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A Mathematica Package Enumerating Hypergroups of Order 3

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We provide a Mathematica package that can be applied to a set of hypergroupoids and sort out the ones which are hypergroups. This, easy to handle program, is also applied to each and every hypergroupoid of order 3 (which are defined through another routine), picks out the ones which are hypergroups and calculates their number. Also, it separates all the hypergroups of order 3 to isomorphic classes and gives their cardinalities. Previous results are confirmed by the above routines.

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1 Introduction

In a non empty set H , a hypercomposition is a function from $H \times H$ to the powerset $P(H)$ of H . This notion was introduced in Mathematics together with the notion of hypergroup by F. Marty [1]. The axioms that endow the pair (H, \cdot) of a nonempty set H and a hypercomposition " \cdot " with the hypergroup structure are:

- i. $a(bc) = (ab)c$ for every $a, b, c \in H$ (associativity)
- ii. $aH = Ha = H$ for every $a \in H$ (reproductivity)

If only (i) is valid then (H, \cdot) is called semihypergroup, while it is called quasi-hypergroup if only (ii) holds. The result $a \cdot b$ is non void, for every $a, b \in H$, Mittas [5]. Yet if " \cdot " is a hypercomposition in a set H and A, B are subsets of H , then $A \cdot B$ signifies the union:

$$A \cdot B = \cup_{(a,b) \in A \times B} a \cdot b.$$

$A \cdot b$ and $a \cdot B$ will have the same meaning as $A \cdot \{b\}$ and $\{a\} \cdot B$ respectively. Also, when nothing opposes it there is no distinction between the elements and their corresponding singletons.

2 The method

A hypergroupoid is a set $H \neq \emptyset$ with a hypercomposition " \cdot " which is not necessarily associative or reproductive. Regarding the notification of the elements of the hypergroupoids of order 3, it can be assumed that they share the set $H = \{1, 2, 3\}$.

The hypercompositions in H are defined through the following table

\cdot	1	2	3	
1	a_{11}	a_{12}	a_{13}	
2	a_{21}	a_{22}	a_{23}	
3	a_{31}	a_{32}	a_{33}	(1)

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where $a_{ij} \subseteq H, i, j = 1, 2, 3$. The elements a_{ij} 's are chosen among the seven element set

$$\Lambda = P(H) \setminus \emptyset = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

If \mathcal{H}_3 denotes the set of all hypergroupoids of third order then for its cardinality holds: $|\mathcal{H}_3| = 7^9 = 40353607$. Migliorato [4] found, by computer, the total number $N_3 = 23192$ of hypergroups of order 3 while Nordo [6] computed using a program written in PASCAL the number $S_3 = 3999$ of non isomorphic hypergroups of the same order.

The mathematica package given in the Appendix is based on two functions, namely `ReproductivityTest[]` and `AssociativityTest[]`, which check out the corresponding properties. Their argument is a set of hypergroupoids in a list and their output is a True/False table.

The reproductivity of the hypercompositions defined in (1) can be checked, through the verification of validity of the equivalent (to this axiom) equalities:

$$\bigcup_{j=1}^3 a_{ij} = H, \text{ for } i = 1, 2, 3 \quad \text{and} \quad \bigcup_{i=1}^3 a_{ij} = H, \text{ for } j = 1, 2, 3. \quad (2)$$

The cases that pass successfully this first test (i.e. the reproductivity's validity test) are going through the associativity's validity test, which is checking all the 27 possible triples $a(bc) = (ab)c$.

