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On Join Hyperrings

Christos G. Massouros, Gerasimos G. Massouros

54, Klious st., GR15561 Cholargos, Athens, Greece
Email: masouros@gmail.com

Abstract. A join hyperring is a hyperringoid whose additive part is a commutative fortified transposition hypergroup. The hyperringoid and the join hyperring derived from the approach of the theory of formal languages and automata from the stand point of hypercompositional algebra. This paper deals with the structure of the join hyperrings. The behaviour of the canonical and attractive elements is analyzed, the characteristic of the join hyperrings is defined and the good homomorphisms are studied.

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1 Introduction

The theory of languages, viewed from the standpoint of hypercompositional algebra, led to the introduction of new hypercompositional structures [13, 14, 15, 22]. Thus the definition of the regular expressions over an alphabet A requires the consideration of subsets $\{x, y\}$ of the free monoid A^* generated by A. This leads to the definition of the hypercomposition $x + y = \{x, y\}$ in A^* that endows A^* with a join hypergroup structure, which was named ***B-hypergroup***. Moreover, the empty set of words and its properties in the theory of the regular expressions lead to the following extension: Let $0 \notin A^*$. On the set $\underline{A}^* = A^* \cup \{0\}$ define a hypercomposition as follows:

$$x + y = \{x, y\} \text{ if } x, y \in \underline{A}^* \text{ and } x \neq y,$$
$$x + x = \{x, 0\} \text{ for all } x \in \underline{A}^*.$$

This structure is called ***dilated B-hypergroup*** and it lead to the definition of a new class of hypergroups, the ***fortified join hypergroups*** [16, 17].

Before going on and for the self-sufficiency of this paper, it is noted that a ***transposition hypergroup*** [3] is a hypergroup which satisfies a postulated property of transposition i.e. $(b\backslash a) \cap (c/d) \neq \emptyset \Rightarrow (ad) \cap (bc) \neq \emptyset$ where $a/b = \{x \in H \mid a \in xb\}$ and $b\backslash a = \{x \in H \mid a \in bx\}$ are the induced hypercompositions (for some recent interesting examples see [1, 2]). When a hypergroup is commutative, the two induced hypercompositions coincide. A commutative transposition hypergroup is called join hypergroup or join space [30]. In what follows $A..B$ denotes the set containing exactly those elements in A that are not in B.

Definition 1.1. A **fortified join hypergroup** (FJH) is a commutative transposition hypergroup (H,+) which contains an element 0, called the *zero element* of H, which satisfies the axioms:

(i) $0 + 0 = 0$;

(ii) $x \in x + 0$, for every $x \in H$;

(iii) for every $x \in H..\{0\}$ there exists one and only one element $-x \in H..\{0\}$, called the *opposite* of x, such that $0 \in x + (-x)$.

Furthermore the binary operation of the word concatenation in the free monoid A^* is bilaterally distributive over the hyperoperation of the B-hypergroup and so, generally:

Definition 1.2. A **hyperringoid** [20] is a non empty set Y equipped with an operation "$\cdot$" and a hyperoperation "$+$" such that:

i) $(Y, +)$ is a hypergroup,

ii) $(Y, \cdot)$ is a semigroup,

iii) the operation "$\cdot$" distributes on both sides over the hyperoperation "$+$".

M. Krasner was the first who introduced and studied hypercompositional structures with an operation and a hyperoperation, giving thus birth to the ***hyperring*** [6] i.e a hyperringoid whose additive part is a canonical hypergroup and the ***hyperfield*** [5]. Thereafter, these structures were extensively studied in depth. See e.g. [7, 8, 9, 10, 11, 12, 23, 24, 25, 27, 28, 29, 31]. Afterward, I. Mittas introduced the ***superring*** and the ***superfield***, in which both, the addition and the multiplication are hypercompositions [26].

