On 2-element Fuzzy and Mimic Fuzzy Hypergroups

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Dedicated to the memory of mathematicians-teachers Gerasimos G. Legatos (1916-2012), Georgios C. Massouros (1920-1990) and Jean D. Mittas (1921-2012), whose struggle for the advancement of mathematical education in Greece greatly influenced and inspired younger scientists

Abstract. This paper deals with the enumeration of the different classes of 2-element fuzzy hypergroups and with the construction of some classes of 2-element mimic fuzzy hypergroups. It also introduces the notion of the pseudo-mimic fuzzy hypergroup.

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1. CRISP AND FUZZY HYPERGROUPS

Hypercompositional algebra was born in 1934, when F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [14]. A (*crisp*) *hypercomposition* or *hyperoperation* in a non-empty set H is a function from $H\times H$ to the power set $P(H)$ of H. A non-void set H endowed with a hypercomposition "$\cdot$" is called *hypergroupoid* if $ab\neq\varnothing$ for any a,b in H; otherwise, it is called *partial hypergroupoid.* Note that, if A, B are subsets of H, then AB signifies the union $\bigcup_{(a,b)\in A\times B} ab$. Since $A\times B=\varnothing \Leftrightarrow A=\varnothing$ or $B=\varnothing$, one observes that, if $A=\varnothing$ or $B=\varnothing$, then $AB=\varnothing$ and vice versa. aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a.

Definition 1. [14] A *hypergroup* is a non-void set H endowed with a hypercomposition, which satisfies the following axioms:

i. $(ab)c=a(bc)$ for every $a,b,c\in H$ (associativity) and

ii. $aH=Ha=H$ for every $a\in H$ (reproduction).

If only (i) is valid, then the hypercompositional structure is called *semi-hypergroup*, while if only (ii) is valid, then it is called *quasi-hypergroup*. The quasi-hypergroups in which the weak associativity is valid, i.e. $(ab)c\cap a(bc)\neq\varnothing$ for every $a,b,c\in H$, were named *H_V-groups* [30]. It is worth mentioning that the result of the hypercomposition of any two elements in a hypergroup or in an H_V-group is always a non-void set (e.g. see [18]). Subsequently, hypergroups were enriched with internal and external hypercompositions and so new hypercompositional structures came into being (e.g.: see [15, 16]).

F. Marty also defined in [14] the two induced hypercompositions (right and left division) resulting from the hypercomposition of the hypergroup, i.e.:

$$\frac{a}{|b}=\{x\in H \mid a\in xb\} \quad\text{and}\quad \frac{a}{b|}=\{x\in H \mid a\in bx\}.$$

It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, a/b or $a:b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b\backslash a$ or $a..b$ is used to denote the left division. The use of these induced hypercompositions led to the definition of transposition and join hypergroups and H_V-groups [9, 19, 20, 21].

As has been proven in [18], Definition 1 is equivalent to the following:

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Definition 2. A *hypergroup* is a non-void (crisp) set H endowed with a (crisp) hypercomposition, which satisfies the following axioms:

i. $(ab)c = a(bc)$ for every $a,b,c \in H$ (associativity) and

ii. $a/b \neq \varnothing$ and $b \backslash a \neq \varnothing$ for every $a,b \in H$.

Linking hypercompositional algebra with fuzzy set theory, one can distinguish three approaches, which were employed in order to link these two topics. One approach is to consider a certain hyperoperation defined through a fuzzy set (P. Corsini [1], P. Corsini - V. Leoreanu, [3], I. Cristea e.g. [4, 5], I. Cristea - S. Hoskova [6], M. Stefanescu - I. Cristea [25], K. Serafimidis et al. [24] etc.). Another is to consider fuzzy hyperstructures in a similar way as Rosenfeld did for fuzzy groups [22] (A. Hasankhani, M. Zahedi [8, 31], B. Davvaz [7] and others). The third approach is employed in the pioneering papers by P. Corsini - I. Tofan [2] and by I. Tofan - A. C. Volf [26, 27], which introduce fuzzy hyperoperations that induce *fuzzy hypergroups*. This approach was further adopted by other researchers (Ath. Kehagias - e.g. [10, 11], V. Leoreanu-Fotea - e.g. [12, 13], K. M. Sen - R. Ameri et al. [23], etc.).

