquad C_x \dagger C_y = \{ C_z \;/\; z \in x + y \}$$

where C_x is the class of an arbitrary element $x \in H$.

Lemma 2.2. *If the hypergroup H is transposition, then H/R is also a transposition hypergroup.*

Proposition 2.4. *Let R be a congruence relation in a hyperringoid Y right compatible to the multiplication. Then the quotient set Y/R becomes a right hypermoduloid over Y.*

From Proposition 2.3. and 2.4. it derives

Corollary 2.2. *If M is a finite strongly distributive hypermoduloid over a hyperringoid Y, then the hypermoduloid Y/T is also finite.*

HYPERMODULES

Suppose that M is a module over a unitary ring P and let G be a subgroup of the multiplicative semigroup $P^*=P\setminus\{0\}$ of P, which satisfies the condition $xG\cdot yG=xyG$, for each $x,y \in P$. In [4] it is proved that the above equality is equivalent to the normality of G in P* only when P* is a group, that is, when P is a division ring. G defines in P a partition, the equivalence classes of which are the cosets xG, $x\in P$. The quotient set of this partition is denoted by P/G and it becomes a hyperring if it is enriched with the following composition and hypercomposition:

$$xG\cdot yG = xyG$$
$$xG \dagger yG = \{(xp+yq)G \mid p,q \in G\}$$

for each xG, yG $\in$ P/G. This hyperring, which was constructed by M. Krasner, was named quotient hyperring [2]. Furthermore, in [5] this construction has been extended to the hypermodules through the introduction of a relation g in a module M, in the following way:

$$(x,y) \in g \Leftrightarrow x=qy,\ q\in G$$

It can easily be proved that g is an equivalence relation. Let x_g signifies the equivalence class of an arbitrary element x and let M_g be the set of the equivalence classes modulo g. M_g becomes a canonical hypergroup, if it is endowed with the hypercomposition:

$$x_g \underset{\bullet}{+} y_g = \{ z_g \in M_g \mid z_g \subseteq x_g + y_g \}$$

i.e. $x_g \underset{\bullet}{+} y_g$ consists of all the classes $z_g \in M_g$ which are contained in the setwise sum of x_g with y_g. Now let P_G be the quotient hyperring of P by G. Then:

Proposition 3.1. *M_g becomes a strongly distributive hypermodule over P_G, if an external operation from $P_G \times M_g$ to M_g is defined as follows: $k_G\ x_g = (kx)_g$, for each $k_G \in P_G$, $x_g \in M_g$*

It is worth mentioning that the elements of this hypermodule are selfopposite, i.e. $x_g \underset{\bullet}{+} x_g = \{0, x_g\}$, when $-1\in G$.

In accordance to the above, suppose that V is a vector space over an ordered field F and suppose that F^+ is the positive cone of F. Since F^+ is a multiplicative subgroup of F*, there exists the quotient hyperfield $F/F^+=\{F^-,0,F^+\}$ of F by F^+. Next let $\overline{V}$ be the hypermodule (vector hyperspace) over F/F^+, which derives from V, using the above described construction. Then the set $\overline{V}$ is exactly what is called “ray join space” in [11]. Next, consider a hypersphere S of V centered at 0. The map $\overline{x} \rightarrow x$ of $\overline{V}$ onto $S\cup\{0\}$ is one to one and the elements of the hypersum $\overline{x}+\overline{y}$, $\overline{x} \neq \overline{y}$ are mapped to the points of the minor arc which has end points x, y and lies on the great circle of the hypersphere that passes through x, y. In this case, the two end points x and y do not belong to the minor arc xy, since $\overline{x}, \overline{y} \notin \overline{x}+\overline{y}$, while $\overline{x}+(-\overline{x})=\{-\overline{x},0,\overline{x}\}$.

Proposition 3.2. *Let M be a strongly distributive hypermodule over a division hyperring $(D,+,\cdot)$. A new commutative hypercomposition is introduced in M, which is defined as follows:*

$$x\dagger y = \begin{cases} x+y \cup\{x, y\}, & \text{if } x, y \neq 0 \text{ and } x \neq -y \\ M, & \text{if } x = -y \\ x, & \text{if } y=0 \end{cases}$$

and a similar one is introduced in D, that is:

$$m\dagger n = \begin{cases} m+n \cup\{m, n\}, & \text{if } m, n \neq 0 \text{ and } m \neq -n \\ D, & \text{if } x = -y \\ m, & \text{if } n=0 \end{cases}$$

Then (D,†,·) is a division hyperring and M endowed with the hypercomposition "†" becomes a hypermodule over (D,†,·), which is not strongly distributive.

Let M be a module over a non commutative field K and let the equivalence relation g be defined as follows:

$$(x,y) \in g \Leftrightarrow x=qy, \; q\in K^*$$

then, according to Proposition 3.1, M_g becomes a strongly distributive hypermodule over the quotient hyperfield K/K*={0,K*}. If the construction which is presented in Proposition 3.2 is applied to this hypermodule and also, if the elements of M_g-{0} are defined as points and the result of the hypercomposition $x_g \dagger y_g = (x_g \underset{\bullet}{+} y_g) \cup \{x_g, y_g\}$ of any two points x_g, y_g with $x_g \neq y_g$, are defined as lines, then an analytic projective geometry is formed. Moreover all analytic projective geometries can derive using this method (see also [11]).

Furthermore, applying the construction of Proposition 3.2 in the vector hyperspace $\overline{V}$, the two participating elements $\overline{x}, \overline{y}$ belong to their hypersum $\overline{x} \dagger \overline{y}$, giving thus closed arcs on the hypersphere S of V. Also $\overline{x} \dagger (-\overline{x}) = \overline{V}$, i.e. any two opposite points generate the whole hypersphere (which derives as the result of their hypercomposition). This construction is very natural, since two opposite points define infinitly many great circles that contain all the points of the sphere. Thus every Euclidian spherical geometry can be described algebraically as a quotient hypermodule.

Proposition 3.3. *Let R be a hyperring, then R^n is a hypermodule over R which is not strongly distributive.*

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