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QUASICANONICAL HYPERGROUPS

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Dedicated to the memory of GEORGE CH. MASSOUROS

ABSTRACT. This paper is a study of the quasicanonical hypergroups. Some methods of constructing quasicanonical hypergroups are presented and the congruence relations, the homomorphisms and some properties of their direct products are studied.

1. INTRODUCTION

A non commutative canonical hypergroup [12] is called quasicanonical hypergroup ([1], [6]). That is a quasicanonical hypergroup satisfies the axioms: (ii)-(v) of the canonical hypergroup

i) $x*y = y*x$

ii) $(x*y)*z = x*(y*z)$

iii) There exists an element $v \in H$ such that $v*x = x*v = x$, for every $x \in H$

iv) For every $x \in H$ there exists one and only one x' such as $v \in x*x' = x'*x$ (x' is written x^-).

v) $z \in x*y \Longrightarrow x \in z*y^- \Longrightarrow y \in x^-*z$

We note that this is exactly the same structure as the one that Comer [2] and Ioulidis [4] call polygroup.

In a quasicanonical hypergroup the relation:

$$x*y \cap z*w \neq \emptyset \Longrightarrow y*w^- \cap x^-*z \neq \emptyset$$

holds. The analogous was proved for the canonical hypergroups by J. Mittas [12]. Also this condition is one of the Join Space axioms [15] (for the relation of the Join Spaces to the canonical hypergroups see [3] and [9], [10]).

As in the case of the canonical hypergroups ([7], [8], [11], [14]) we can construct quasicanonical hypergroups using other hypergroups or even groups. Indeed let $(H, \circledast)$ be a quasicanonical hypergroup. We introduce in H a new hypercomposition $*$ as follows:

- $x*y = x\#y \cup \{x,y\}$, if $x \neq y^-$, $x,y \neq v$
- $x*x^- = x^-*x = H$, if $x \neq v$
- $x*v = v*x = x$ for every $x \in H$

Then $(H,*)$ becomes a quasicanonical hypergroup (see also [6], [11]) . If $(H,\#)$ is a group then $(H,*)$ becomes again a quasicanonical hypergroup ($x*y=\{x,y,x\#y\}$). Now if H is a group and Z its center, H becomes a quasicanonical polysymmetrical hypergroup (see [13]) if we define in it a hypercomposition $*$ as follows :

$$x*y=\{ xyz \mid z \in Z \}, \text{ if } x,y \neq 1,$$

and $\quad x*1 = 1*x = x.$

Also if we define in H a hypercomposition $*$ in the following way:

- $x*y = [\bigcup_{z\in Z}\{xyz\}] \cup \{x,y\}$, if $x \neq y^{-1}$, $x,y \neq 1$
- $x*x^{-1} = x^{-1}*x = H$, if $x \neq 1$
- $x*1 = 1*x = x$, for every $x \in H$

then $(H,*)$ becomes again a quasicanonical polysymmetrical hypergroup.

2. CONGRUENCE RELATIONS

Let ξ be a binary relation between two quasicanonical hypergroups H, K. Then ξ will be called <u>**homomorphic relation**</u> if:

i) $(v,v') \in \xi$, where v and v′ are the neutral elements of H and K

ii) if $(a,b) \in \xi$, then $(a^-,b^-) \in \xi$

iii) if (a_1,b_1), $(a_2,b_2) \in \xi$, then for every $y \in a_1*a_2$ and for every $z \in b_1*b_2$ is valid:

$$[\{y\} \times (b_1*b_2)] \cap \xi \neq \emptyset \quad \text{and} \quad [(a_1*a_2) \times \{z\}] \cap \xi \neq \emptyset$$