A *fuzzy hypercomposition* maps the pairs of elements of the Cartesian product $H \times H$ to fuzzy subsets of *H*. Thus, if we denote the collection of all fuzzy subsets of H by $F(H)$, then a *fuzzy hypercomposition* is the map $\circ : H \times H \to F(H)$. Hence, if $\circ$ is a fuzzy hyperoperation, then $a \circ b$ is a function and the notation $(a \circ b)(x)$ denotes the value of $a \circ b$ at the element x. The definition of the fuzzy hyperoperation subsumes the relevant definition of the crisp hyperoperation as a special case, since the latter results from the former through the use of the characteristic function.

Definition 3. [10, 11] If $\circ$: H×H → F(H) is a fuzzy hypercomposition, then, for every a∈H, B∈F(H), the fuzzy sets $a \circ B$ and $B \circ a$ are defined respectively by

$$(a \circ B)(z) = \vee_{y \in H} \left(\left[(a \circ y)(z) \right] \wedge B(y) \right) \text{ and } (B \circ a)(z) = \vee_{y \in H} \left(\left[(y \circ a)(z) \right] \wedge B(y) \right).$$

Per Definition 3, if $a,b,c \in H$:

$$\big(a \circ (b \circ c)\big)(z) = \vee_{y \in H} \left[(a \circ y)(z) \wedge (b \circ c)(y) \right] \text{ and } \big((a \circ b) \circ c\big)(z) = \vee_{y \in H} \left[(y \circ c)(z) \wedge (a \circ b)(y) \right].$$

Definition 4. [10, 11] If $\circ$: *H×H → F(H)* is a fuzzy hypercomposition, then, for every *A,B∈F(H)*, the fuzzy set $A \circ B$ is defined by $(A \circ B)(z) = \vee_{x,y \in H} \left(\left[(x \circ y)(z) \right] \wedge A(x) \wedge B(y) \right)$.

Definition 5. [2, 27] If $\circ$: *H×H → F(H)* is a fuzzy hypercomposition, then H is called a *fuzzy hypergroup*, if the following two axioms are valid:

i. $(a \circ b) \circ c = a \circ (b \circ c)$ for every $a,b,c \in H$ (associativity),

ii. $a \circ H = H \circ a = X_H$ for every $a \in H$ (reproduction).

where X_H is the characteristic function of *H*. If only (i) is valid, then *H* is called a *fuzzy semi-hypergroup* [23], while if only (ii) is valid, then *H* is called a *fuzzy quasi-hypergroup*.

If H is a non-void set endowed with a fuzzy hypercomposition $\circ$, then two new induced fuzzy hypercompositions "/" and "\" can be defined as follows:

$$(a/b)(x) = (x \circ b)(a) \text{ for every } a,b,x \in H \text{ and } (b \backslash a)(x) = (b \circ x)(a) \text{ for every } a,b,x \in H.$$

As in the case of crisp hypercompositions, the two induced fuzzy hypercompositions were named *fuzzy right division* and *fuzzy left division* respectively [18].

As has been proven in [18], if we replace the second axiom of Definition 5 with

$$a/b \neq 0_H \text{ and } a \backslash b \neq 0_H \quad \text{for every } a,b \in H,$$

then the resulting structure is not a fuzzy hypergroup (as in the case of crisp hypergroups), but a new structure, which was called *mimic fuzzy hypergroup* (*fuzzy*$_M$*- hypergroup*) [18]. Thus, the following definitions result:

Definition 6. If $\circ$: *H×H → F(H)* is a fuzzy hypercomposition, then *H* is called a *mimic fuzzy hypergroup* (*fuzzy*$_M$*- hypergroup*), if the following two axioms are valid:

i. $(a \circ b) \circ c = a \circ (b \circ c)$ for every $a,b,c \in H$ (associativity),

ii. $a/b \neq 0_H$ and $a \backslash b \neq 0_H$ for every $a,b \in H$.

