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(Editors)

**Advances in Generalized
Structures, Approximate
Reasoning and Applications**

*Dedicated to Professor Horia-Nicolai Teodorescu
at his 50-year anniversary, for his contributions
to fuzzy systems, chaos theory, and applications.*

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Looking at Certain Hypergroups and their Properties

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The canonical hypergroup, the quasicanonical hypergroup, the join hypergroup (join space), the fortified join hypergroup, the transposition hypergroup and the fortified transposition hypergroup are so much related to each other, that they can be viewed as members of the same "family". Here appears the genesis of these hypergroups, certain properties that they have, moreover some of their more important applications.

Keywords: hypergroup, hyperfield, field, multiplicative subgroup

According to the first postulate of Euclid [8]:

"*ΗΙΤΗΣΘΩ ΑΠΟ ΠΑΝΤΟΣ ΣΗΜΕΙΟΥ ΕΠΙ ΠΑΝ ΣΗΜΕΙΟΝ ΕΥΘΕΙΑΝ
ΓΡΑΜΜΗΝ ΑΓΑΓΕΙΝ*"

(Let the following be postulated: to draw a straight line from any point to any point.)

So, any pair of points (a,b) can be mapped to the segment of the straight line ab . This segment always exists and it is a nonempty set of points. In fact, it is a multivalued result of the composition of two elements (i.e. of the two endpoints), which is exactly the definition of the hypercomposition: A *hypercomposition*, in a set H , is a function from $H \times H$ to the powerset $P(H)$ of H . Marty [18] introduced this notion in Mathematics, together with the notion of the *hypergroup*, during the 8th congress of the Scandinavian Mathematicians, held in Stockholm, in 1934. Frederic Marty was a French mathematician who was born in 1911. He introduced the hypergroup in connection to his thesis on meromorphic functions (that was written under the direction of Paul Montel). He died 29 years old, while he was in military duty, during World War II, when his airplane was hit over the Baltic Sea.

The axioms that endow the pair (H, \cdot) , where H is a nonempty set and " \cdot " is a hypercomposition in H , with the hypergroup structure are:

- i. $a(bc) = (ab)c$ for every $a, b, c \in H$ (associativity)
- ii. $aH = Ha = H$ for every $a \in H$ (reproductivity)

It can be proved that in a hypergroup, the result of the hypercomposition is always a nonempty set. Indeed, let $ab = \emptyset$. Then $H = aH = a(bH) = (ab)H = \emptyset H = \emptyset$, which is absurd.

Also F. Marty defined the two *induced hypercompositions* (the left and the right division) that derive from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{b} = \{x \in H \mid a \in xb\} \text{ and } \frac{a}{b|} = \{y \in H \mid a \in by\}$$

For the sake of simplicity of the notation, these hypercompositions are also denoted by $a:b$ or a/b and $a..b$ or $b\backslash a$ respectively. When " \cdot " is commutative, $a:b = a..b$. Besides, $a:b \neq \emptyset$ and $a..b \neq \emptyset$, since, from the reproductivity $H = Hb = bH$, it derives that, for $a \in H$, there exists at least one $x \in H$ and one $y \in H$ such that $a \in xb$ and $a \in by$. Moreover it has been proved that the nonempty result of the induced hypercompositions is equivalent to the reproductivity axiom [23, 25].

Coming back to Euclid and to the above mentioned hypercomposition of the points, we observe that the nonempty result of the induced hypercompositions is given by his second postulate [8]:

"*ΚΑΙ ΠΕΠΕΡΑΣΜΕΝΗΝ ΕΥΘΕΙΑΝ ΚΑΤΑ ΤΟ ΣΥΝΕΧΕΣ ΕΠ' ΕΥΘΕΙΑΣ ΕΚΒΑΛΕΙΝ*"
(To produce a finite straight line continuously in a straight line)

Therefore the set of the points of the plane and, more generally, the set of the points of any n -dimensional vector space V over an ordered field F becomes a hypergroup with hypercomposition:

$$ab = \{ka + mb \mid k, m \in F, k, m > 0, k+m=1\} \quad (1)$$

This hypergroup is called *attached hypergroup* to the vector space. In fact, several hypergroups can be attached to a vector space [44]. Thus, many geometrical notions can be described with the use of the hypergroups and therefore many results that are reached in the hypergroup theory can be transferred into Geometry. W. Prenowitz [46, 47, 48, 49] firstly introduced the use of the hypergroups in the study of geometry.