If ξ is a homomorphic relation between the quasicanonical hypergroups H, K and if k is a sub-hypergroup of K, then the subset h of H which consists of all these elements $a \in H$, for which there is $b \in k$ such as $(a,b) \in \xi$, forms a sub-hypergroup of H. Also if ξ, ζ are homomorphic relations between the quasicanonical hypergroups H,K and K,T respectively, then the relation $\xi\zeta$ which is defined by the following way

$$(a,b) \in \xi\zeta \iff (a,d) \in \xi \text{ and } (d,b) \in \zeta \text{ for some } d \in K$$

is a homomorphic relation between H and T. Indeed, $(v,v) \in \xi\zeta$. Next suppose that $(a,b) \in \xi\zeta$. Then there is $d \in K$ such as $(a,d) \in \xi$ and $(d,b) \in \zeta$. Thus $(a^-.d^-) \in \xi$, $(d^-,b^-) \in \zeta$ and so $(a^-,b^-) \in \xi\zeta$. Now let (a_1,b_1) , $(a_2,b_2) \in \xi\zeta$. Then there are d_1, $d_2 \in K$ such as (a_1,d_1), $(a_2,d_2) \in \xi$ and (d_1,b_1), $(d_2,b_2) \in \zeta$. So for every $w \in a_1*a_2$ there is $y \in d_1*d_2$ such as $(w,y) \in \xi$. Also for every $y \in d_1*d_2$ there is $z \in b_1*b_2$ such that $(y,z) \in \zeta$. Thus for every $w \in a_1*a_2$ there is $z \in b_1*b_2$ such as $(w,z) \in \xi\zeta$. Similarly one can prove that for every z in b_1*b_2 there is $w \in a_1*a_2$ such as $(w,z) \in \xi\zeta$ and so $\xi\zeta$ is a homomorphic relation.

If ξ is a reflexive homomorphic relation in a quasicanonical hypergroup H, then ξ is an equivalence relation. Indeed let

$(a,b) \in \xi$. Since ξ is reflexive: (a,a), $(b,b) \in \xi$ and since ξ is homomorphic, $(a^-,b^-) \in \xi$. Therefore for every $x \in a*a^-$ there is $w \in a*b^-$ such as $(x,w) \in \xi$. So $(v,w) \in \xi$. But $(b,b) \in \xi$ and therefore for every $y \in w*b$ we have $[(v*b) \times \{y\}] \cap \xi \neq \emptyset$ or $(b,y) \in \xi$ for every $y \in w*b$. But $w \in a*b^-$ thus $a \in w*b$ and so $(b,a) \in \xi$. Therefore ξ is symmetric. Next let (a,b), $(b,d) \in \xi$. Since $(b^-,b^-) \in \xi$ we have $[\{y\} \times (d*b^-)] \cap \xi \neq \emptyset$ for every $y \in b*b^-$ and so $(v,w) \in \xi$ for some $w \in d*b^-$. Next because $(a,b),(v,w) \in \xi$ we have that $[(v*a) \times \{z\}] \cap \xi \neq \emptyset$ for every $z \in w*b$. But $w \in d*b^-$, so $d \in w*b$, thus $(a,d) \in \xi$ and therefore ξ is transitive.

A homomorphic relation which is also an equivalance relation, is called congruence relation.

If ξ is a congruence relation in a quasicanonical hypergroup H, then ξ defines a partition in H and the set H/ξ of the classes mod ξ becomes a quasicanonical hypergroup, if the hypercomposition is defined via the H's hypercomposition, i.e. $x\xi \# y\xi = \{ w\xi \mid w \in x*y \}$. The class of v (mod ξ), where v is the neutral element of H, is a quasicanonical syb-hypergroup of H and if h is a quasicanonical sub-hypergroup of H, which belongs to H/ξ for some congruence ξ, then $h = v\xi$.

Next suppose that H is quasicanonical hypergroup and let h be quasicanonical sub-hypergroup of H. Then h defines a partition in H and the set of classes mod h, $H_h = H/h$, becomes a hypergroup, if we define the hypercomposition # of any two classes ah, bh as follows: $ah\#bh = \{ wh \in H_h \mid wh \subseteq ah*bh \}$

Afterwards we shall look for a condition for h which will make H_h a quasicanonical hypergroup. We observe that h is a right neutral element of H_h. Consequently, if H_h is quasicanonical, h has to be a left neutral element as well. Thus it must be valid that $h*xh \subseteq xh$ for every $x \in$ H. From this relation derives that we must have $h*x \subseteq x*h$, for every $x \in H$, which, since H is quasicanonical, gives $h*x = x*h$. Thus one can prove that:

Proposition 2.1 If h is a quasicanonical sub-hypergroup of H, which satisfies the condition $h*x \subseteq x*h$ for every $x \in H$, then H_h is a quasicanonical hypergroup and $xh\#yh = \bigcup_{z \in x*y}\{zh\}$. Also if h is a quasicanonical sub-hypergroup of H, then $h*x \subseteq x*h$ if and only if for every $t \in h*x$ is valid $x^-*t \cap h \neq \emptyset$.