If only (ii) is valid, then *H* is called a *mimic fuzzy quasi-hypergroup* (*fuzzy*$_M$*- quasi-hypergroup*), while, if the weak associativity is valid, instead of (i), *H* is called a *mimic fuzzy* H_v*-group* (*fuzzy*$_{MHv}$*-group*).

Definition 7. A fuzzy$_M$-hypergroup *H* will be called *commutable fuzzy*$_M$*-hypergroup,* if $a \circ H = H \circ a$ for any $a \in H$.

2. ON THE CLASSES OF TWO-ELEMENT FUZZY AND FUZZY$_M$-HYPERGROUPS

If the set $H=\{a,b\}$ is endowed with a fuzzy hypercomposition, then the following eight results are generated:

$$(a\circ a)(a),\ (a\circ a)(b),\ (a\circ b)(a),\ (a\circ b)(b),\ (b\circ a)(a),\ (b\circ a)(b),\ (b\circ b)(a),\ (b\circ b)(b).$$

Proposition 1. *There are ten different classes of 2-element fuzzy hypergroups, as illustrated in the following table:*

TABLE (1)

	i	ii	iii	iv	v	vi	vii	viii	ix	x
$(a\circ a)(a)$	1	1	1	1	1	1	1	1	1	1
$(a\circ a)(b)$	1	1	t	x	z	n	h_1	h_2	p_1	p_2
$(a\circ b)(a)$	1	1	1	1	1	m	k_1	k_2	q_1	q_2
$(a\circ b)(b)$	1	1	1	1	1	1	1	1	1	1
$(b\circ a)(a)$	1	1	1	y	1	1	j_1	j_2	r_1	r_2
$(b\circ a)(b)$	1	1	1	1	1	1	1	1	1	1
$(b\circ b)(a)$	1	1	1	1	w	1	1	1	1	1
$(b\circ b)(b)$	1	t	1	1	1	1	1	1	s_1	s_2

where $t, y, x, w, z, m, n, j_1, k_1, h_1, j_2, k_2, h_2, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2 \neq 1$ *and for* (iv) $x \le y$, *for* (v) $z \le w$, *for* (vi) $n \le m$, *for (vii)* $h_1 \le k_1 \le j_1$, *for (viii)* $h_2 \le j_2 \le k_2$, *for (ix)* $p_1 \le q_1 \le r_1 \le s_1$ *and for (x)* $p_2 \le r_2 \le q_2 \le s_2$.

Remarks. ***(a)*** Concerning the variables of Table 1, the obvious limitations for values 0 and 1 are in effect, so as not to have the same fuzzy hypergroups in two different classes. For example, if t=1, then (i) results from (ii) and (iii). Classes (vii), (viii) and (ix), (x) contain common elements when there are equalities in their variables.
(b) Each of the above classes has one isomorphic class. For example, class (vii) is isomorphic to the following:

TABLE (2)

$(a\circ a)(a)$	$(a\circ a)(b)$	$(a\circ b)(a)$	$(a\circ b)(b)$	$(b\circ a)(a)$	$(b\circ a)(b)$	$(b\circ b)(a)$	$(b\circ b)(b)$
1	1	1	j_1	1	k_1	h_1	1

(c) The classes (v), (ix) and (x) are called ***main***. If the parameters of Table 1 are all equal to 0, then we take the eight non-isomorphic 2-element crisp hypergroups [29]. The non-isomorphic 3-element hypergroups are enumerated in [28] but the enumeration of the different classes of the 3-element fuzzy hypergroups remains an open question.

If we replace 1 by another number of the interval (0,1) in (i) of Table 1, then reproduction is not valid; however, the left and the right divisions are non-zero. Therefore, the resulting structure is a fuzzy$_M$-hypergroup.