A subset h of a hypergroup H is called *semi-subhypergroup* if $ab \subseteq h$ for every $a, b \in h$. h is called *subhypergroup* of H if $ah = ha = h$, for every $a \in h$. Also, a subhypergroup h is a *right closed* subhypergroup of H if $ah \cap h = \emptyset$, for every $a \in H \setminus h$. Analogous is the definition of the *left closed* subhypergroup of H . A subhypergroup is called *closed* if it is closed from the right and from the left. It has been proved that h is

right closed in H if and only if $a:b \subseteq h$ and left closed, if and only if $a..b \subseteq h$, for every $a, b \in h$ [23, 25]. The notation $[A]$ (respectively $\langle A \rangle$) is used to signify the semi-subhypergroup (resp. the closed subhypergroup), which derives from an arbitrary nonempty subset A of H . The intersection of two subhypergroups of a hypergroup, if it is not void, it is a semi-subhypergroup, while the non void intersection of two closed subhypergroups is always a closed subhypergroup. Another category of subhypergroups is the *invertible* ones: A subhypergroup h of H is called *right invertible* if $xh \cap yh = \emptyset$, for every $x, y \in H$ with $xh \neq yh$. Analogous is the definition of the *left invertible* subhypergroup of H . From its definition, it derives that every invertible subhypergroup is also closed, but the opposite is not valid [38].

Several geometrical notions can be described with the use of the above Definitions. Let's take the convexity for instance. It is known that a figure is called *convex*, if it always contains the segment that joins each pair of its points. It is already mentioned that the hypercomposition (1) endows the set of the points of the plane, as well as the set of the points of any vector space V over an ordered field, with a hypergroup structure. From this point of view, i.e. with the use of the hypercomposition (1), a subset E of V is convex if $ab \subseteq E$, for every $a, b \in E$. But a subset E of a hypergroup that has this property is a semi-subhypergroup. Thus:

Proposition 1. *The convex sets of a vector space are the semi-subhypergroups of its attached hypergroup.*

Consequently, the properties of the convex sets of a vector space can be derived as applications of the properties of the semi-subhypergroups, or the subhypergroups of a hypergroup, and more precisely, of its attached hypergroup. So, this approach, except from the fact that it leads to remarkable results, it also gives the opportunity to generalize the already known theorems of the vector spaces in sets with fewer axioms. Let us see some examples, starting with the notion of the affine dependence.

Proposition 2. *In a vector space V over an ordered field F , the elements $a_i, i=1, \dots, k$ are affinely dependent if and only if there exist distinct integers $s_1, \dots, s_n, t_1, \dots, t_m$ that belong to $\{1, \dots, k\}$ such that*

$$[a_{s_1}, \dots, a_{s_n}] \cap [a_{t_1}, \dots, a_{t_m}] \neq \emptyset$$

in its attached hypergroup.

In a hypergroup now, the following theorem is valid [23, 25]:

Theorem 1. *Let H be a hypergroup in which every set with cardinality greater than n has two disjoint subsets A, B such that $[A] \cap [B] \neq \emptyset$. If $(Y_i)_{i \in I}$ with $\text{card } I \geq n$ is a finite family of semi-subhypergroups of H , in which the intersection of every n elements is non void, then all the sets Y_i have a non void intersection.*

The combination of Proposition 2, and Theorem 1 gives the Corollary:

