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TRANSPOSITION POLYSYMMETRICAL HYPERGROUPS

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ABSTRACT. This paper presents various examples and properties of Transposition Polysymmetrical hypergroups, which are hypergroups that satisfy a postulated property of transposition, have a nonscalar identity and contain at least one symmetric element for each one of their nonidentity elements. This type of hypergroups initially appeared, in their commutative case, during the study of the theory of Languages and Automata from the standpoint of hypercompositional structures theory.

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1. INTRODUCTION – HYPERGROUPS RESULTING FROM AUTOMATA

The operational details of the automaton and the intention to describe its structure, led to the introduction of the several types of associated (or attached) hypergroups, e.g. the associated hypergroups of the order, of the grade, of the paths, of the operation [9,10], some of which have enriched the class of the hypergroups with new types of hypergroups. Such hypergroups are: the **Join Polysymmetrical Hypergroup**, which is analyzed below, and the **Fortified Join Hypergroup**. Before proceeding to the definitions of these hypergroups, let us recall that a transposition hypergroup (see [2]) is a hypergroup that satisfies a postulated property of transposition i.e. $(b\backslash a) \cap (c/d) \neq \varnothing \Rightarrow (ad) \cap (bc) \neq \varnothing$, where $a/b = \{x \in H \mid a \in xb\}$ and $b\backslash a = \{x \in H \mid a \in bx\}$ are the induced inverse hypercompositions [5]. A join space, also join hypergroup, is a commutative transposition hypergroup. Now a **Fortified Transposition Hypergroup** is a transposition hypergroup H, having an identity or neutral element e such that ee=e,

$x \in ex=xe$, for every $x \in H$ and also for every $x \in H-\{e\}$ there exists a unique element $x' \in H-\{e\}$ such that $e \in xx'$, and, furthermore, x' satisfies $e \in x'x$ [3]. If H is a join hypergroup, then we have the **Fortified Join Hypergroup** [8,13,14]. This last hypergroup resulted from the theory of Languages [7,8,10,11].

Indeed, the necessity of the introduction of the "null word" as an element "0" bilaterally absorbing with regard to the multiplication (concatenation of the words) in the free monoid A* which is generated by the alphabet A has lead to the introduction of the **dilated B-hypergroup** [11,8,10] i.e. a fortified join hypergroup for which $x+y = \{x, y\}$ if $x \neq y$ and $x+x = \{x, 0\}$. Furthermore, the concatenation of the words in A* which is bilaterally distributive over the hypercomposition of the B-hypergroup, leads, in a natural way, to the definition of the **hyperringoid** and the **linguistic hyperringoid** [15]. Moreover, the associated hypergroups of the order and of the grade are directly connected to the minimization of the automaton. The construction of these hypergroups involves the notions of the order of a state and of the grade of a state respectively. **Order of a state** is the minimum of the lengths of all the words that lead to this state from the conventional start state. **Grade of a state** s_i is the set of the elements of the language over an alphabet A which lead from s_i to a final state of the automaton [10]. With the use of the notion of the grade, there can be introduced an equivalence relation in the set of the states, named **grade equivalence**, as follows:

s_i R s_j when the grade of s_i is equal to the grade of s_j

Considering that the automaton has only one final state s_F (real or conventional), the set of its states can be equipped with a hypercomposition in the following way

$$s_i + s_j = \begin{cases} C_{si}^R \cup C_{sj}^R & \text{if } C_{si}^R \neq C_{sj}^R \text{ and } s_i, s_j \neq s_F \\ C_{si}^R \cup \{s_F\} & \text{if } C_{si}^R = C_{sj}^R \end{cases}$$

where C_{si}^R is the equivalence class of s_i. With this hypercomposition, which is derived from the notion of the grade of a state, the set of the states of the automaton becomes a hypergroup. Moreover this hypergroup is an example from a whole new class of hypergroups, the transposition polysymmetrical ones.