The new hypercompositional structures that arose from the theory of languages and automata were given names according to the terminology by Krasner and Mittas [5, 6, 26]. Thus, provided that $(Y, +)$ is a join hypergroup, $(Y, +, \cdot)$ is called ***join hyperringoid***. The join hyperringoid that derives from a B-hypergroup is called ***B-hyperringoid*** and the special B-hyperringoid that appears in the theory of languages is the ***linguistic hyperringoid***. If the additive part of a hyperringoid is a fortified join hypergroup whose zero element is bilaterally absorbing with respect to the multiplication, then, this hyperringoid, is named ***fortified join hyperringoid*** or ***join hyperring (JH)***. A ***join hyperdomain*** is a join hyperring which has no divisors of zero. A ***proper join***

hyperring, is a join hyperring which is not Krasner's hyperring. A join hyperring K is called ***join hyperfield*** if $K^* = K..\{0\}$ is a multiplicative group.

Since the additive part of a hyperringoid is a fortified join hypergroup it must be mentioned that such a hypergroup consists of two types of elements, the ***canonical (c-elements)*** and the ***attractive (a-elements)*** [4, 17]. An element x is called a c-element if $0 + x$ is the singleton $\{x\}$, while it is called an a-element if $0 + x$ is the biset $\{0, x\}$. The set of the canonical element is denoted by C and the set of the attractive elements is denoted by A. Moreover, another distinction between the elements of the FJH stems from the fact that the equality $-(x - x) = x - x$ is not always valid. So, those elements that satisfy the above equality are called ***normal***, while the rest are called ***abnormal*** [16, 17]. A join hyperring in which the additive hypergroup consists only of normal elements, is called ***normal***.

2 Algebra of subhypergroups of the additive hypergroup of a join hyperring. Hyperideals

The additive hypergroup of a join hyperring is a fortified join hypergroup. Thus regarding its subhypergroups [18], it has ***join*** ones and others that are not join, i.e. subhypergroups which satisfy the transposition axiom inside them (because they are stable with regard to the induced hypercomposition) and others that do not. It is proved that the join subhypergroups are the closed ones [18] and that they are part of a bigger class of subhypergroups, the class of the symmetric subhypergroups. ***Symmetric*** is a subhypergroup h of a FJH, for which $-x \in h$ for every $x \in h$. Of course, in a FJH there also exist non symmetric subhypergroups.

In what follows some important compositions that can be defined in the collection of subhypergroups of the additive hypergroup of a join hyperring are investigated. Since the set of the symmetric subhypergroups and the set of the join subhypergroups of a fortified join hypergroup are complete lattices [18], two of these compositions are the intersection and the subhypergroup, symmetric or join, generated by a collection of symmetric or join subhypergroups respectively. More precisely if Y_1 and Y_2 are symmetric subhypergroups, the symmetric subhypergroup $[Y_1 \cup Y_2]$ generated by Y_1 and Y_2 coincides with the set $Y_1 + Y_2$ of sums $y_1 + y_2$, y_1 in Y_1, y_2 in Y_2. On the other hand if Y_1 and Y_2 are join subhypergroups, the join subhypergroup $\langle Y_1 \cup Y_2 \rangle$ generated by Y_1 and Y_2 coincides with the set $(Y_1 + Y_2) \cup (Y_1 \div Y_2) \cup (Y_2 \div Y_1)$ where "$\div$" is the induced hypercomposition, i.e. $y_1 \div y_2 = \{x \in Y; y_1 \in x + y_2\}$.

Now it will be introduced the third important composition on subhypergroups of the additive hypergroup of a join hyperring. Let Y_1 and Y_2 be symmetric subhypergroups, then the product $Y_1 Y_2$ is defined to be the symmetric subhypergroup generated by all the products $y_1 y_2$, y_1 in Y_1, y_2 in Y_2. If the join hyperring Y is an integral hyperdomain then Y_1Y_2 coincides with the union of finite hypersums

$$P = \bigcup_{x_i \in Y_1, y_i \in Y_2} (x_1y_1 + x_2y_2 + \ldots + x_ny_n).$$

Indeed, it is clear that P contains all the products from Y_1Y_2 and that P is contained in any symmetric subhypergroup that contains all of these products. Also it is clear that P is stable under hypercomposition and that 0 is in P. Finally, since the integral join hyperrings are normal [20] $-(x_1y_1 + x_2y_2 + \ldots + x_ny_n) = -x_1y_1 - x_2y_2 - \ldots - x_ny_n$, is valid and since Y is an integral domain it holds that $-x_1y_1 - x_2y_2 - \ldots - x_ny_n = (-x_1)y_1 + (-x_2)y_2 + \ldots + (-x_n)y_n$. Hence P is a symmetric subhypergroup and $P = Y_1Y_2$.