2.1 Classes of isomorphism

A hypergroupoid of order 3, is isomorphic with another 5 hypergroupoids. This derives from interchanges among the elements of the set H . More precisely

- (i) keep 1 interchange 2,3
- (ii) keep 3 interchange 1,2
- (iii) change 1 by 2 change 2 by 3 change 3 by 1
- (iv) change 1 by 3 change 2 by 1 change 3 by 2
- (v) keep 2 interchange 1,3

So for the above matrix of hypercomposition (1) there derive the following five isomorphic hypercompositions:

(i)	1	2	3	(ii)	1	2	3	(iii)	1	2	3	(iv)	1	2	3	(v)	1	2	3	
	1	\bar{a}_{11}	\bar{a}_{13}	\bar{a}_{12}		\bar{a}_{22}	\bar{a}_{21}	\bar{a}_{23}		\bar{a}_{33}	\bar{a}_{31}	\bar{a}_{32}		\bar{a}_{22}	\bar{a}_{23}	\bar{a}_{21}		\bar{a}_{33}	\bar{a}_{32}	\bar{a}_{31}
	2	\bar{a}_{31}	\bar{a}_{33}	\bar{a}_{32}		\bar{a}_{12}	\bar{a}_{11}	\bar{a}_{13}		\bar{a}_{13}	\bar{a}_{11}	\bar{a}_{12}		\bar{a}_{32}	\bar{a}_{33}	\bar{a}_{31}		\bar{a}_{23}	\bar{a}_{22}	\bar{a}_{21}
	3	\bar{a}_{21}	\bar{a}_{23}	\bar{a}_{22}		\bar{a}_{32}	\bar{a}_{31}	\bar{a}_{33}		\bar{a}_{23}	\bar{a}_{21}	\bar{a}_{22}		\bar{a}_{12}	\bar{a}_{13}	\bar{a}_{11}		\bar{a}_{13}	\bar{a}_{12}	\bar{a}_{11}

where \bar{a}_{ij} are the subsets that derive from the transposition of the corresponding a_{ij} 's of the original matrix and the proper replacement of their elements.

3 Examples and results

Let's assume that it must be verified whether or not the following three hypergroupoids are hypergroups.

·	1	2	3	·	1	2	3	·	1	2	3
1	{1}	{2}	{1, 2, 3}		{1, 3}	{3}	{2}		{1, 2, 3}	{3}	{2, 3}
2	{2}	{1, 2, 3}	{1, 2, 3}		{2}	{1, 2, 3}	{2}		{1}	{3}	{3}
3	{1, 2, 3}	{1, 2, 3}	{1, 3}		{2}	{1}	{1, 2, 3}		{3}	{1, 2}	{1, 3}

we write:

```
In[1]:=HyperGroupTest.m
In[2]:=h1={{ {1}, {2}, {1, 2, 3}}, { {2}, {1, 2, 3}, {1, 2, 3}}, { {1, 2, 3}, {1, 2, 3}, {1, 3}}};
In[3]:=h2={{ {1, 3}, {3}, {2}}, { {2}, {1, 2, 3}, {2}}, { {2}, {1}, {1, 2, 3}}};
In[4]:=h3={{ {1, 2, 3}, {3}, {2, 3}}, { {1}, {3}, {3}}, { {3}, {1, 2}, {1, 3}}};
In[5]:=HyperGroupTest[{h1, h2, h3}]
Out[5]= {True, False, False}
```

From the last line (Out[5]) it derives that only the first hypercomposition defines a hypergroup in H . It is obvious though, that the second hypercomposition satisfies the reproductivity axiom and so the corresponding hypergroupoid is a quasi-hypergroup. For performing the test only to the second hypergroupoid, it must be written:

```
In[5]:=HyperGroupTest[{h2}]
Out[5]= {False}
```