Corollary 1. (Helly's Theorem) *In a finite family $(C_i)_{i \in I}$ of convex sets in R^d , with $d+1 \leq \text{card } I$, if any $d+1$ of the sets C_i have a nonempty intersection, then all the sets C_i have nonempty intersection.*

In order to use more effectively the hypergroup theory into Geometry, W. Prenowitz introduced properly defined hypergroups, known as join spaces. Using these hypergroups he studied the classical geometries, descriptive geometries, spherical geometries and projective geometries. Now, the term *join space* or *join hypergroup* signifies a commutative hypergroup that satisfies the geometrically motivated transposition property:

$$(a/b) \cap (c/d) \neq \emptyset, \text{ implies } ad \cap bc \neq \emptyset$$

In the following, in order to show the strong relation that exists between the semi-subhypergroups and the convex sets, we will give two theorems that are valid in the join hypergroups and that also lead to two very well known results in the theory of the vector spaces.

Theorem 2. *Let A, B be two disjoint semi-subhypergroups of a join hypergroup (H, \cdot) and let x be an idempotent element of H (i.e. $xx=x$), which does not belong to the union $A \cup B$. Then $[A \cup \{x\}] \cap B = \emptyset$, or $[B \cup \{x\}] \cap A = \emptyset$ (for the proof see [23]).*

With the use of this Theorem, we get:

Theorem 3. *Let H be a join hypergroup, every element of which is idempotent. Also let A, B be two disjoint semi-subhypergroups. Then there exist disjoint semi-subhypergroups X, Y , such that $A \subseteq X$, $B \subseteq Y$ and $H = X \cup Y$ (for the proof see [23]).*

Since the attached hypergroup to the vector space over an ordered field is a join one, those two Theorems give directly, as Corollaries, the Lemma of Kakutani and the Theorem of Stone respectively:

Corollary 2. (Kakutani's Lemma) *If A, B are two disjoint convex sets in a vector space V and if x is a point not in their union, then either the convex envelope of $A \cup \{x\}$ and B are disjoint, or else the convex envelope of $B \cup \{x\}$ and A are disjoint.*

Corollary 3. (Stone's Theorem) *If A, B are two disjoint convex sets in a vector space V , then there exist disjoint convex sets X and Y such that $A \subseteq X$, $B \subseteq Y$ and $V = X \cup Y$.*

If the join hypergroup is endowed with certain other axioms, then there appear hypergroups in which we get theorems that also generalize other known theorems of the

convex sets, as for instance the theorems of Randon, of Caratheodory, of Steinitz etc. [for example, see 11, 22, 29, 31].

In the join spaces domain, there is also the work of J. Jantosciak. Besides his work with W. Prenowitz [50, 51], J. Jantosciak, has published a series of papers [10, 11, 13] that are important contributions to the development of this area. Especially in his paper [13] he removes the commutativity from the axioms of the join hypergroup introducing thus the *transposition hypergroup*. More precisely, a transposition hypergroup is a hypergroup, which satisfies the axiom:

$$b \vee a \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset$$

In [13], J. Jantosciak develops the algebra of the transposition hypergroups and he obtains the generalizations of the isomorphism theorems and the Jordan-Holder theorem of group theory. The join hypergroups have also been studied by P. Corsini and V. Leoreanu, who produced interesting results on their part [17].

The join hypergroups though, are not connected only to Geometry. G. Massouros, analyzing the theory of Languages and Automata with the use of the hypercompositional algebra proved that the join hypergroups are closely related to the set of the words over an alphabet and to the regular expressions which describe the languages that the automata accept [34, 36, 41]. Also, in order to introduce the "null word" in the join hypergroup that he had attached to the set of the words, as well as other reasons that have to do with the electronic realization of the automaton, G. Massouros introduced a non scalar neutral element in the join hypergroup, defining thus the *fortified join hypergroup* [34, 37]. In particular, the fortified join hypergroup $(H, +)$ is a join hypergroup that satisfies the following two axioms:

- (FJ₁) There exists a unique neutral element 0 in H such that $x \in 0+x$, for every $x \in H$, and $0+0=0$.
- (FJ₂) For every $x \in H \setminus \{0\}$ there exists one and only one $x' \in H \setminus \{0\}$, such that $0 \in x+x'$.