The associative law $(Y_1Y_2)Y_3 = Y_1(Y_2Y_3)$ can easily be established; for either of these subhypergroups is the totality of finite hypersums of the form $\Sigma x_iy_iz_i$, x_i in Y_1, y_i in Y_2, z_i in Y_3. Also the distributive laws $Y_1(Y_2 + Y_3) = Y_1Y_2 + Y_1Y_3$ and $(Y_1 + Y_2)Y_3 = Y_1Y_3 + Y_2Y_3$ holds. Let us prove the first of these. First note that $Y_1(Y_2+Y_3)$ is the symmetric subhypergroup generated by all products xw, x in Y_1 and $w \in y + z, y$ in Y_2, z in Y_3. Since $x(y + z) = xy + xz \subseteq Y_1Y_2 + Y_1Y_3$, it holds that $Y_1(Y_2 + Y_3) \subseteq Y_1Y_2 + Y_1Y_3$. On the other hand $xy \in x(y + 0) \subseteq Y_1(Y_2 + Y_3)$. Hence $Y_1Y_2 \subseteq Y_1(Y_2 + Y_3)$. Similarly $Y_1Y_2 \subseteq Y_1(Y_2 + Y_3)$. But then $Y_1Y_2 + Y_1Y_3 \subseteq Y_1(Y_2 + Y_3)$. Therefore $Y_1(Y_2 + Y_3) = Y_1Y_2 + Y_1Y_3$. Evidently this same argument applies to the other distributive law.

A subhypergroup T of the additive hypergroup determines a subhyperringoid if and only if T is stable under multiplication. The condition for this can be expressed in terms of multiplication as follows: $T^2 \subseteq T$. If T is symmetric or join then the symmetric subhyperring and the join subhyperring is determined respectively. The conditions that a subhypergroup I be an hyperidealoid are that $YI \subseteq I$ (L) and $IY \subseteq I$ (R). If I is a subhypergroup such that (L) is valid, then I is called left hyperidealoid and if (R) holds, then I is a right hyperidealoid. If I is join or symmetric then the join hyperideal and the symmetric hyperideal is defined respectively.

Since $a \div a = A$, for any attractive element a [4, 17] it derives that $A \cup \{0\}$ is the minimal join subhypergoup [18]. More precisely it holds:

Proposition 2.1. *In a join hyperring Y, the union $A^\wedge = A \cup \{0\}$ of the a-elements with the zero element is the minimal bilateral join hyperideal of Y and furthermore it is the minimal join subhyperring of Y.*

Proposition 2.2. *If Y is join hyperring and I is a symmetric hyperideal in Y, then the relation $(m, n) \in R \Leftrightarrow (m - n) \cap I \neq \emptyset$ is a congruence relation.*

Proof. In [21] it has been proved that if I is a hyperidealoid in a hyperringoid Y, and "$\div$" is the induced hypercomposition, i.e. $m \div n = \{x \in Y | m \in x + n\}$, then the relation R defined as follows: $(m, n) \in R$ if $(m \div n) \cap I \neq \emptyset$ and $(n \div m) \cap I \neq \emptyset$, is a homomorphic relation. In fortified join hypergroups it is known that if $m \neq n$, then $m \div n \cup n^{-1} = m - n$. Therefore the above relation is homomorphic. Next it will be proved that R is an equivalence relation. Indeed $(m, m) \in R$, for all $m \in Y$, because $0 \in (m - m) \cap I$. Now let $(m, n) \in R$. Then $(m - n) \cap I$ is non void, so there exists $x \in (m - n) \cap I$ or $-x \in (n - m) \cap I$. Thus the intersection $(n - m) \cap I$ is non void and so the relation R is symmetric. Next let $(m, n) \in R$ and $(n, s) \in R$. Then the intersections $(m - n) \cap I$ and $(n - s) \cap I$ are non void. Thus $m \in n + I$ and $n \in s + I$. Therefore $m \in s + I$ and so R is transitive. $\square$