Definition 1.1. A **Transposition Polysymmetrical Hypergroup (TPH)** is a transposition hypergroup $(H,\cdot)$ that contains an element e which satisfies the axioms:

TP1. $ee=e$

TP2. $x \in xe = ex$, for every $x \in H$

TP3. for every $x \in H-\{e\}$ there exists at least one element $x' \in H-\{e\}$, a **symmetric** of x, such that $e \in x x'$, and $e \in x'x$

The set of the symmetric elements of x is denoted by S(x) and it is called the **symmetric set** of x. S(e) contains only the element e. A commutative transposition polysymmetrical hypergroup is called **Join Polysymmetrical Hypergroup (JPH)**.

In a TPH the identity need not be unique. This becomes apparent from the Example 2.3, which is presented later on in the context of the next section.

Example 1.1. Let K be a field and G a subgroup of its multiplicative group. In K we define a hypercomposition as follows:

$$x \dagger y = \{ xp+yq \mid p,q \in G \}$$

Then (K, †) is a join polysymmetrical hypergroup having the 0 of K as its neutral element. The symmetric set of an element x of K is $S(x) = \{-xp \mid p \in G \}$

The construction of the above Example was inspired by the construction of M. Krasner's quotient hyperfields [4]. The next Example gives a construction of a JPH which is a generalization of the associated (or attached) grade hypergroup:

Example 1.2. Let E be a nonempty set and R an equivalence relation in it. If e is an element of E and its class is $C_e^R = \{e\}$, then (E, +) becomes a JPH where the hypercomposition "+" is defined as follows:

$$x + y = \begin{cases} C_x \cup C_y, & \text{if } C_x \neq C_y \\ C_x \cup C_e, & \text{if } C_x = C_y \end{cases}$$

Regarding the hypercompositional structures in automata, there exists also the paper [1], where the authors introduce a hypergroup which is associated to the states of an automaton. This hypergroup is based on the hypercomposition $s \circ t = \delta(s,A^*) \cup \delta(t,A^*)$ where δ is the transition function, s,t are states of the automaton and A* is the free monoid of words over the alphabet A. Using this hypergroup the authors of [1] reached very interesting results in automata theory.

The TPH is not the only example of polysymmetrical hypergroups. J. Mittas has dealt extensively with certain polysymmetrical hypergroups (see [17,19]) and so did S. Ioulidis [18] and C. Yatras [21]. In his paper [17], J. Mittas, starting with a remark regarding algebraically closed fields, was led to a special type of completely regular hypergroup, which he named polysymmetrical and subsequently, C. Yatras called it **M-polysymmetrical hypergroup.** A M-polysymmetrical hypergroup (H,+) is a commutative hypergroup which also satisfies the axioms:

M-P1. $(\exists 0 \in H)\ (\forall x \in H)\ [x \in 0+x]$
M-P2. $(\forall x \in H)\ (\exists x' \in H)\ [x+x'=0]$
M-P3. for every $x,y,z \in H$, $x' \in S(x)$, $y' \in S(y)$, $z' \in S(z)$ we have $z \in x+y \Rightarrow z' \in x'+ y'$

Another type of polysymmetrical hypergroup was studied by J. Mittas and S. Ioulidis in [18]. This hypergroup satisfies the axioms:

P1. $(\exists e \in H)\ (\forall x \in H)\ [x = ex = xe]$

P2. $(\forall x \in H)\ (\exists x' \in H)\ [e \in xx' \cap x'x]$

Next, J. Mittas in his paper [19], enriched the commutative case of the above hypergroup with the axiom:

CP. $z \in xy \Rightarrow (\exists\, x' \in S(x))\ [y \in zx']$

and thus he defined the **canonical polysymmetrical hypergroup**.

It is known that transposition holds in the case of canonical hypergroups. So the question arises: Does transposition hold in the case of canonical polysymmetrical hypergroups? The answer is negative as is shown by the following example:

