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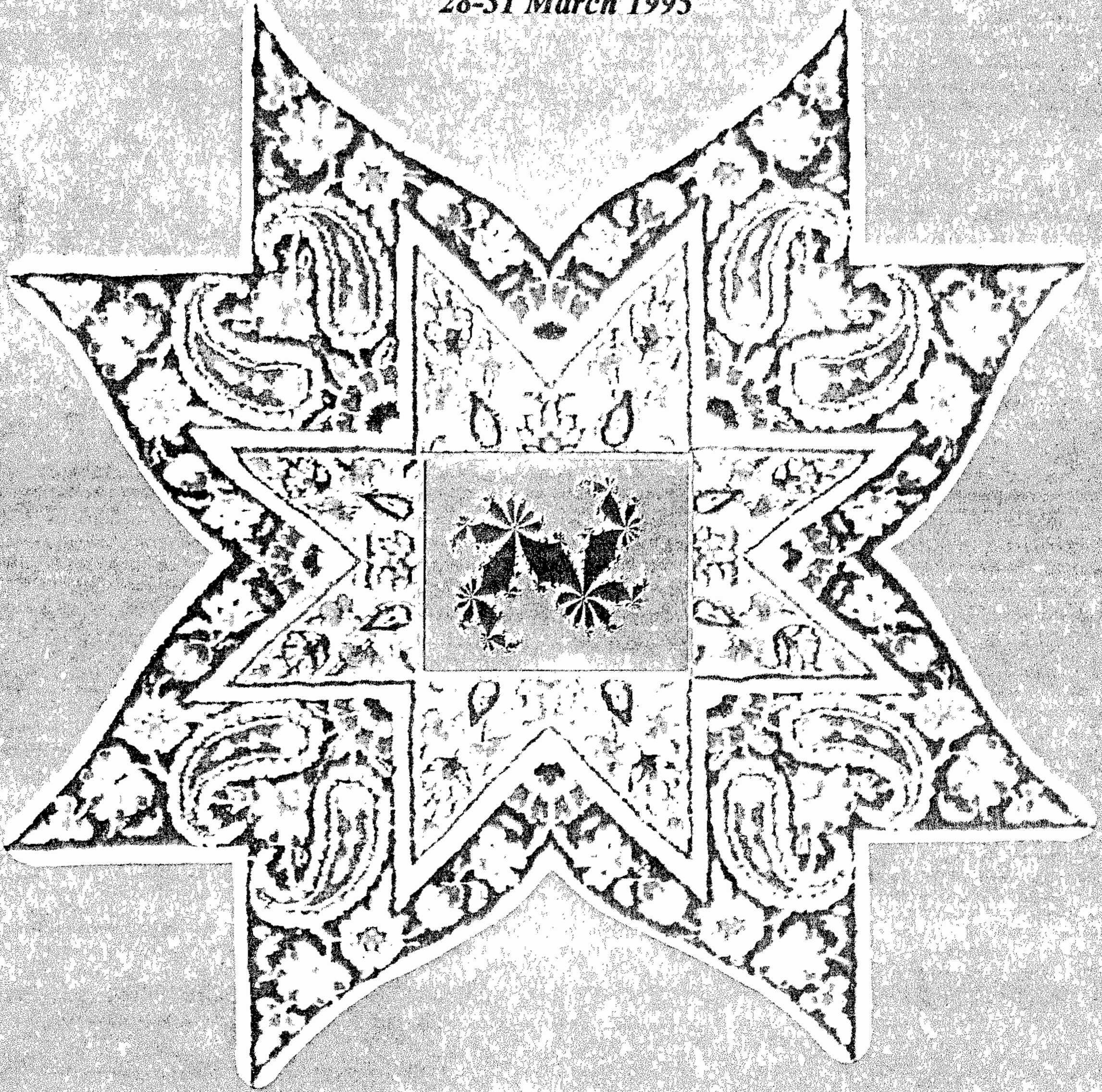

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Application of the Hypercompositional Structures Into Geometry

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ABSTRACT

This paper presents a study of geometrical topics with the use of algebraic methods from theory of the hypercompositional structures. In the beginning there appears the close relation between the Euclide's postulates and the axioms of the hypergroup. Next, with the use of certain hypergroups, there derive results on the convex sets. Lastly there appears a description of certain geometries through the hypercompositional structures.