In order to evaluate N_3 all the 40-million hypergroupoids must be checked. First Λ is formed and then all the $7^3 = 343$ triads (in variable a3) and $7^6 = 343^2$ hexads (in variable a6) with elements from this set. Their combination in one and two rows respectively forms hypercomposition matrices of the form (1). Thus the memory requirements of a 7^9 length list containing the description of hypercompositions can be overcome. This can be done writing the lines below where variable HyperGroups3 collects the Hypergroups we find.

```
In[6]:=lambda = Drop[Subsets[{1, 2, 3}], 1];
In[7]:=a6=Tuples[lambda, 6];
In[8]:=a3 = Tuples[lambda, 3];
In[9]:=HyperGroups3 = {};
In[10]:=Do[
  temp={Partition[Join[a3[[j1]], a6[[j2]]], 3]};
  If[HyperGroupTest[temp][[1]],HyperGroups3 = Join[HyperGroups3, temp]]
  , {j1, 1, 343}, {j2, 1, 343^2}];
In[11]:=Print[Length[HyperGroups3]]
Out[11]=23192
```

A function that gives the six hypercompositions which form isomorphic hypergroupoids is given by simply accounting the observations in subsection (2.1).

```
IsomorphTest[a_List] := Module[{},
q1=ReplaceAll[{{a[[1,1]],a[[1,3]],a[[1,2]]},{a[[3,1]],a[[3,3]],a[[3,2]]},
  {a[[2,1]],a[[2,3]],a[[2,2]]}},{(2)->(3),(3)->(2),(1,2)->(1,3),(1,3)->(1,2)}];
q2=ReplaceAll[{{a[[2,2]],a[[2,1]],a[[2,3]]},{a[[1,2]],a[[1,1]],a[[1,3]]},
  {a[[3,2]],a[[3,1]],a[[3,3]]}},{(1)->(2),(2)->(1),(2,3)->(1,3),(1,3)->(2,3)}];
q3=ReplaceAll[{{a[[3,3]],a[[3,1]],a[[3,2]]},{a[[1,3]],a[[1,1]],a[[1,2]]},
  {a[[2,3]],a[[2,1]],a[[2,2]]}},{(1)->(2),(2)->(3),(3)->(1),(1,2)->(2,3),
  (2,3)->(1,3),(1,3)->(1,2)}];
q4=ReplaceAll[{{a[[2,2]],a[[2,3]],a[[2,1]]},{a[[3,2]],a[[3,3]],a[[3,1]]},
  {a[[1,2]],a[[1,3]],a[[1,1]]}},{(1)->(3),(2)->(1),(3)->(2),(1,2)->(1,3),
  (2,3)->(1,2),(1,3)->(2,3)}];
q5=ReplaceAll[{{a[[3,3]],a[[3,2]],a[[3,1]]},{a[[2,3]],a[[2,2]],a[[2,1]]},
  {a[[1,3]],a[[1,2]],a[[1,1]]}},{(1)->(3),(3)->(1),(1,2)->(2,3),(2,3)->(1,2)}];
  {a, q1, q2, q3, q4, q5}
];
```

In order to count the number of the different non isomorphic classes of hypergroups of order 3, a 6 digit array, called cardinalities is used by the program. Each time the routine encounters a non isomorphic class, it drops it from HyperGroups3.

```
In[12]:=cardinalities = {0, 0, 0, 0, 0, 0};
In[13]:=While[Length[HyperGroups3] > 0,
  temp = Flatten[Table[Position[HyperGroups3, IsomorphTest[HyperGroups3[[1]][[j1]]],
  1], {j1, 1, 6}], 1]; len = Length[Union[temp]];
  cardinalities[[len]] = cardinalities[[len]] + 1;
  HyperGroups3 = Delete[HyperGroups3, temp]
]
In[14]:=Total[cardinalities];
Out[14]=3999
In[15]:=Print[cardinalities];
Out[15]={6, 10, 244, 0, 0, 3739}
```

So we found that $S_3 = 3999$, and it is confirmed by the cardinalities of the isomorphic classes that $6 \cdot 1 + 10 \cdot 2 + 244 \cdot 3 + 3739 \cdot 6 = N_3$.