Theorem 2.1 If h is a quasicanonical sub-hypergroup of a quasicanonical hypergroup H, then h is a member of an appropriate quotient system of H by a congruence ξ, if and only if $x*h = h*x$ for all $x \in H$.

P r o o f. Let h be a member of H/ξ for some congruence ξ. Then $h = v\xi$. For $v\xi$ now we have $(v\xi)*x \subseteq v\xi*x\xi = x\xi$. Then for every $y \in x\xi$ we have $y\xi = x\xi$ thus $v\xi \in y\xi*(x\xi)^-$ and so $v\xi \in \{z\xi \mid z \in y*x^-\}$ Therefore there is $w \in v\xi \cap y*x^-$. So $y \in w*x$, from which we get

$x\xi \subseteq v\xi * x$ and thus $v\xi * x = x\xi$. Also $x * v\xi \subseteq x\xi$. Therefore $x * v\xi \subseteq v\xi * x$ and since H is quasicanonical this last relation becomes an equality. Now suppose that h is a quasicanonical sub-hypergroup of H such that $x * h = h * x$. We define in H a relation ξ as follows:

$$(x,y) \in \xi \iff x * y^- \cap h \neq \emptyset$$

Firstly we shall prove that ξ is a homomorphic relation. Indeed $v \in h$, thus $(v,v) \in \xi$. Next suppose that $(a_1,b_1),(a_2,b_2) \in \xi$. Then $(a_1 * b_1^-) \cap h \neq \emptyset$ and $(a_2 * b_2^-) \cap h \neq \emptyset$. So there exists $s \in h$ such that $s \in a_1 * b_1^-$. Thus $a_1 \in s * b_1$ and $b_1 \in a_1 * s^-$. From these last relations we get that $a_1 \in h * b_1$ (1), $b_1 \in a_1 * h$ (2). Similarly there is $t \in h$ such that $t \in a_2 * b_2^-$. Thus $a_2 \in t * b_2$, $b_2 \in a_2 * t^-$ and so we have $a_2 \in h * b_2$ (3), $b_2 \in a_2 * h$ (4). From (1) and (3) derives that $a_1 * a_2 \subseteq (b_1 * b_2) * h$. So for every $a \in a_1 * a_2$ there is $b \in b_1 * b_2$ such that $a \in b * h$ or equivalently $(a * b^-) \cap h \neq \emptyset$ which, because of the definition of ξ, gives $(a,b) \in \xi$. Similarly from (2) and (4) derives that for every $b \in b_1 * b_2$ we have $[(a_1 * a_2) \times \{b\}] \cap \xi \neq \emptyset$. Therefore ξ is a homomorphic relation. Now, in order to show that ξ is an equivalence relation, according to what we have mention above, we just have to prove that ξ is reflexive. Indeed, $v \in (a * a^-) \cap h \neq \emptyset$ thus $(a,a) \in \xi$ and so the Theorem.

A quasicanonical sub-hypergroup h of a quasicanonical hypergroup H is called <u>**normal**</u> if and only if it is a member of an appropriate quotient system of H by some congruence relation ξ. A quasicanonical hypergroup H is the <u>**direct product**</u> of its sub-hypergroups $A_1, A_2, \ldots, A_n$ if:

i) the $A_1, A_2, \ldots, A_n$ are normal in H

ii) $H = A_1 * A_2 * \ldots * A_n$

iii) $A_i \cap A_1 * A_2 * \ldots * A_{i-1} * A_{i+1} * \ldots * A_n = \{v\}$, $i = 1,2,\ldots,n$