According to the first postulate of Euclid:
"*Ηι τησθω απο παντος σημειου επι πᾶν σημ εῖον ευθεῖαν γραμμην αγαγεῖν*"

(Let the following be postulated: to draw a straight line from any point to any point).

So, to any pair of points (a,b), the segment of the straight line ab can be mapped. This segment always exists and it is a nonempty set of points. In fact it is a multivalued result of the composition of two elements, which is exactly the definition of the hypercomposition:

A **hypercomposition** in a set E is a function from $E \times E$ to the power set $P(E)$ of E. It has been introduced in mathematics, along with the introduction of the notion of the hypergroup, by F. Marty [5]. **Hypergroup** is a nonempty set H having a hypercomposition ".", for which the following axioms are valid:
i. $a(bc) = (ab)c$ for every $a, b, c \in H$ (associativity)
ii. $aH = Ha = a$ for every $a \in H$ (reproductivity)
It has been proved that always $ab \neq \emptyset$ in a hypergroup, for every $a, b \in H$.

Moreover F. Marty has defined the two **induced hypercompositions** (the left and the right division) that derive from the hypercomposition of the hypergroup, i.e. $\frac{a}{|b} = \{x \in H | a \in xb\}$ and $\frac{a}{b|} = \{y \in H | a \in by\}$. These hypercompositions, for simplicity of the notation, are also denoted by $a : b$ and $a..b$ respectively. When "." is commutative, $a : b = a..b$. It has been proved that in a hypergroup $a : b \neq \emptyset$ and $a..b \neq \emptyset$ as well as that the nonempty result of the induced hypercompositions is equivalent to the reproductive axiom [9]. Here we remark that the nonempty of the induced hypercomposition of the above mentioned hypercomposition of the points, is exactly the second Euclid's postulate:

"*Και πεπερασμενμν ευθεῖαν κατα το συν εχες επ' ευθειας εκβαλεῖον*"

(To produce a finite straight line continuously in a straight line).

Therefore the set of the points of the plane and, more generally, the set of the points of any n-dimensional vector space V over a valuated field F becomes a hypergroup with hypercomposition:

$$ab = \{\kappa a + \lambda b \mid \kappa, \lambda \in F,\ \ \kappa, \lambda > 0,\ \ \kappa + \lambda = 1\}$$

This hypergroup is called **attached hypergroup** to the vector space and from its study there derive results on the vector space. Such, for instance, is the case of the convex sets. A figure is called convex if the segment which joins any two of its points is entirely in it. On the other hand **semi-subhypergroup** h of H is a subset of the hypergroup H for which $ab \subseteq h$, for every $a, b \in h$. (h is a **subhypergroup** if $ah = ha = h$, for every $a \in h$). Thus the convex sets of the vector space V correspond to the semi-subhypergroups of the attached hypergroup of the vector space and therefore the study of those semi-subhypergroups gives results for the convex sets (e.g. [9], [10]). Another example comes from the affine subspaces of a vector space. Considered as subsets of the attached hypergroup of the vector space, they correspond to a special type of subhypergroups, the closed subhypergroups. **Closed subhypergroup** h of a hypergroup H is the subhypergroup for which $ah \cap h = \emptyset$, for every $a \in H \backslash h$ [3]. It has been proved that a subhypergroup h is closed in H if and only if it is divisionally closed both from the left and the right [10]. The notation [A] will be used to signify the semi-subhypergroup which is generated by A.