The study of this hypergroup has shown that its elements have several interesting properties [33, 37]. Initially it has been proved that for every element x , the inclusion $0+x \subseteq \{0,x\}$ holds. This property motivated the introduction of the notion of the strong neutral element of the hypergroup, i.e. an element e with the property $x \in ex = xe \subseteq \{e,x\}$, for all the elements x of the hypergroup. We remind that a neutral element of a hypergroup H is called scalar if $x = ex = xe$ for all x in H. Also, the elements of the fortified join hypergroup can be separated into different categories, according to certain properties that they have. Such a separation is based on the validity or no validity of the equality $-(x-x) = x-x$. The elements that satisfy this equality are called *normal*, while the ones that do not satisfy it, are called *abnormal*. Furthermore, another separation of the elements of the fortified join hypergroup arises from their behavior in their hypersum with 0. Indeed

in such a hypergroup, there exist elements whose hypersum with 0 gives as a result the element itself, and yet, there exist other elements for which the same hypersum gives as a result the biset that consists of the two addends. The first ones are called *canonical* elements and the others are called *attractive*, since they attract 0 to the result of the hypercomposition. The classification to canonical and attractive elements and the properties of these elements, lead to a structure theorem for the fortified join hypergroups [33], which will be presented shortly, after the presentation of the canonical hypergroups.

Apart from the different kinds of elements, the fortified join hypergroups have a variety of subhypergroups that are studied in [39]. It is proved that the attractive elements and the zero element constitute the minimum closed subhypergroup. Also very important are the symmetrical subhypergroups i.e. subhypergroups h for which $-x$ belongs to h for every x in h . Moreover, the study of the homomorphisms of these hypergroups is very interesting [28], since they contain different kinds of elements, moreover different types of subhypergroups. The mapping of the attractive elements to the canonical ones and the opposite, as well as the fact that if x is an attractive element and x belongs to the kernel of a normal homomorphism does not imply that $-x$ belongs also to the kernel of this homomorphism, leads to the introduction of the *complete homomorphism* i.e. a homomorphism φ for which $x \in \ker\varphi$ implies that $-x \in \ker\varphi$. If $\varphi(0)=0$, the complete homomorphism maps symmetrical subhypergroups to symmetrical subhypergroups and its kernel is also a symmetrical subhypergroup.

The use of these hypergroups and generally of the hypercompositional structures in the theory of languages and automata resulted in the proof of several theorems, special cases of which are theorems, like the ones of Kleene [35] and of Myhill-Nerode [36].

J. Jantosciak and Ch. Massouros (in [14]), using strong identities, have achieved the fortification of the transposition hypergroups. So, the fortified transposition hypergroup is defined and shown to contain a unique strong identity. Moreover, each nonidentity element is shown to have unique nonidentity left and right inverses that are identical. The symmetrical subhypergroups of this hypergroup are remarkable, because they can define partitions. Indeed, if H is a fortified transposition hypergroup of attractive elements, x an element of H and K a nonempty symmetrical subhypergroup, then the x_K , i.e. the double coset of K determined by x , is given by:

$$x_K = \begin{cases} K, & \text{if } x \in K \\ K \setminus (x/K) = (K \setminus x)/K, & \text{if } x \notin K \end{cases}$$

and $x_K \cap y_K \neq \emptyset$ implies $x_K = y_K$. It is shown that the family $H:K$ of double cosets forms a fortified transposition hypergroup, in which K is the strong identity and every member of this family is attractive.