Proposition 2. *If the fuzzy hypercomposition is commutative, then* $(H,\circ)$ *is a fuzzy semihypergroup when* $(a\circ a)(a) \ge (a\circ b)(b)$, $(b\circ b)(b) \ge (a\circ b)(a)$ *and*

$$(a\circ a)(b) \le max\{(a\circ b)(a),(a\circ b)(b)\} \text{ and } (b\circ b)(a) \le min\{(a\circ b)(a),(a\circ b)(b)\}$$

$$\text{or} \quad (b\circ b)(a) \le max\{(a\circ b)(a),(a\circ b)(b)\} \text{ and } (a\circ a)(b) \le min\{(a\circ b)(a),(a\circ b)(b)\}.$$

Proposition 2 reveals a class of commutative fuzzy$_M$-hypergroups when the left and the right divisions are non-zero. Another class of commutative fuzzy$_M$-hypergroups results from the following proposition:

Proposition 3. *If the fuzzy hypercomposition is commutative and* $(a\circ b)(a)=(b\circ b)(a)$, $(a\circ b)(b)=(b\circ b)(b)$, *then* $(H,\circ)$ *is a fuzzy semihypergroup when* $max\{(a\circ a)(a),(a\circ a)(b)\} \ge max\{(b\circ b)(a),(b\circ b)(b)\}$.

Table 3 below presents five fuzzy semihypergroups:

TABLE (3)							
$(a\circ a)(a)$	$(a\circ a)(b)$	$(a\circ b)(a)$	$(a\circ b)(b)$	$(b\circ a)(a)$	$(b\circ a)(b)$	$(b\circ b)(a)$	$(b\circ b)(b)$
0,11	0,6	0,5	0,9	0,7	0,8	0,4	0,7
0,11	0,6	0,5	0,9	0,7	0	0	0,7
0,11	0,6	0	0,9	0,7	0,8	0	0,7
0,11	0,6	0,5	0,9	0	0,8	0	0,7
0,11	0,6	0	0,9	0	0	0	0

The first two are fuzzy$_M$-hypergroups, while the next two are not, since $a/b=0_H$ is valid for the third one and $b\backslash a=0_H$ is valid for the fourth one. Nevertheless, $x\circ H\neq 0_H$ and $H\circ x\neq 0_H$, $x\in\{a,b\}$ is true for the third and the fourth ones, while $b\circ H=0_H$ is valid for the last one. This property leads us to the following definition:

Definition 8. A fuzzy semihypergroup $(H,\circ)$ will be called *pseudo fuzzy$_M$-hypergroup,* if $a\circ H\neq 0_H$ and $H\circ a\neq 0_H$ for any $a\in H$.

The enumeration of the different classes of 2-element fuzzy$_M$-hypergroups and pseudo fuzzy$_M$-hypergroups remains an open question.

