On Open and Closed Hypercompositions

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Abstract. This paper presents hypercompositions which either contain the two participating elements into their result (closed hypercomposition) or not (open hypercomposition). Moreover it introduces and studies the inaccessible subsets of a hypergroup, which are the concept antipodal of its semi-sub-hypergroups.

1. INTRODUCTION

Let H be a non-empty set. A composition in H is a map from $H \times H$ to H, while a hypercomposition in H is a map from $H \times H$ to the power-set $P(H)$ of H. Hence the composition is a partial case of the hypecomposition. In 1934 F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [8]. An algebraic structure (H, ·), $H \neq \varnothing$, which satisfies the axioms

i. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associative axiom) and

ii. $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductive axiom)

is called *group* if « · » is a composition and *hypergroup* if « · » is a hypercomposition [12].

F. Marty defined in [8] the two induced hypercompositions in a hypergroup, the right and the left division, which derive from the hypercomposition of the hypergroup:

$$\frac{a}{|\,b} = \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b\,|} = \{x \in H \mid a \in bx\}.$$

It is obvious that if the hypercomposition is commutative, then the right and the left division coincide. For the sake of notational simplicity, a / b or $a : b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b \backslash a$ or $a..b$ is used to denote the left division [5, 10]. Obviously, if H is a group, then $a / b = ab^{-1}$ and $b \backslash a = b^{-1}a$. In the theory of hypergroups the following principle of duality is valid [5, 6]:

Given a theorem, the dual statement which results from the interchanging of the order of the hypercomposition «·» (and necessarily interchanging of the left and the right division), is also a theorem.

A hypercomposition is called *closed* (or *containing*; sometimes also called *extensive* [17]) if the two participating elements are in the result of the hypercomposition. A hypercomposition is called *right closed* if $a \in ba$ for all $a, b \in H$ and *left closed* if $a \in ab$ for all $a, b \in H$. A composition can be right or left closed but it cannot be closed. A hypercomposition is called *right open* if $a \notin ba$ for all $a, b \in H$ with $b \neq a$ while it is called *left open* if $a \notin ab$ for all $a, b \in H$ with $b \neq a$. A hypercomposition is called *open* if it is both right and left open. Right closed compositions are left open and left closed compositions are right open. The composition in a group is neither open nor closed.

Example 1.1. Let H be a non-void set. The *B-hypercomposition* in H [13], that is $a \bullet b = \{a, b\}$ for all

$a,b \in H$ and the *total hypercomposition* in H, that is $a*b = H$ for all $a,b \in H$ are both closed hypercompositions. Moreover if "$\circ$" is any other closed hypercomposition in H, then $a \bullet b \subseteq a \circ b \subseteq a*b$ for all $a,b \in H$. For example, such a closed hypercomposition is the one of the *monogene hypergroup*, where $ab = \{a,b\}$ for all $a \neq b$, and $aa = H$ for all $a \in H$.

Example 1.2. If $(H,\cdot)$ is a semigroup and $a \circ b = \{a,b,a \cdot b\}$ for all $a,b \in H$, then "$\circ$" is a closed hypercomposition. Moreover if $(H,\cdot)$ is a hypergroup, then $a \circ b = \{a,b\} \cup a \cdot b$ is a closed hypercomposition as well.

Example 1.3. Let F be a field and G be a subgroup of its multiplicative group. Then, per Theorem 3.1 [11], the Krasner's hypercomposition $xG \dot{+} yG = \{zG \mid z \in xp + yq,\ p,q \in G\}$ [7], provides examples of closed hypercompositions. E.g. if the index of G is 5, $charF \neq 2,3$ and the order of G is greater than 23 then the hypercomposition is closed, while if the order of G is less than 23 then the hypercomposition is not closed.

Example 1.4. Let A be a fortified transposition hypergroup in which every element is attractive. Then, per Theorem 7 [6] and Proposition 2.1.ii [14], $\{a,b\} \subseteq ab$ for each $a,b \in A$. Therefore, the hypercomposition is closed in all fortified transposition hypergroups which are consist only of attractive elements.

Example 1.5. A quasi-ordering hypergroup [1,2,3] is a hypergroup H, endowed with a hypercomposition "$\circ$" which satisfies the following conditions:

(i) $a \circ b = a^2 \cup b^2$

(ii) $a \in a^2 = a^3$

for all $a,b \in H$. The hypercomposition of the quasi-ordering hypergroup is a closed hypercomposition.