A hypergroup closely related to the above is the *canonical hypergroup*. This hypergroup appeared in 1956, in M. Krasner's paper [15], as the additive part of another hypercompositional structure, the hyperfield, which M. Krasner has used as the proper

algebraic tool, in order to define a certain approximation of a complete valued field by sequences of such fields. Later on, J. Mittas separated this hypergroup from the hyperfield, gave it the name canonical and studied it in depth as an independent mathematical structure [42, 43]. A canonical hypergroup is a hypercompositional structure $(H, +)$, that for every $x, y, z \in H$ satisfies the axioms:

CH₁. $x+y = y+x$ (commutativity)

CH₂. $x+(y+z) = (x+y)+z$ (associativity)

CH₃. there exists an element $0 \in H$ for which $0+x = x$

CH₄. for every $x \in H$ there exists one and only one element $x' \in H$, denoted by $-x$, such that $0 \in x+(-x)$.

CH₅. $z \in x+y \Rightarrow x \in z+(-y)$ (reversibility)

It has been proved that axiom CH₅ is equivalent to:

CH'₅. $(\forall (x, y) \in H^2) [-(x+y) = -x-y]$ as well as to:

CH''₅. $(\forall (x, y, z, w) \in H^4) [(x+y) \cap (z+w) \neq \emptyset \Rightarrow (z-x) \cap (y-w) \neq \emptyset]$

Many researchers have worked on the canonical hypergroups. Among them, P. Corsini analyzed certain types of canonical hypergroups [7], while R.L. Roth dealt with the character and the conjugacy class hypergroups of a finite group. In [52] he proved that the conjugacy classes of a group G becomes a canonical hypergroup if the hypercomposition of two conjugacy classes is defined to be the set of conjugacy classes contained in their setwise product and that the set of irreducible complex characters of G is also a canonical hypergroup, the hypercomposition of two characters being the set of distinct irreducible components in their product. Also he gave many results regarding the group G and the above two associated with G canonical hypergroups. Special interest also appears in the study of the valuation of the canonical hypergroups [30]. J. Mittas developed this theory.

If from the axioms of the canonical hypergroups, the commutativity is left out, then, there derives another class of hypergroups, the *quasicanonical* ones (as they were named by Bonansinga and Corsini) or polygroups (by Comer). Bonansinga [1, 2], Corsini [7] and S. Ioulidis [9] studied the structure of these hypergroups and their subhypergroups, while Ch. Massouros ([24]) studied their congruence relations and their homomorphisms. Comer has established a dual equivalence between the category of polygroups and the category of complete atomic integral relation algebras in [3], where interesting examples of polygroups are given. Also in [6], the notion of conjugacy for polygroups is given and it has been proved that the conjugacy relations of a polygroup form a complete lattice with a very rich structure. The lattices are related to some very difficult problems involving groups. Moreover in [5] the notion of a partial multivalued loop is given. This notion is like a "hypercategory" and it has been proved that a polygroup can be obtained as a special case. Furthermore, De Salvo and Corsini have presented studies on the feebly

quasicanonical hypergroups, which are a generalization of the quasicanonical ones [7]. Yet, in [53], one can find the definition of the fuzzy subpolygroups and some of their properties.

As mentioned above, the canonical hypergroup is closely related to the join one. Thus, apart from the fact that every canonical hypergroup is a join one, it can also be proved that if a join hypergroup has a scalar neutral element, then it is canonical [23]. Moreover, if a transposition hypergroup has a scalar neutral element, then it is a quasicanonical hypergroup [13]. Furthermore, the quotient space of a transposition hypergroup modulo a reflexive closed subhypergroup forms a quasicanonical hypergroup [13].

Having dealt with the canonical and the quasicanonical hypergroups, it is already time to come back to the structure theorem of the fortified join hypergroups, that we mentioned above. It has been proved that every fortified join hypergroup can be obtained by the following construction:

Let $(K, +_1)$ be a canonical hypergroup and let $(E, +_2)$ be a fortified join hypergroup in which all the non-zero elements are attractive and such that $K \cap E = \{0\}$. Define, in $H = K \cup E$ a hypercomposition and by:

$$x+y = \begin{cases} x +_2 y, & \text{if } (x,y) \in E^2 \\ x +_1 y, & \text{if } (x,y) \in K^2 \text{ and } y \neq -x \\ (x +_1 y) \cup E, & \text{if } (x,y) \in (K \setminus \{0\})^2 \text{ and } y = -x \\ y, & \text{if } x \in E \text{ and } y \in K \setminus \{0\} \end{cases}$$

Then $(H, +)$ is a fortified join hypergroup.

