NATIONAL TECHNICAL UNIVERSITY

DEPARTMENT OF SCIENCES

HYPERGROUPS AND THEIR APPLICATIONS

by

CHRISTOS G. MASSOUROS

DOCTORAL THESIS

SUBMITTED TO THE DEPARTMENT OF SCIENCES OF THE NATIONAL TECHNICAL UNIVERSITY OF ATHENS

ATHENS 1988

S U M M A R Y

In a congress of Scandinavian Mathematicians held in Stokholm, in 1934, F. Marty introduced a new mathematical structure, which he named hypergroup. Lots of papers based either on the research of the algebraic structure of hypergroups, or on the study and the development of the applications that can possibly derive from them, have been published since then. Yet, since the hypergroup can be considered as one of the most general structures of the abstruct Algebra, very often it is endowed with axioms which vary in strength, but which are always defined each time by a specific mathematical need.

This dissertatiton is a contribution to the study of the algebraic properties of the hypergroup, as well as to the study of its applications. It consists of five parts which refer to Hypergroups and Join hypergroups, Homiogene hypergroups, Strongly homiogene hypergroups, Applications of hypergroups, and Hyperfields, respecively.

CHAPTER I. Hypergroups in general as well as a special category of hypergroups called **join hypergroups** are studied here.

At first the properties of the semisubhypergroups and the closed subhypergroups of a hypergroup are examined and studied in this Chapter. Also in the same Chapter the **monogene semisubhypergroups** of a hypergroup are studied in conection with the arbitrary semisubhypergroups. It is also shown how the latter derive from the former.

Next the **dependent**, the **independent**, the **correlated** and the **non correlated elements** are introduced and studied. These elements, together with the **fundamental** ones are essential for the generation of the hypergroups. This Chapter also refers to the **generators**, the **generating set** and the **generating number** <**gennum**> of a hypergroup.

In addition it is shown that every linear space can define through its operations a join hypergroup which is called **attached hypergroup.** The convex sets of the linear space correspond to the semisubhypergroup of its attached hypergroup, while the linear subspaces correspond to the closed subhypergroup of the attached hypergroup. This correlation of the linear spaces with the hypergroups lead to significant applications which are presented in the fourth Chapter.

CHAPTER II. In this Chapter two new axioms are introduced to the hypergroup and this new hypercompositional structure is called **homiogene hypergroup.** Mainly the join homiogene hypergroup is studied here.

It is proved that the independent elements are the generators of the closed subhypergroups. Moreover a number of propositions presents the properties of the generating sets of the closed subhypergroups.

Here is also studied when a hypergroup **covers** another one. Finally a relation is proved, which connects the generating numbers of two closed subhypergroups with the generating numbers of their intersection and of the closed subhypergroup which is spanned by their union.

CHAPTER III. In this Chapter the homiogene hypergroup is endowed with two new axioms and so the **strongly homiogene hypergroup** is introduced. In these hypergroups it is proved that the dependent and the correlated elements are the same. This affects the relevant propositions and theorems of previous Chapters. A number of properties which connect mainly the closed subhypergroups of these hypergroups with the monogene subhypergroups are proved.

Most of the applications of the linear spaces presented in the next Chapter are derived from the conclusions of this Chapter.

CHAPTER IV. A relation which connects the generating number of an attached hypergroup with the dimention of the respective linear space is proved.

Next well known Theorems of the linear spaces (such as Kakutani, Stone, Helly, Randon, Caratheodory, Steinitz) are proved as Corollaries and special cases of some Theorems of the previous Chapters.

Also in this Chapter a Theorem is proved which connects the two types of hypermodules. In addition relations which connect the hypermodules with some Geometries (such as Spherical Geometries, Progective Geometries etc.) derive and it is shown how the hypermodules can describe such Geometries.

CHAPTER V. This chapter deals with the hyperfields. The notion of the hyperfield was introduced by M. Krasner in 1956, in order to define a certain approximation of a complete valued field by sequences of such fields. These hyperfields were named residual hyperfields. Later on the class of the quotient hyperfields (which contains the residual ones) was constructed by M. Krasner. In the second paragraph of this Chapter a hyperfield, which is not a quotient hyperfield is presented. The other parts of this Chapter present all the known constructions of hyperfields and therefore a review on this area is presented.