Example 1.3. Let H be a set totally ordered and symmetric around a center, denoted by $0 \in H$ with regard to which a partition $H = H^- \cup \{0\} \cup H^+$ can be defined, such that $x<0<y$ for every $x \in H^-$, $y \in H^+$ and $x \le y$ implies $-y \le -x$ for every $x,y \in H$ (where $-x$ is the symmetric of x with regard to 0). Then H with hypercomposition

$$x+y = \begin{cases} y, & \text{if } |x| \le |y| \text{ for every } x,y \in H^- \cup \{0\} \text{ or } x,y \in \{0\} \cup H^+ \\ [x,y], & \text{if } x \in H^- \text{ and } y \in H^+ \end{cases}$$

becomes a canonical polysymmetrical hypergroup (see [19]). Suppose now that $x,y,a,b \in H^+$ and $x<y<a<b$. Then $x/y = a/b = H^-$. Thus $x/y \cap a/b \neq \varnothing$. But $x+b = \{b\}$, $y+a = \{a\}$ and so $x+b \cap y+a = \varnothing$ i.e. transposition is not valid.

2. The Reversibility and other Properties of the TPH

The axiom CP of the canonical polysymmetrical hypergroups, or the axiom M-P3 of the M-polysymmetrical hypergroups is known as the axiom of reversibility. This axiom appears to the definition of other hypergroups as well (e.g. canonical hypergroups). The basic concept of reversibility is the transformation of the relation $z \in xy$ with the use of their symmetric elements. For example, in the case of quasicanonical hypergroups, we have that $z \in xy$ implies $y \in x^{-1}z$ and $x \in zy^{-1}$ [6] (an extensive analysis of the reversibility is given in [19]). Keeping in mind the general definition of the completely regular hypergroups (according to Marty's definition) [5] the property of reversibility in the case of TPHs can be formulated as follows:

In the relation $z \in xy$ the reversibility holds if there exists $x' \in S(x)$, or $y' \in S(y)$ such that either $y \in x'z$ or $x \in zy'$.

This type of reversibility is called **partial**, since reversibility may hold in a stronger way, that is $z \in xy$ implies that there exist $x' \in S(x)$ and $y' \in S(y)$ such that $y \in x'z$ and $x \in zy'$. If reversibility is satisfied in this way, then it is called **complete**. Moreover, if in a TPH H reversibility holds for every relation $z \in xy$ with $x,y,z \in H$, then we say that the reversibility holds in H.

The hypergroup of Example 1.1. is a TPH in which complete reversibility holds. There exist though TPHs in which partial reversibility holds and others in which reversibility is not valid at all. This becomes evident from the following examples:

Example 2.1. a) Let (G,+) be any commutative group and let P be a subgroup of G. Then G becomes a JPH if it is equipped with the hypercomposition:

$$x \dagger y = x+y+P \quad \text{and} \quad 0 \dagger 0 = 0$$

Indeed, it is obvious that $(G,\dagger)$ is a P-hypergroup of G [20] which is modified in the definition of $0\dagger 0$. Next, for the induced inverse hypercomposition it holds: $x/y=x-y+P$. Regarding the proof of the transposition axiom, let $x/y \cap z/w \neq \varnothing$. Then $(x-y+P)\cap(z-w+P) \neq \varnothing$. So there exist $a,b \in P$ such that $x-y+a=z-w+b$, from where it derives that $x+w+a=z+y+b$ and therefore $(x+w)\cap(z+y)\neq\varnothing$. The nonscalar neutral element is 0, since $x \in x\dagger 0=x+P$. Also, if $x\neq 0$ is an arbitrary element of G, then $S(x)=-x+P$ (note that $-P=P$). Now if $x \in P$, then $0 \in x+0$, but $x \notin 0+0$. Hence partial reversibility is valid in $0 \in x+0$, while complete reversibility holds in all the other cases. Indeed, if $x \in y\dagger w$ then $x=y+w+c$ for some element c from P. Thus, if $y'=-y-c$, then y' belongs to S(y) and $w \in x\dagger y'$. Similarly, there exists $w' \in S(w)$ such that $y \in x\dagger w'$.

b) From the above JPH we can construct another one, equipping G with the hypercomposition: $x \star y = (x\dagger y) \cup \{x, y\}$. In this JPH, partial reversibility holds for the relations $x \in x\star y$, when $x \notin x\dagger y$, or $x=0$, while complete reversibility holds for all the other cases.