Analogous results regarding the structure of the fortified transposition hypergroups have also been achieved in [14]. It has been proved that a transposition hypergroup H containing a strong identity e is isomorphic to the expansion (as defined in [12]) of the quasicanonical hypergroup $C \cup \{e\}$ by the transposition hypergroup A of all attractive elements through the idempotent element e .

The above hypergroups can also appear as parts of other hypercompositional structures [16, 35, 40]. As mentioned above, the canonical hypergroup itself is the additive part of the *hyperfield*. The hyperfield is a triplet $(H, +, \cdot)$, where H is a nonvoid set, (H, \cdot) is an almost-group i.e. the union of a multiplicative group H^* with a bilaterally absorbing element, denoted by 0 , $(H, +)$ is a canonical hypergroup and also, the distributive axiom is valid: $z(x+y) = zx + zy$, $(x+y)z = xz + yz$.

The certain type of hyperfield that firstly appeared in Algebra, was the *residual hyperfield*. So naturally M. Krasner, who introduced it, asked the question:

Are there other hyperfields than the residual ones?

Later on, Krasner himself constructed a more general class of hyperfields, which contains the residual ones. These are the *quotient hyperfields* and their construction is as follows [16]:

Let F be a field and G a subgroup of the multiplicative group of the field. Then the quotient set F/G becomes a hyperfield if we define:

$$xG + yG = \{(xp+yq)G \mid p,q \in G\} \text{ and } xG \cdot yG = xyG$$

Similarly, he constructed the *quotient hyperrings* and then he posed the question [16]:

Are there hyperrings and hyperfields, other than the quotient ones?

The answer to this question, which is positive, was given independently by Ch. Massouros (in [19, 20, 21, 27]) and A. Nakassis (in [45]). A. Nakassis has worked with hyperfields in which the hypersum of two different to each other and non opposite elements does not contain the two addends [45]. In these hyperfields the differences $x-x$ have interesting properties. Thus, for every selfopposite element x of such a hyperfield H it holds: $\text{card}(x-x) = 2$. If x is not selfopposite, then $\text{card}(x-x) = 3$. Also $(x-x) \cap (y-y) = \{0\}$, if $y \neq -x$, x and $\bigcup_{x \in H} (x-x) = H$. A. Nakassis started his construction with a multiplicative group T^* which has more than three elements and he considered one more element 0 which is multiplicatively absorbing in $T = T^* \cup \{0\}$, i.e. $a \cdot 0 = 0 \cdot a = 0$ for every $a \in T$. Next, he endowed T with a hyperfield structure introducing the hypercomposition:

$$\begin{array}{ll} a+0 = 0+a = a & \text{for every } a \in T \\ a+a = \{0, a\} & \text{for every } a \in T^* \\ a+b = b+a = T \setminus \{0, a, b\} & \text{for every } a, b \in T^*, \text{ with } a \neq b \end{array}$$

Then, having worked with the properties of the differences $x-x$, he has proved that one can choose either the cardinality or the structure of the group T^* of this construction in such a way that $(T, +, \cdot)$ does not become isomorphic to a quotient hyperfield. For this proof he has used the following important Proposition, which allowed him to calculate the number of the elements that are contained in the result of the hypersum of two elements.