It must be mentioned though that apart from the above hypercomposition, other hypercompositions can be introduced, resulting to the appearence of other attached hypergroups each one of which having its own properties and therefore its own results on V (see [13]).

The first one who has introduced the hypergroups into Geometry is W.Prenowitz. In order to do so, he introduced a special type of hypergroup, which he named **join space** and using it he has proceeded to a significant new consideration of several topics in Geometry [14]. The axioms of the join space are

(i) $ab = ba$,

(ii) $aa = a$,

(iii) $a/a = a$ and

(iv) if $(a:b) \cap (c:d) \neq \emptyset$, then $ad \cap bc \neq \emptyset$.

A commutative hypergroup that satisfies only the last one of the above axioms is called **join hypergroup** [9].

In order to show the relation of the semi-subhypergroups with the convex sets we will give two theorems that hold in the join hypergroups, which lead to two very well known results in the theory of the vector spaces.

Theorem 1. Let A, B be two disjoint semi-subhypergroups of a join hypergroup $(H,.)$ and let X be an idempotent element of H, which does not belong to the union $A \cup B$. Then $[A \cup \{X\}] \cap B = \emptyset$; or $[B \cup \{X\}] \cap A = \emptyset$.

Proof. Let $[A \cup \{X\}] \cap B = \phi$, or $[B \cup \{X\}] \cap A = \phi$. Then there exists $a \in A$ such that $Xa \cap B \neq \phi$, from where it derives that $X \in B : a$. Similarly there exsits $b \in B$, such that $X \in A : b$. Thus $B : a \cap A : b \neq \phi$, from where $Bb \cap Aa \neq \phi$ and since $Bb \subseteq B$, $Aa \subseteq A$ we have that $A \cap B \neq \phi$, which contradicts the supposition.

With the use of this Theorem, we get the next Theorem:

Theorem 2. Let H be a join hypergroup, every element to which is idempotent. Also let A, B be two disjoint semi-subhypergroups. Then there exist disjoint semi-subhypergroups X, Y, such that $A \subseteq X$, $B \subseteq Y$ and $H = X \cup Y$.

And since the attached hypergroup to the vector space that appears here is a join one, those two Theorem give directly as Corollaries the Lemma of Kakutani and the Theorem of Stone respectively:

Corollary 1. If A, B are two disjoint convex sets in a vector space V and if X is a point in their union, then either the convex envelope of $A \cup \{X\}$ and B are disjoint, or else the convex envelope of $B \cup \{X\}$ and A are disjoint (**Kakutani's Lemma**).

Corollary 2. If A, B are two disjoint convex sets in a vector space V, then there exist disjoint convex sets X and Y such that $A \subseteq X$, $B \subseteq Y$ and $V = X \cup Y$ (**Stone's Theorem**).

The hypergroup, as a very general algebraic structure can be endowed with more axioms leading to a big variety of special types of hypergroups. So W. Prenowitz has additionally introduced the following axiom into the join spaces: if $c \in \langle a, b \rangle$ and $c \neq a$, then $\langle a, b \rangle = \langle a, c \rangle$ creating thus the exchange space. In this hypergroup he has developed dimension theory and a theory of linear independence, similar to the ones of the classical Geometry. Also D. Freni has presented an analogous theory in a more general hypergroup, which he named **Cambiste** [2]. A hypergroup

H is called cambiste if for every $X, Y \in H$ and for every $A \subseteq H$ holds:

$$[X \in < A \cup \{Y\} >, X \notin A] \Rightarrow [Y \in < A \cup \{X\} >]$$

Freni has proved that these hypergroups have at least one basis, i.e. a free set of generators, and that all the bases have the same cardinality, which he named **dimension**. The dimension of H is denoted by dim H. A special type of the cambiste hypergroups is the **homiogene** hypergroup [8]. A hypergroup H is called homiogene if from the relation $X \in < Y >$, where X is not the scalar neutral element of H (if there exists one), derives that $< X > = < Y >$¿. The homiogene hypergroup has given many interesting results in Geometry, like the one deriving from the following theorem in the **strongly homiogene** hypergroup:

Definition 1. Let H be a join hypergroup which satisfies the following axioms
(i) H is a homiogene hypergroup
(ii) $[X] = < X >$ for every $X \in H$
(iii) $< X, Y > = < X > \cup < Y > \cup < X > . < Y > \cup < X > : < Y > \cup < Y > : < X >$ for every $X, Y \in H$
Then H will be called strongly homiogene hypergroup.