Example 2.2. Let Δ_i, $i \in I$ (card $I \geq 2$) be a family of totally ordered sets which have a common minimum element e. The set $\Delta = \cup_{i\in I}\Delta_i$ with hypercomposition:

$$xy = \begin{cases} [\, \min\{x, y\}\, ,\, \max\{x, y\}\,] & \text{if } x, y \in \Delta_i,\ i \in I \\ [\, e, x\,] \cup [\, e, y\,] & \text{if } x \in \Delta_i,\ y \in \Delta_j \text{ and } i \neq j,\ i, j \in I \end{cases}$$

becomes a JPH with neutral element e. Indeed, For the proof of the associativity we have the cases:

i) if $x,y,w \in \Delta_i$, then $(x+y)+w = x+(y+w) = [\, \min\{x,y,w\}, \max\{x,y,w\}\,]$

ii) if $x,y \in \Delta_i$ and $w \in \Delta_j$ with $i \neq j$, then

$$(x+y)+w = x+(y+w) = [\, 0, \max\{x,y\}\,] \cup [\, 0, w\,]$$

iii) if x,y,w belong to different sets, then

$$(x+y)+w = x+(y+w) = [\,0, x\,] \cup [\,0, y\,] \cup [\,0, w\,]$$

We note that in this hypergroup, for the induced hypercomposition, it holds:

$$x/y = \begin{cases} \Delta & \text{if } x=y \\ [\,e, x\,] \cup (\cup_{i\neq j} \Delta_i) & \text{if } x,y \in \Delta_j \text{ and } x<y \\ \{\, w \in \Delta_j \mid w \geq x \,\} & \text{if } x,y \in \Delta_j \text{ and } x>y \\ \{\, w \in \Delta_j \mid w \geq x \,\} & \text{if } x \in \Delta_j \text{ and } y \in \Delta_i \end{cases}$$

Therefore $x/y \neq \varnothing$ and consequently the hypercompositional structure is a hypergroup. The transposition axiom's validity can be verified in all the cases, e.g. if x,y,w,z belong to the same ordered set and x<y<w<z, then $x/y \cap w/z \neq \varnothing$, from where $(x+z) \cap (w+y) = [y, w] \neq \varnothing$. Moreover, if $x \in \Delta_j$, then $S(x) = \cup_{i\neq j} \Delta_i$. Thus, regarding the reversibility, for every $x,y \in \Delta_i$ with $x<y$, we have $z \in xy \Rightarrow x \in zy'$ for every $y' \in S(y)$, if $z \neq y$, while for every $x' \in S(x)$ we have $y \notin x'z$. Therefore partial reversibility holds. Similarly, partial reversibility holds in all the other cases.

Example 2.3. Let R^2 be the Cartesian plane and let the result of the hypercomposition xy of two elements $x,y \in R^2$ be the closed line segment that they define, if $x \neq y$ and $xx=\{x\}$. Considering $e=(0, 0)$ to be the neutral element, R^2 becomes a JPH in which the reversibility is not valid. Indeed, for every nonzero element x, the set S(x) is the opposite open half-line of x, starting from (0, 0). From figure 1, it becomes obvious that the reversibility is not valid, since neither x belongs to wS(y), nor y belongs to wS(x).

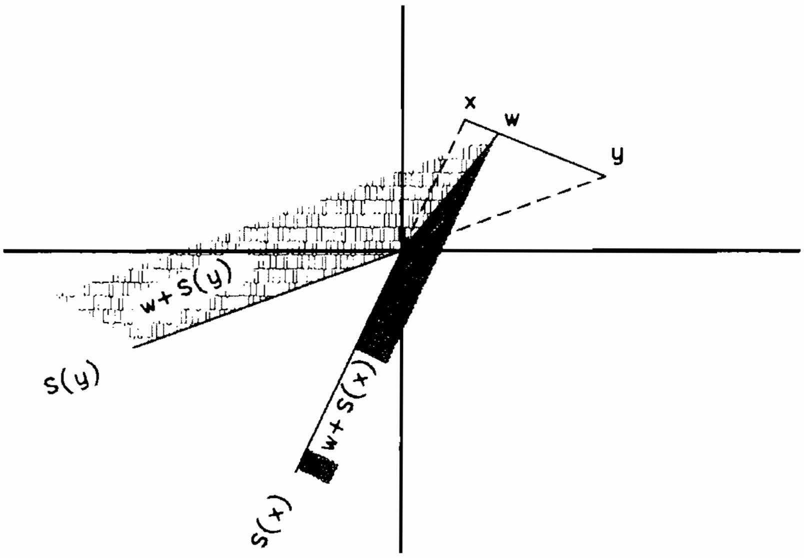

Figure 1

On the contrary, the Cartesian line equipped with the same hypercomposition:

$$xy = [\min(x, y), \max(x, y)]$$

becomes a JPH in which the partial reversibility is valid.