Proposition 4. *Let R be a ring and P an equivalence relation that induces a hyperring structure in R/P with $P(0) = \{0\}$. Assume that a hyperring H is embeddable in the partition hyperring R/P and assume that there exist two elements a and b in H , such that $[c+(-c)] \cap [b+(-b)] = \{0\}$ for every c in $a+b$. Then the cardinality of the isomorphic image of b (viewed as a subset of R) cannot exceed the cardinality of $a+b$.*

Contrarily to Nakassis' hyperfields, in [19, 20, 21, 27] are used hyperfields in which the hypersum of two elements contains the two addends and it is proved that certain classes of such hyperfields contain elements that are not quotient hyperfields. The construction of these hyperfields starts with a multiplicative group, which is then equipped with a multiplicatively absorbing element and a hypercomposition. Thus:

Construction I.

Let (Θ, \cdot) be a commutative multiplicative group with more than two elements and let 0 be a bilaterally multiplicatively absorbing element. We introduce in the almost-group $\Theta_0 = \Theta \cup \{0\}$ a hypercomposition “+” as follows:

$$\begin{array}{ll} x+0 = 0+x = x & \text{for every } x \in \Theta_0 \\ x+x = \Theta_0 \setminus \{x\} & \text{for every } x \in \Theta \\ x+y = y+x = \{x, y\} & \text{for every } x, y \in \Theta, \text{ with } x \neq y \end{array}$$

Then, $H(\Theta) = (\Theta_0, +, \cdot)$ is a hyperfield.

Construction II.

Let (Θ, \cdot) be a commutative multiplicative group with more than two elements and let (Θ^\wedge, \cdot) be its direct product with the multiplicative group $\{-1, 1\}$. We consider the almost-group $(\Theta_0^\wedge, \cdot) = (\Theta^\wedge \cup \{0\}, \cdot)$ and we introduce in Θ_0^\wedge a hypercomposition “+” as follows:

$$\begin{array}{ll} (x,i)+(x,i) = \Theta_0^\wedge \setminus \{(x,i), (x,-i), 0\} & \text{for every } (x,i) \in \Theta^\wedge \\ (x,i)+(x,-i) = \Theta_0^\wedge \setminus \{(x,i), (x,-i)\} & \text{for every } (x,i) \in \Theta^\wedge \\ y+0 = 0+y = y & \text{for every } y \in \Theta_0^\wedge \\ (x,i)+(w,j) = \{(x,i), (w,j), (x,-i), (w,-j)\} & \text{for every } (x,i), (w,j) \in \Theta^\wedge \\ & \text{with } (w,j) \neq (x,-i), (x,i) \end{array}$$

Then, $K(\Theta) = (\Theta_0^\wedge, +, \cdot)$ is a hyperfield.

Theorem 5. *If Θ is a periodic group, then the hyperfields $H(\Theta)$ and $K(\Theta)$ do not belong to the class of the quotient hyperfields (for the proof see [20, 21, 27]).*

In the process of solving the problem of the existence or non existence of non quotient hyperfields, Ch. Massouros constructed the class of the *monogene hyperfields*, i.e. hyperfields $(H, +, \cdot)$ that have the property $x \cdot x = H$ for every nonzero element x of H [19, 26]. Then, A. Nakassis proved that there exist monogene hyperfields that are isomorphic to the quotient ones [19, 26]. The question though of *whether there exist monogene hyperfields that are not quotient hyperfields* is still open. The study of this problem has shown that if a monogene hyperfield is isomorphic to a quotient hyperfield F/G , then the equality $G \cdot G = F$ is valid. This gave birth to another problem in the theory of fields [19, 26]:

Which fields can be written as a difference of a subgroup of their multiplicative group from itself, and which are these subgroups?

Ch. Massouros proved the Theorem:

Theorem 6. [27, 32] Let F be a finite field and G a subgroup of the multiplicative group of F of order m and index n . Then $G \cdot G = F$, provided that

$n=2$ and $m > 2$.

$n=3$ and $m > 5$.

$n=4$, $-1 \in G$ and $m > 11$.

$n=4$, $-1 \notin G$ and $m > 3$.

$n=5$ and $m=16$, or $m > 23$.

However, the thorough solution of the above mentioned problem remains still unknown.

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