3. HOMOMORPHISMS OF HYPERGROUPS

Let A, B be two hypergroups and let P(B) be the power set of B. Then a function $\psi: A \longrightarrow P(B)$ is called **homomorphism** if $\psi(x * y) \subseteq \psi(x) * \psi(y)$ (1), for every $x,y \in A$. ψ is called **strong homomorphism** if the above relation holds as an eqality, i.e. $\psi(x * y) = \psi(x) * \psi(y)$ (2). A function $\psi: A \longrightarrow B$ is called **strict homomorphism** if (1) is valid. $\psi: A \longrightarrow B$ is called **normal homomorphism** if (2) is valid. These definitions of the homomorphisms have been introduced by M. Krasner [5]. We shall give two examples of homomorphisms

i) Let f,g be two functions from the hypergroup A to the hypergroup B. We define the hypercomposition $f * g$ to be the set $\{h: A \longrightarrow B \mid (\forall x \in A)[h(x) \in f(x) * g(x)]\}$. Thus for every $x \in A$ we have $(f * g)(x) \subseteq B$. So we can say that $f * g$ is a function from A to P(B) of the form: $(f*g)(x) = \bigcup_{h \in f*g}\{h(x)\}$. For this function it can be proved that the equality $(f*g)(x)=f(x)*g(x)$ holds, and that if f,g are normal homomorphisms and B is commutative then $f*g$ is a homomorphism. Also one can see that if f,g are strict homomorphisms then $f*g$ is also a homomorphism (see [6]).

ii) Let φ,ψ be two homomorphisms from A to $P(B)$. Then we define a new function from A to $P(B)$, the union of φ and ψ, as follows: $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$.

Proposition 3.1 Let B, A, A', C be hypergroups, f, s, t, g be functions as follows:

$$B \xrightarrow{\ f\ } A \xrightarrow{\ s,t\ } A' \xrightarrow{\ \rho,g\ } C$$

and φ, ψ be homomorphisms from B to $P(A)$ and from A' to $P(C)$ respectively. Then the relations hold:

i) $g^\circ(s*t) = g^\circ s * g^\circ t$ if g is a normal homomorphism

ii) $(s*t)^\circ f = s^\circ f * t^\circ f$

iii) $\rho^\circ(s*t) = \rho^\circ s * \rho^\circ t$ if ρ is a strict homomorphism

iv) $\psi^\circ(s*t) \subseteq \psi^\circ s * \psi^\circ t$

v) $(s*t)[\varphi(x)] \subseteq s(\varphi(x)) * t(\varphi(x))$ for every $x \in B$.

Proof. i) Let $h \in s*t$ and $x \in A$, then $h(x) \in s(x)*t(x)$, and $(g^\circ h)(x) = g(h(x)) \in g(s(x))*g(t(x)) = (g^\circ s)(x)*(g^\circ t)(x)$ for every $x \in A$. Thus $g^\circ h \in g^\circ s * g^\circ t$ and so $g^\circ(s*t) \subseteq g^\circ s*g^\circ t$. Next if $h \in g^\circ s*g^\circ t$. then for every $x \in A$ we have: $h(x) \in g(s(x))*g(t(x)) \Longleftrightarrow$ $\Longleftrightarrow h(x) \in g((s*t)(x))$ which means that there is $k_x \in s*t$ such as $h(x) \in (g^\circ k_x)(x)$ for every $x \in A$. Now we define a function as follows: $k(x) = k_x(x)$ for every $x \in A$. Then $k \in s*t$, so $g^\circ s*g^\circ t \subseteq g^\circ(s*t)$ and (i) is proved.

ii) Let $h \in s*t$ and $x \in B$, then $(h^\circ f)(x) \in (s*t)(f(x)) \Longleftrightarrow$ $\Longleftrightarrow (h^\circ f)(x) \in s(f(x))*t(f(x))$ thus $(s*t)^\circ f \subseteq s^\circ f*t^\circ f$. Next let $h \in s^\circ f*t^\circ f$. Then $h(x) \in (s*t)(f(x))$. So there is a function $k_{f(x)}$ in $s*t$ such as $h(x) = k_{f(x)}(f(x))$ for every $x \in B$. Now we define the function $k: A \longrightarrow A'$ as follows :

$$k(y) = \begin{cases} k_{f(x)}(f(x)) & \text{if } y = f(x) \\ k(y) \in (s*t)(y) & \text{if } y \in A \setminus f(B) \end{cases}$$

which belongs to $s*t$. Thus there is $k \in s*t$ such that $k^\circ f = h$ and so $s^\circ f*t^\circ f \subseteq (s*t)^\circ f$. Analogous is the proof of the other relations.