From this example it also derives that in a TPH the identity element e need not be unique.

Proposition 2.1. *If in a TPH (H,·) the sets S(x), x ∈ H define a regular partition in H, then the quotient set H:S = {S(x) | x ∈ H}, equipped with the hypercomposition:*

$$S(x) \odot S(y) = \{ S(w) \mid w' \in x' \cdot y', \text{ with } w' \in S(w), x' \in S(x), y' \in S(y) \}$$

becomes a fortified transposition hypergroup.

P r o o f. Since S is regular, the quotient set H:S is a transposition hypergroup. Next we remark that S(S(x)) is a class. Indeed, if $x', x'' \in S(x)$, then $x \in S(x')$ and $x \in S(x'')$. Thus $S(x') \cap S(x'') \neq \varnothing$ and so $S(x') = S(x'')$, since the symmetric sets make a partition in H. Moreover e is the neutral element of H:S, while S(S(x)), which is a class, is the symmetric of S(x).

Example 2.4. For the JPH E of the Example 1.2. it holds $S(x)=C_x$, $x \in E$. Thus $E:R=E:S=\{S(x) \mid x \in E\}$. Also it is clear that the partition defined by R is a regular one and that for the hypercomposition "⊙" we have:

$$S(x) \odot S(y) = \{ S(x), S(y) \} \quad \text{if} \quad S(x) \neq S(y)$$
$$S(x) \odot S(x) = \{ S(x), e \}$$

The fortified transposition hypergroup E:R that derives is a dilated B-hypergroup.

3. ATTRACTIVE AND NON ATTRACTIVE ELEMENTS IN THE TPH, SYMMETRIC SUBHYPERGROUPS.

As it derives from the above Examples 2.1 - 2.3, it is possible for the neutral element e either to belong or not to belong in the result of the hypercomposition ex. More precisely we have the Proposition:

Proposition 3.1. *For every element x of a TPH (H,·) holds* $ex \subseteq \{e\} \cup S(S(x))$.

P r o o f. Let $y \neq e$ and $y \in ex = xe$, then $x \in e \backslash y$ and $x \in y/e$. Moreover for every $x' \in S(x)$ holds $x \in e/x'$ and $x \in x' \backslash e$. Consequently $e/x' \cap e\backslash y \neq \varnothing$ and $y/e \cap x'\backslash e \neq \varnothing$, so $ee \cap yx' \neq \varnothing$ and $ee \cap x'y \neq \varnothing$, that is $e \in yx'$ and $e \in x'y$. Thus $y \in S(x') \subseteq S(S(x))$.

As in the case of the fortified transposition hypergroups, an element x of a TPH H, will be called **attractive** if $e \in xe$, while a nonidentity element x will be called **non attractive** if $e \notin xe$ [3,13]. We denote by A the set of the attractive elements and by C the set of non attractive elements. Then $H = A \cup C$ and $A \cap C = \varnothing$. The hypergroup of Example 1.1 is a TPH in which all the nonidentity elements are non attractive, while the TPH of Example 2.1.a consists of attractive and non attractive elements (P is the set of the attractive elements). Moreover the TPH of Examples 2.1.b, 2.2. and 2.3. consist only of attractive elements.

Regarding the attractive elements and the reversibility we remark that in a TPH which contains a nonidentity attractive element x, the complete reversibility is not valid, since $e \in xe$ but $x \notin ee=e$.