REFERENCES

1. P. Corsini, "Join spaces, power sets, fuzzy sets", Algebraic Hyperstructures and Applications, Proc. of the 5th AHA, Hadronic Press 1994, pp. 45-52.
2. P. Corsini, I. Tofan, "On fuzzy hypergroups" *Pure Math Appl 8*, 1997, pp. 29-37.
3. P. Corsini, V. Leoreanu, "Applications of Hyperstructures Theory", Kluwer Academic Publishers, 2003.
4. I. Cristea, "Hyperstructures and fuzzy sets endowed with two membership functions", *Fuzzy Sets and Systems* 160 (8), 2009, 1114-1124
5. I. Cristea, "About the fuzzy grade of the direct product of two hypergroupoids", *Iran. J. Fuzzy Syst.* 7, no. 2, 2010, 95-108.
6. I. Cristea, S. Hoskova, "Fuzzy Pseudotopological Hypergroupoids", *Iran. J. of Fuzzy Systems* vol. 6, no. 4, 2009, 11-19.
7. B. Davvaz, "Fuzzy H_V –groups", *Fuzzy Sets and Systems* 101, 1999, 191-195.
8. A. Hasankhani, M.M. Zahedi, "Fuzzy sub-F-polygroups", *Ratio Mathematica* 12, 1997, pp. 35-44.
9. J. Jantosciak, Ch. G. Massouros, "Strong Identities and fortification in Transposition hypergroups", *Journal of Discrete Mathematical Sciences & Cryptography* 6, 2003, no 2-3, pp. 169-193.
10. Ath. Kehagias, K. Serafimidis, "The L-Fuzzy Nakano Hypergroup", *Information Sciences*, vol. 169, 2005, 305-327.
11. Ath. Kehagias, "The Construction of Fuzzy-valued t-norms and t-conorms", *Studies in Comput. Intell.* (SCI) 67, 2007, 361-370.
12. V. Leoreanu, "About hyperstructures associated with fuzzy sets of type 2", *Ital. J. of Pure and Appl. Math.*, 17, 2005, 127-136.
13. V. Leoreanu-Fotea, "Fuzzy join n-ary spaces and fuzzy canonical n-ary hypergroups", *Fuzzy Sets and Systems* 161, 2010, 3166–3173
14. F. Marty, "Sur un généralisation de la notion de groupe". 8ième Congrès des mathématiciens Scand., Stockholm 1934, pp. 45-49.
15. Ch. G. Massouros, "On the theory of hyperrings and hyperfields", *Algebra and Logic* 24, no 6, 1985, pp. 728-742.
16. Ch. G. Massouros, "Constructions of hyperfields", *Mathematica Balcanica*, vol. 5, no. 3, 1991, pp. 250 – 257.
17. Ch. G. Massouros, Ch. Tsitouras, Enumeration of hypercompositional structures defined by binary relations, *Italian Journal of Pure and Applied Mathematics*, no. 28, 2011, pp. 43-54.
18. Ch. G. Massouros, G. G. Massouros, "On certain fundamental properties of hypergroups and fuzzy hypergroups – Mimic fuzzy hypergroups", *International Journal of Risk theory* (to appear)
19. Ch. G. Massouros, A. Dramalidis, "Transposition H_v-groups", *Ars Combinatoria*, vol. 106, 2012, pp. 143-160.
20. G. G. Massouros, "The Subhypergroups of the Fortified Join Hypergroup", *Italian Journal of Pure and Applied Mathematics*, no 2, 1997, pp. 51-63.
21. G. G. Massouros, Ch. G. Massouros and J. Mittas, Fortified join hypergroups, *Annales Math Blaise Pascal*, vol. 3 n. 2 (1996) 155-169
22. A. Rosenfeld, Fuzzy Groups, *J. Math. Anal. Appl*. 35, 1971, pp. 512-517.
23. M. K. Sen, R. Ameri, G. Chowdhury, Fuzzy hypersemigroups, *Soft Comput.* 12, 2008, pp. 891-900.
24. K. Serafimidis, Ath. Kehagias, M. Konstantinidou, "The L-Fuzzy Corsini Join Hyperoperation", *Ital. J. of Pure and Appl. Math.*, no 12, 2002, pp.83-90.
25. M. Stefanescu, I. Cristea, "On the fuzzy grade of hypergroups", *Fuzzy Sets and Systems* 159 (9) , 2008, pp. 1097-1106.
26. I. Tofan, A. C. Volf, "On some connections between hyperstructures and fuzzy sets", *Ital. J. of Pure and Appl. Math.*, 7, 2000, 63-68.
27. I. Tofan, A. C. Volf, "Algebraic aspects of information organization", Systematic Organisation of Information in Fuzzy Systems, P. Melo-Pinto et al. (Eds) IOS Press, pp. 71-87, 2003.
28. Ch. Tsitouras, Ch. G. Massouros, "On enumeration of hypergroups of order 3", *Computers and Mathematics with Applications* vol. 59, 2010, pp. 519-523.
29. T. Vougiouklis, "Cyclic Hypergroups" PhD Thesis, 1980, Democritous Univ.of Thrace, Greece.
30. T. Vougiouklis, "The fundamental relation in hyperrings. The general hyperfield". Proc. of the 4th AHA, World Scientific 1991, 209-217
31. M. M. Zahedi, A. Hasankhani, "F-Polygroups", Int. J. Fuzzy Math., vol. 4, 1996, pp.533 -548.