Proposition 3.2 Let B, A, A', C be hypergroups, φ, ψ be homomorphisms from A to $P(A)$, k, ρ be functions from A' to C and g be a function from B to A. Then the relations hold

i) $k[(\varphi*\psi)(x)] = k[\varphi(x)] * k[\psi(x)]$ for every $x \in A$, if k is a normal homomorphism.

ii) $\rho[(\varphi*\psi)(x)] \subseteq \rho[\varphi(x)] * \rho[\psi(x)]$ for every $x \in A$, if ρ is a strict homomorphism.

iii) $(\varphi*\psi)^\circ g = \varphi^\circ g * \psi^\circ g$

Proposition 3.3 Let φ be a normal homomorphism from a quasicanonical hypergroup E to a quasicanonical hypergroup H. Then $\ker\varphi = \varphi^{-1}(\varphi(\nu))$ is a normal sub-hypergroup of E.

Proof. With no essential difficulty it can be prooved that $\ker\varphi$ is a quasicanonical sub-hypergroup of E. In order to prove that it is a normal sub-hypergroup of E, according to proposition 2.1, we must show that for every $t \in (\ker\varphi)*a$ derives that $(a^{-}*t) \cap \ker\varphi \neq \emptyset$. Indeed, let $t \in (\ker\varphi)*a$. Then there is $s \in \ker\varphi$ such as $t \in s*a$ from where $\varphi(t) \in \varphi(s*a)$ or $\varphi(t) \in \varphi(s)*\varphi(a)$ or $\varphi(t) = \varphi(a)$ so $\varphi(a^{-}*t) = \varphi(a^{-})*\varphi(t) = \varphi(a^{-})*\varphi(a) \ni \varphi(v)$, thus $a^{-}*t \cap \ker\varphi \neq \emptyset$.

<u>Remark.</u> If $\varphi(a) = v$ for some $a \in E$, then $\varphi(v) = v$. Indeed $v = \varphi(a) = \varphi(a*v) = \varphi(a)*\varphi(v) = v*\varphi(v) = \varphi(v)$. Also if $\varphi(v) = v$, then $\varphi(a^{-}) = (\varphi(a))^{-}$. Indeed $\varphi(v) \in \varphi(a^{-}*a) \Longleftrightarrow v \in \varphi(a^{-})*\varphi(a) \Longleftrightarrow (\varphi(a))^{-} = \varphi(a^{-})$.

Theorem 3.1 If $\varphi: E \longrightarrow H$ is a normal homomorhpism of quasicanonical hypergroups, then $E / \ker\varphi \approx \mathrm{Im}\varphi$.

Proof. Since $\ker\varphi$ is a normal sub-hypergroup of E the $E / \ker\varphi$ is a quasicanonical hypergroup. We define the function $\Phi : E / \ker\varphi \longrightarrow H$ as follows: $\Phi(a\ker\varphi) = \varphi(a)$. Φ is a normal homomorphism because, if $a\ker\varphi = b\ker\varphi$ then there exists $x \in \ker\varphi$ such that $a \in b*x$ from where we get the sequence of the relations: $\varphi(a) \in \varphi(b*x)$, $\varphi(a) \in \varphi(b)*\varphi(x)$, $\varphi(a) \in \varphi(b)*\varphi(v)$, $\varphi(a) \in \varphi(b*v)$, so $\varphi(a) = \varphi(b)$ and therefore $\Phi(a\ker\varphi) = \Phi(b\ker\varphi)$. Thus Φ is a function from $E / \ker\varphi$ to H. Now we will prove that Φ is a homomorphism. Indeed, $\Phi(a\ker\varphi * b\ker\varphi) = \Phi[\, U_{x\in a*b}\{x\ker\varphi\}\,] = [\, U_{x\in a*b}\{\Phi(x\ker\varphi)\}\,] = U_{x\in a*b}\{\varphi(x)\} = \varphi[\, U_{x\in a*b}\{x\}\,] = \varphi(a*b) = \varphi(a)*\varphi(b) = \Phi(a\ker\varphi)*\Phi(b\ker\varphi)$. Consequently Φ is a normal homomorphism. Next we will prove that Φ is an one to one function. Let $\Phi(a\ker\varphi) = \Phi(b\ker\varphi)$, then we have: $\varphi(a) = \varphi(b)$, $\varphi(b^{-})*\varphi(a) = \varphi(b^{-})*\varphi(b)$, $\varphi(b^{-})*\varphi(a) = \varphi(b^{-}*b)$, $\varphi(v) \in \varphi(b^{-})*\varphi(a)$, $\varphi(v) \in \varphi(b^{-}*a)$, $b^{-}*a \cap \ker\varphi \neq \emptyset$, thus $a\ker\varphi = b\ker\varphi$. Moreover $\mathrm{Im}\Phi = \mathrm{Im}\varphi$ since if $a \in E$ we have: $\Phi(a\ker\varphi) \in \mathrm{Im}\Phi$ and $\Phi(a\ker\varphi) = \varphi(a) \in \mathrm{Im}\varphi$. So Φ is an isomorphism from $E / \ker\varphi$ to $\mathrm{Im}\varphi$.