Example 1.6. If «$\circ$» is a hypercomposition in a non-empty set H, then the hypercomposition $a*b = H - a \circ b$ is called complement hypercomposition of «$\circ$» [4]. Thus, the complement hypercompositions of those of example 1.1 are $a*b = H - \{a,b\}$ and $a*b = H - \{a,b,a \cdot b\}$ which are open hypercompositions. In [9] the following hypercomposition was used for the construction of non-quotient hyperfields over a set H with $card\ H > 3$:

$a \circ b = \{a, b\}$, for all $a,b \in H$ with $a \neq b$ and $a \circ a = H - \{a\}$, for all $a \in H$.

Its complement hypercomposition is:

$a*b = H - \{a,b\}$, for all $a,b \in H$ with $a \neq b$ and $a*a = \{a\}$, for all $a \in H$.

This is an open hypercomposition and it is essentially the one used by A. Nakassis, in order to prove the existence of non-quotient hyperrings [16].

Example 1.7. Let V be a vector space over a field F. Then, per Proposition 1 [15], the hypercomposition:

$$x \dot{+} y = \{\kappa x + \lambda y \mid \kappa + \lambda = 1,\ \ \kappa > 0,\ \ \lambda > 0\}$$

is an open hypercomposition.

More examples of open and closed hypercomposition can be found in polysymmetrical hypergroups [18].

2. PROPERTIES OF THE OPEN AND THE CLOSED HYPERCOMPOSITIONS

Proposition 2.1. [5, 10] *In any hypergroup, the following are valid:*

i. $(a/b)/c = a/(cb)$ *and* $c \backslash (b \backslash a) = (bc) \backslash a$ *(mixed associativity),*

ii. $(b \backslash a)/c = b \backslash (a/c)$,

iii. $b \in (a/b) \backslash a$ *and* $b \in a/(b \backslash a)$.

Proposition 2.2. *The hypercomposition in a hypergroup H is right closed if and only if $a/a = H$ for all $a \in H$, while it is left closed if and only if $a \backslash a = H$ for all $a \in H$.*

P r o o f. Suppose that the hypercomposition is right closed. Then $a \in xa$ for all $x \in H$. Hence $x \in a/a$ for all $x \in H$. Therefore $H = a/a$. Conversely now. Let $H = a/a$ for all $a \in H$. Then $a \in ba$ for all $a,b \in H$. Thus the hypercomposition is right closed.

Proposition 2.3. *The hypercomposition in a hypergroup H is right open if and only if $a/a = a$ for all $a \in H$, while it is left open if and only if $a \backslash a = a$ for all $a \in H$.*

P r o o f. Suppose that the hypercomposition is right open. Let a be an arbitrary element of H. Then $a \notin ba$ for all $b \in H$ with $b \neq a$. Hence $b \notin a/a$ for all $b \in H$ with $b \neq a$. Moreover, per reproductive axiom, $a \in Ha$, thus $a \in aa$. Therefore $a = a/a$. Conversely now. Let $a/a = a$ for all $a \in H$. Then $b \notin a/a$ for all $b \in H$ with $b \neq a$. So $a \notin ba$, for all $b \in H$ with $b \neq a$, i.e. the hypercomposition is right open.

Proposition 2.4. *If a hypercomposition in a hypergroup H is right or left open, then all its elements are idempotent.*

P r o o f. Suppose that the hypercomposition is right open and that for some $a \in H$ there exists $b \neq a$, such that $b \in aa$. Then, $a/b \subseteq a/aa$. Per Propositions 2.1.i and 2.3, $a/(aa) = (a/a)/a = a/a = a$. Thus, $a/b = a$. Therefore, $a \in ab$, which contradicts the assumption. Hence, $aa = a$ for all $a \in H$.

3. INACCESSIBLE ELEMENTS

A non-empty subset S of a hypergroup H is called *semi-subhypergroup* if it is stable under the hypercomposition, i.e. if it has the property $xy \subseteq S$ for all $x, y \in S$. S is a subhypergroup of H, if it satisfies the axiom of reproduction, i.e. if the equality $xS = Sx = S$ is valid for all $x \in S$.

Definition 3.1. Let Q be a non-empty subset of a hypergroup H. Then an element $a \in Q$ is called *Q-inaccessible* or *inaccessible in Q* if a is never contained in the result of the hypercomposition of two distinct elements of Q. If every element of Q is inaccessible in Q, then Q is called *inaccessible subset* of H.

One can realize that the inaccessible subsets of a hypergroup are the concept antipodal of its semi-sub-hypergroups.