Theorem 3. Let Ω be a strongly homiogene hypergroup and let $x_1, x_2, \ldots, x_{n+1}$ be elements of one of its closed sub-hypergroups, H, for which dim $H = n$. Then there exists non void and disjoint subsets of $\{x_1, x_2, \ldots, x_{n+1}\}$ such as

$$[x_{u1} . \ldots . [x_{uv}] \cap [x_{\lambda 1}] . \ldots . [x_{\lambda u}] \neq \emptyset$$

with $1 \leq u1 < \cdots < uv \leq n+1$, $1 \leq \lambda 1 < \cdots < \lambda u \leq n+1$.

Corollary 3. Let $M = \{x_1, \ldots, x_n\}$ be a finite set of n points from R^d such that $n \geq d+2$. Then there exist subsets M_1 and M_2 of M with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$ such that the convex hull of M_1 and the convex hull of M_2 have nonempty intersection (**Randon's Theorem**).

Also the hypergroups are being used as a basis for the creation of other algebraic structures, one of which is the hyperring, introduced by M. Krasner. The hyperring is a triplet $(P, +, .)$, where P is a nonempty set, "+" a hypercomposition and "." a composition, that satisfy the axioms:

a. $(P, +)$ is a commutative hypergroup for which:
   (i) $(\exists\, 0 \in P)(\forall\, x \in P)[x + 0 = x]$
   (ii) for every $x \in P$ there exists one and only one $x' \in P$ (denoted by $-x$) such that $0 \in x + x'$.
   (iii) $z \in x + y \Rightarrow x \in z - y$.
b. $(P, .)$ is a multiplicative semigroup in which 0 is a bilaterally absorbing element, i.e. $0x = x0 = 0$.
c. The multiplication is bilaterally distributive as for the hyperaddition.

The hypergroup $(P, +)$ has been named canonical by J. Mittas and it has been studied in great depth by himself (e.g. [11]).

Another hypercompositional structure is the hypermodule. M is called a left hypermodule over P (or left P-hypermodule), where P is a unitary hyperring, if $(M, +)$ is a canonical hypergroup endowed with an external operation $(\lambda, a) \to \lambda a$ from $P \times M$ to M, which satisfies the conditions:
(i) $\lambda(a + b) = \lambda a + \lambda b$,
(ii) $(\lambda + k)a = \lambda a + ka$,
(iii) $(\lambda k)a = \lambda(ka)$.
(iv) $1a = a$ and $0a = 0$, for every $a, b \in M$ and $\lambda, k \in P$.

Let's remark in this point that in the definition of the hypermodule there also exists another version of axiom (ii):
(ii^*) $(\lambda + k)a \subseteq \lambda a + ka$
which is being studied in [12], while in [7] appears a study of the hypermodules as they are defined with the initial axiom. Similar are the definitions of the right P-hypermodule.

Proposition 1. Let M be a hypermodule over a division hyperring $(D, +, .)$ and let "+"

be its hyercomposition. If in M and in D the new hypercomposition:

$a \dagger b = (a+b) \cup \{a, b\} \quad u \dagger \lambda = (u+\lambda) \cup \{u, \lambda\}$

for every $a \neq -b$ belonging to $M \backslash \{0\}$ and $u \neq -\lambda$ from $D \backslash \{0\}$

$a \dagger (-a) = M$ for every $a \in M \backslash \{0\}$, $a \dagger 0 = 0 \dagger a = a$ for every $a \in M$

$u \dagger (-u) = D$ for every $u \in D \backslash \{0\}$, $u \dagger 0 = 0 \dagger u = u$ for every $u \in D$

is defined, then $(D, \dagger, .)$ becomes a division hyperring and $(M, \dagger)$ a hypermodule over D satisfying the axiom (ii^*).