The following Theorems and Proposition are valid for the quasicanonical hypergroups (see also [6]).

Theorem 3.2 If A, B are quasicanonical sub-hypergroups of a quasicanonical hypergroup H and if A is normal in $\langle A \cup B\rangle$ then $A \cap B$ is a quasicanonical sub-hypergroup of B and $\langle A\cup B\rangle/A \approx B/A\cap B$.

Proposition 3.4 Let H, E be two quasicanonical hypergroups and let A, B be two normal sub-hypergroups of H, E respectively. If φ is a normal homomorphism from H to E such that $\varphi(A) \subseteq B$, then the function $\Phi(xA) = \varphi(x)B$ is a normal homomorphism from H / A to E / B .

Theorem 3.3 Let A, B be two normal sub-hypergroups of a quasicanonical hypergroup H. If $A \subseteq B$, then $H/B \approx (H/A)/(B/A)$.

Next we will present five Propositions which concern the direct product of the hypergroups and the homomorphisms.

Proposition 3.5 Let AxB be the direct product of two quasicanonical hypergroups. Then there are four normal homomorphisms ε_i, ρ_i, $(i = 1,2)$ such that $\rho_1(a_1,a_2)=a_1$, $\rho_2(a_1,a_2)=a_2$, $\varepsilon_1(a_1)=(a_1,\nu)$, $\varepsilon_2(a_2)=(\nu,a_2)$ which satisfy the following conditions:
$\rho_1\varepsilon_1 = 1_A$, $\rho_1\varepsilon_2 = \nu$, $\rho_2\varepsilon_1 = \nu$, $\rho_2\varepsilon_2 = 1_B$, $\rho_1\varepsilon_1+\rho_2\varepsilon_2 = 1_{A\times B}$

Proposition 3.6 Let three quasicanonical hypergroups B, A_i, $(i = 1,2)$ and two strict homomorphisms $\tau_i : B \text{ ---> } A_i$ $(i = 1,2)$. Then there exists a uniquely defined homomorphism $\tau : B \text{ ---> } A_1 \times A_2$ such as $\rho_1 \circ \tau = \tau_1$ and $\rho_2 \circ \tau = \tau_2$.

Proposition 3.7 Let three quasicanonical hypergroups B, A_i, $(i = 1,2)$ and two homomorphisms τ_i from B to A_i $(i = 1,2)$. Then there exists a maximum homomorphism τ from B, to $A_1 x A_2$ such as $\rho_1(\tau(x)) = \tau_1(x)$ and $\rho_2(\tau(x) = \tau_2(x)$.

Proposition 3.8 Let three quasicanonical hypergroups B, A_i, $(i = 1,2)$ and two homomorphisms $\sigma_i : A_i \text{ ---> } B$ $(i = 1,2)$. Then there exists a maximum homomorphism $\sigma : A_1 \times A_2 \text{ ---> } B$ such as $\sigma(\varepsilon_1(x)) = \sigma_1(x)$ and $\sigma(\varepsilon_2(x)) = \sigma_2(x)$.

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