Proposition 3.2. *If x is an attractive element of a TPH, then S(x) consists of attractive elements.*

P r o o f. Let $e \in ex$. Then $x \in e\backslash e$. Moreover, if x' is an arbitrary element from S(x), then $e \in xx'$. Therefore $x \in e/x'$. Consequently $e\backslash e \cap e/x' \neq \varnothing$, from where $(ee) \cap x'e \neq \varnothing$ and so $e \in x'e$. Thus x' is an attractive element.

Corollary 3.1. *If x is a non attractive element, then S(x) consists of non attractive elements.*

Proposition 3.3. *Let $x \neq e$, then $S(x) \cup \{e\} = x\backslash e \cap e/x$, if x is attractive and $S(x) = x\backslash e \cap e/x$ if x is non attractive.*

Corollary 3.2. *If X is nonempty, $e \notin X$ and X contains an attractive element, then $S(X) \cup \{e\} = X\backslash e \cap e/X$, while $S(X) = X\backslash e \cap e/X$ if X consists of non attractive elements.*

Proposition 3.4. $x \in x/e = e\backslash x$.

Corollary 3.3. *If X is nonempty, then* $X \subseteq X/e = e\backslash X$.

Proposition 3.5. $A = e/e = e\backslash e$.

P r o o f. If x belongs to A, then $e \in xe$, and so $x \in e/e$. Also if $x \in e/e$, then $e \in xe$. Thus $A=e/e$. Dually, $A=e\backslash e$.

Proposition 3.6. *If x is an attractive element of a TPH, then all the elements of ex are attractive. Also if x is a non attractive element, then ex contains only non attractive elements.*

P r o o f. According to Proposition 3.2, if x is an attractive element, then S(x) consists of attractive elements and so S(S(x)) consists of attractive elements as well. Next, because of Proposition 3.1, $ex \subseteq \{e\} \cup S(S(x))$, thus ex contains only attractive elements. Similarly if we assume that x is a non attractive elements, then, from Corollary 3.1. and Proposition 3.1, we deduce that ex consists of non attractive elements.

Proposition 3.7. *In a TPH if x is an attractive element and y is a non attractive element, then xy and yx consists of non attractive elements.*

P r o o f. Assume that $a \in xy$ and a is attractive. Then, $x \in e \backslash e$ (Prop. 3.5.) and $x \in a/y$. So transposition gives $ea \cap ye \neq \varnothing$, contrary to Proposition 3.6. Thus xy consists of non attractive elements. Similarly, yx also consists of non attractive elements.

Corollary 3.4. *If x, y are attractive elements, then $x/y \subseteq A$ and $y \backslash x \subseteq A$*

Proposition 3.8. *The result of the hypercomposition of two attractive elements of a TPH contains only attractive elements.*

P r o o f. Let x, y be two attractive elements and suppose that z is a non attractive element which belongs to xy. Then $z \in xy$ implies $x \in z/y$. Moreover if $x' \in S(x)$, then $x \in x' \backslash e$. Thus $z/y \cap x' \backslash e \neq \varnothing$. Therefore $(ey) \cap (x'z) \neq \varnothing$ (1). But x is an attractive element, so x′ is also attractive (Prop. 3.2) and therefore $x'z$ does not contain the neutral element and it consists of non attractive elements (Prop. 3.7). Moreover y is an attractive element and therefore ey consists only of attractive elements (Prop. 3.6). Consequently the intersection $(ey) \cap (x'z)$ is the empty set, which contradicts (1).

Recall that a subhypergroup h of H (i.e a subset of H for which xh=hx=h, for every $x \in h$) is right closed (resp. left closed) if for every $a \in H-h$ it holds: $(ah) \cap h = \varnothing$ (resp. $(ha) \cap h = \varnothing$). h is closed if it is right and left closed (for more details see [12]).

Proposition 3.9. *The set A of the attractive elements of a TPH (H, ·) is a closed subhypergroup of H.*

P r o o f. According to Proposition 3.8, if $x \in A$, then $xA \subseteq A$. Next, let y be an arbitrary element of A. We shall prove that $y \in xA$. Indeed, if x is an element of A, then its symmetric set is a subset of A (Prop. 3.2). Thus if $x' \in S(x)$, then $x'y \subseteq A$ and $y \in ey \subseteq (xx')y = x(x'y)$. Therefore there exists $z \in x'y$ such that $y \in xz \subseteq xA$. Thus xA = A. Similarly Ax = A, hence A is a subhypergroup of H. Now if w belongs to H-A, i.e. if w is a non attractive element, then, because of Propositions 3.7 and 3.6, we have that $wA \subseteq H-A$. Therefore $(wA) \cap A = \varnothing$ and so A is right closed. Similarly A proves to be left closed and thus A is a closed subhypergroup of H.