Example 3.1. Let $(\mathbb{Z},+)$ be the additive group of integers. Then any subset of the odd numbers and the odd numbers themselves, are inaccessible subsets of $\mathbb{Z}$.

Proposition 3.1. *A non-empty subset Q of H is inaccessible if $xy \cap Q = \varnothing$ for any two distinct elements x, y in Q.*

Proposition 3.2. *If the hypercomposition in a hypergroup H is right, left or bilaterally closed, then Q-inaccessible elements do not exist, for any subset Q of H.*

Proposition 3.3. *If the hypercomposition in a hypergroup H is right, left or bilaterally closed, then H does not have inaccessible subsets.*

Proposition 3.4. *If the hypercomposition in a hypergroup H is open, then the bisets in H are inaccessible subsets.*

Proposition 3.5. *If S, T are semi-subhypergroups or inaccessible subsets of a hypergroup H and $S \cap T \neq \varnothing$, then $S \cap T$ is a semi-subhypergroup or an inaccessible subset of H respectively.*

In what follows we will consider hypergroups with open hypercompositions.

Definition 3.2. An element a of a semi-subhypergroup S is called *interior element* of S if for each $x \in S$ there exists $y \in S$ such that $a \in xy$. An element of S which is not an interior element is called *frontier element* of S.

Proposition 3.6. *Let a be an interior element and b a frontier element of a semi-subhypergroup S of H. Then ab and ba consists only of interior elements of S.*

P r o o f. Let c be an element of ab and x an arbitrary element of S. Since a is an interior element of S it derives that there exists $z \in S$ such that $a \in xz$. Hence $c \in (xz)b = x(zb)$. Therefore, there exists $y \in zb \subseteq S$ such that $c \in xy$. Thus c is an interior element of S. Dually ba consists of interior elements and the Proposition is established**.**

Proposition 3.7. *If a semi-subhypergroup S of a hypergroup H consists only of interior elements, then S is a subhypergroup of H.*

P r o o f. Suppose that a is an arbitrary element of S. Since S is a semi-subhypergroup, $aS \subseteq S$ is valid. Next let y be any element of S. Since y is an interior element, there exists $x \in S$ such that $y \in ax$. Hence $S \subseteq aS$. Therefore $S = aS$. Dually, $S = Sa$ holds.

Proposition 3.8. *If S is a semi-subhypergroup of a hypergroup H and an element $a \in S$ is S-inaccessible, then it is a frontier element* of S.

Proposition 3.9. *Q is a maximal inaccessible set of a hypergroup H if and only if*

$$H = Q \cup QQ \cup Q / Q \cup Q \setminus Q \ .$$

P r o o f. Suppose that Q is an inaccessible set such that $H = Q \cup QQ \cup Q / Q \cup Q \setminus Q$. Let $c \in H - Q$. Consider the set $P = Q \cup \{c\}$. If $c \in QQ$, then there exist $x, y \in Q$ such that $c \in xy$. Thus $xy \cap P \neq \varnothing$. Therefore P is not inaccessible set. If $c \in Q / Q$, then there exist $x, y \in Q$ such that $c \in x / y$. Hence $x \in cy$. Thus $cy \cap P \neq \varnothing$. Consequently P is not inaccessible set. Dually, P is not inaccessible set, when $c \in Q \setminus Q$. Therefore Q is not properly contained in any inaccessible set, so by definition Q is a maximal inaccessible set. Conversely now, suppose that Q is a maximal inaccessible set of H. Then for all $c \in H - Q$, the set $V = Q \cup \{c\}$ is not inaccessible. Thus $V \cap \left[\bigcup_{\substack{x,y \in V \\ x \neq y}} xy \right] \neq \varnothing$. That is $V \cap [cQ \cup Qc \cup QQ] \neq \varnothing$. Since the hypercomposition is open, it derives that $c \notin cQ \cup Qc$. Thus, if $c \in V \cap [cQ \cup Qc \cup QQ]$, then $c \in QQ$. If $c \notin V \cap [cQ \cup Qc \cup QQ]$, there exists $x \in Q$ such that $x \in V \cap [cQ \cup Qc \cup QQ]$. Since Q is inaccessible, it derives that $x \notin QQ$. So $x \in cQ$ or $x \in Qc$. Thus $c \in Q / Q$ or $c \in Q \setminus Q$. Consequently $H = Q \cup QQ \cup Q / Q \cup Q \setminus Q$.

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