Let's construct now a P-hypermodule:

Let M be a P-module, where P is a unitary ring and let G be a subgroup of the multiplicative semigroup of P, which satisfies the condition $aG.bG = abG$, for every $a, b \in P$. It is worth mentioning that this condition is equivalent to the normality of G in $P^* = P \backslash \{0\}$ only when P^* is a group [6]. Then G defines a partition in $P, P_G = P/G$, the classes of which are of the type $xG, x \in P$, and also a hyperring structure over P_G, with the following multiplication and hyperaddition:

$xG.yG = xyG$

$xG \dagger yG = \{(xp + yp)G \mid p, q \in G\}$, for every $xG, yG \in P_G$

This hyperring is called **quotient hyperring** and it has been constructed by M. Krasner [4]. In the following and for the simplicity of the notation, the elements $xG \in P_G$ will be denoted by xG.

Next let's introduce in M the equivalence relation g:

$$x \; g \; y \iff x = qy, \quad q \in G$$

and let M_g be the set of the equivalence classes of M module g. Then M_g endowed with the hypercomposition "#":

$$a_g \# b_g = \{w_g \in M_g \mid w_g \subseteq a_g + b_g\}$$

becomes a canonical hypergroup.

Furthermore, with the external composition from $P_G \times M_g$ to M_g:

$\lambda_g . a_g = (\lambda a)_g$ for every $\lambda_G \in P_G$, $a_g \in M_g$

M_g becomes a P_G-hypermodule. M_g is called **quotient hypermodule.**

The hypermodule that has just been constructed is closely related to several geometries. We mention here its relation to the spherical geometry. Let's start with a vector space V over an ordered field F. Let F^+ be the positive cone of F. Since F^+ is a multiplicative subgroup of F, the quotient hypermodule (more precisely the vector hyperspace) over the hyperfield $K = F/F^+ = \{F^-, 0, F^+\}$ can be constructed. Let's denote this quotient hyperspace with $\underline{V}$ and its elements with $\underline{a}$. An image of such a hyperspace can be given with the vector space R^3. The element of the vector hyperspace $\underline{R}^3$ are the open halflines with endpoints at 0 and the hypersum of two halflines $\underline{a} + \underline{b}$ contains all the halflines between $\underline{a}$ and $\underline{b}$. Now let's suppose that V is a real vector space with inner product and S a hypersphere of V, centered at 0. The function $\underline{x} \to x$ of $\underline{V}$ onto $S \cup \{0\}$ is one to one and $\underline{a} + \underline{b}$, $\underline{a} \neq \underline{b}$ is mapped to the open minor arc $\hat{ab}$ of the great circle defined from a and b, since the two participating elements $\underline{a}$ and $\underline{b}$ do not belong to the hypersum $\underline{a} + \underline{b}$. In this case $\underline{a} + (-\underline{a}) = \{-\underline{a}, 0, \underline{a}\}$. On the contrary, the hypersum $\underline{a} \dagger \underline{b}$, deriving from the hypercomposition " $\dagger$ " of Proposition 1, contains the elements $\underline{a}$ and $\underline{b}$. Consequently $\underline{a} \dagger \underline{b}$ is mapped to the closed minor arc $\hat{ab}$. Now $\underline{a} \dagger (-\underline{a}) = \underline{V}$. This is a more realistic approach, because the whole hypersphere is generated by two opposite points, since infinitely many great circles containing all the points of S pass through them. Therefore the quotient hypermodules can describe every Euclidean spherical geometry.

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