Proposition 3.10. *If $x \in C$, then $A \subseteq xC \cap Cx$.*

P r o o f. Let x be a non attractive element of H. Then from the reproductive axiom (i.e. xH=Hx=H), it derives that if y is an attractive element, then there exist elements $z,w \in H$ such that $y \in xz$ and $y \in wx$. This elements z,w can not be attractive, because if they were such, then (Prop. 3.7) the sets xz and wx would contain only non attractive elements and so they could not contain y which is attractive. Thus z,w are non attractive elements and so the Proposition.

Corollary 3.5. *The set C of the non attractive elements of a TPH $(H,\cdot)$ is not stable under the hypercomposition.*

From Propositions 3.7, 3.9 and Corollary 3.5 it derives that:

Proposition 3.11. *The set A of the attractive elements of a TPH $(H,\cdot)$ is the minimum (in the sense of inclusion) closed subhypergroup of H.*

In the fortified transposition hypergroups, subhypergroups with special interest are the symmetric ones [3,13,14,16]. Respective interest appears in the symmetric subhypergroups of the TPH.

Definition 3.1. A subhypergroup h of a transposition polysymmetrical hypergroup is called **symmetric**, if $x \in h$ implies $S(x) \subseteq h$.

Subsequent to the definition we remark that {e} and A are symmetric subhypergroups.

Proposition 3.12. *A nonempty subset h of a TPH $(H,\cdot)$ is a symmetric subhypergroup of H if and only if $xS(y) \subseteq h$ and $S(y)x \subseteq h$, for every $x,y \in h$.*

P r o o f. The above condition is obviously valid when h is a symmetric subhypergroup of H. Conversely now, suppose that x belongs to h. Then $xS(x) \subseteq h$ and so $e \in h$, which implies $eS(x) \subseteq h$ and so $S(x) \subseteq h$. Next, for the proof of the reproductive axiom we consider an arbitrary element y of h. Then it holds $S(y) \subseteq h$ and $y \in S(y')$ for every y′ in S(y). So for every $x \in h$, $xy \subseteq xS(y') \subseteq h$ is valid. Thus $xh \subseteq h$. By duality $hx \subseteq h$. Also $S(x)y \subseteq h \Rightarrow xS(x)y \subseteq xh \Rightarrow ey \subseteq xh \Rightarrow y \in xh$ That is $h \subseteq xh$. Dually $h \subseteq hx$. Hence $xh = hx = h$, for every $x \in h$.

Proposition 3.13. *h is a symmetric subhypergroup if and only if $hh=h$ and $S(h)=h$.*

Proposition 3.14. *The intersection of two symmetric subhypergroups is a symmetric subhypergroup.*

P r o o f. Let h_1, h_2 be two symmetric subhypergrous of H. Then $e \in h_1 \cap h_2$ and $S(x) \subseteq h_1 \cap h_2$ for every $x \in h_1 \cap h_2$. Next let x be an element of $h_1 \cap h_2$. Then $x(h_1 \cap h_2) \subseteq xh_1 = h_1$ and $x(h_1 \cap h_2) \subseteq xh_2 = h_2$. Therefore $x(h_1 \cap h_2) \subseteq h_1 \cap h_2$. Now let $y \in h_1 \cap h_2$ and $x' \in S(x)$. Then $y \in ey \subseteq (xx')y = x(x'y) \subseteq x(h_1 \cap h_2)$, so $h_1 \cap h_2 \subseteq x(h_1 \cap h_2)$ and therefore $h_1 \cap h_2 = x(h_1 \cap h_2)$. Similarly $h_1 \cap h_2 = (h_1 \cap h_2)x$ and so the Proposition.

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