4 Conclusion

Generally speaking, few things are known about the construction of finite hypergroups. For example it is known that if (H, \cdot) is a group or a hypergroup, then the (H, \diamond) with $a \diamond b = a \cdot b \cup \{a, b\}$ is a hypergroup [2]. Thus using Cayley's theorem a family of finite hypergroups can be constructed based on finite groups. From the above analysis it derives that there are $7^9 = 40353607$ hypergroupoids of order 3, 23192 of these are hypergroups. The group of order 3 is among them, as well as the corresponding hypergroup constructed as above. The set of 23192 hypergroups is partitioned in 3999 equivalence classes. The 3739 of the above classes consists of 6 members, the 244 consists of 3 members, the 10 have 2 members and the last 6 are one member classes. The total hypergroup, that is the hypergroup in which the result of the hypercomposition consists always of all the elements of the hypergroup, is in the set C_1 of the six, one member, classes. In the same set belongs the B-hypergroup, i.e. the hypergroup in which the result of the hypercomposition consists only of the two elements which participate to the hypercomposition [3]. Relative to the B-hypergroup are two other non isomorphic hypergroups in which the hypercomposition is defined as follows (see [2])

$$ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H & \text{if } a = b \end{cases} \quad \text{and} \quad ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H \setminus \{a\} & \text{if } a = b \end{cases}$$

These hypergroups, when they have order 3, belong also to C_1 . But the observation of the last two one member equivalence classes leads to a general construction of two hypergroups. More precisely:

Proposition: *Let H be an arbitrary set with more than 2 elements. Then the hypercompositions*

$$ab = \begin{cases} H & \text{if } a \neq b \\ a & \text{if } a = b \end{cases} \quad \text{and} \quad ab = \begin{cases} \{a, b\} & \text{if } a \neq b \\ H \setminus \{a\} & \text{if } a = b \end{cases}$$

define in H two non isomorphic hypergroups.

It is worth mentioned that the number of the classes of hypergroups that can be constructed by the known Propositions and Theorems is very small comparing to the existing 3999 classes of hypergroups with three elements. Also one can notice that the ratio of hypergroups to hypergroupoids is exceptionally small since we meet on hypergroup in every 1740 hypergroupoids.

5 Appendix

The Mathematica package that implements the two basic properties (associativity and reproductivity) for testing if a hypergroupoid is indeed a Hypergroup follows.

```
BeginPackage["HyperGroupTest`"]; Clear["HyperGroupTest`*"];

HyperGroupTest::usage = "HyperGroupTest[LookUpTable] tests if
hypergroupoid operation given LookUpTable forms a HyperGroup"

Begin["`Private`"]; Clear["HyperGroupTest`Private`*"];

HyperGroupTest[LookUpTable_List] :=
Table[If[ReproductivityTest[LookUpTable[[j]]],
If[AssociativityTest[LookUpTable[[j]]], True,
False], {j, 1, Length[LookUpTable]}];

AssociativityTest[LookUpTable1_List] :=
Module[{i, j, k, test},
i = 1; j = 1; k = 1; test = True;
While[test && i <= 3,
test = Union[Flatten[Union[Extract[LookUpTable1,
Distribute[{{LookUpTable1[[i, j]], {k}}, List]]]]] ==
Union[Flatten[Union[Extract[LookUpTable1,
Distribute[{{i}, LookUpTable1[[j, k]], List]]]]];
k = k + 1; If[k > 3,
k = 1; j = j + 1;
If[j > 3, i = i + 1; j = 1];
];
];
Return[test]
```

```
];

ReproductivityTest[LookUpTable1_List] :=
  Union[Apply[Union, LookUpTable1, 1]] == {{1,2,3}} &&
  Union[Apply[Union, Transpose[LookUpTable1], 1]] == {{1,2,3}};

End[];
EndPackage[];
```

In the package above the function `AssociativityTest[]` is implemented by using `While`. In the most of the tested hypercompositions the property of associativity failed after the first 2 or 3 checks. Consequently it was not necessary to go through all 27 cases for hypergroups of order 3. Contrarily the function `ReproductivityTest[]` tested all rows and columns simultaneously according to property (2), since this does not increase computational time.

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