Proposition 2.3. *If Y is join hyperring and I is a symmetric hyperideal in Y, then the quotient Y/I becomes a join hyperring if a hypercomposition and a composition are defined as follows:*

$$(x + I) + (y + I) = \{w + I; w \in x + y\} \textit{ and } (x + I)(y + I) = xy + I.$$

Proof. Let $(x + I) \div (y + I) \cap (z + I) \div (w + I) \neq \emptyset$. Then there exists $x' \in x + I, y' \in y + I, z' \in z + I, w' \in w + I$, such that $x' \div y' \cap z' \div w' \neq \emptyset$, which implies that $x' + w' \cap z' + y' \neq \emptyset$, because Y is a join hyperring. Thus $(x + I) + (w + I) \cap (z + I) + (y + I) \neq \emptyset$. $\square$

An important symmetric subhypergroup of the FJH's is the one which consists of the unions of sums of the type:

$$(x_1 - x_1) + \cdots + (x_n - x_n),$$

where $x_i, i = 1, \ldots, n$ belong to a set of normal elements X. This hypergroup is denoted by $\Omega(X)$.

Proposition 2.4. *Let X be a non empty subset of a join hyperring Y, which*

i) *is multiplicatively closed,*

ii) *consists of normal elements,*

iii) *the elements of $-X \cup X$ are not divisors of zero.*

Then $\Omega(X)$ is a symmetric subhyperring of Y. If X is also multiplicatively absorbing, then $\Omega(X)$ is a symmetric hyperideal.

Corollary 2.5. *$\Omega(Y)$ is a symmetric hyperideal of Y.*

Proposition 2.6. *In any join hyperring Y the totality Yx of left multiples yx, y in Y is a symmetric left hyperideal. In a similar manner xY is a symmetric right hyperideal. If Y contains canonical elements, then the above hyperideals are join.*

Proof. Obviously $YYx \subseteq Yx$. Next let yx be an element of Yx. Then $Yx + yx = (Y + y)x = Yx$. Thus Yx is a subhypergroup of Y, and since $-yx$ belongs to Yx, for every yx in Yx it follows that Yx is symmetric. According to Proposition 2.1, if Y consists only of attractive elements, then Y has no proper join subhyperrings. Suppose now that Y contains canonical elements. In this case the product of two attractive elements or the product of an attractive with a canonical element is the

0 [20]. So, if x is an attractive element, then $Yx = \{0\}$, while if x is a canonical element, then Yx is a join subhypergroup. Indeed, let $(y_1x \div y_2x) \cap (y_3x \div y_4x) \neq \emptyset$. Then the elements y_ix are either canonical elements, or 0. If they are canonical elements, then $(y_1x \div y_2x) \cap (y_3x \div y_4x) = (y_1x - y_2x) \cap (y_3x - y_4x)$ ([17] prop.2.9). Let u be an element in $(y_1x - y_2x) \cap (y_3x - y_4x)$. Then $u - u \subseteq (y_1x - y_2x) - (y_3x - y_4x) = (y_1x + y_4x) - (y_3x + y_2x)$. Hence $0 \in (y_1x + y_4x) - (y_3x + y_2x)$ and so $(y_1x + y_4x) \cap (y_3x + y_2x) \neq \emptyset$. Next suppose that some of the y_ix's are 0. For instance let

(a) $y_1x = 0$. Then $(y_1x \div y_2x) \cap (y_3x \div y_4x) = (0 \div y_2x) \cap (y_3x \div y_4x) = \{-y_2x\} \cap (y_3x \div y_4x)$ and therefore $-y_2x \in y_3x \div y_4x \Leftrightarrow y_3x \in y_4x - y_2x \Leftrightarrow y_4x \in y_3x + y_2x \Leftrightarrow (y_1x + y_4x) \cap (y_3x + y_2x) \neq \emptyset$

(b) $y_1x = y_2x = 0$. Since $0 \div 0 = A^{\wedge}$, and the intersection $(y_1x \div y_2x) \cap (y_3x \div y_4x)$ is non void, it must be $y_3x \div y_4x = 0 \div 0$. But in this case the implication $(0 \div 0) \cap (0 \div 0) \neq \emptyset \Leftrightarrow (0 + 0) \cap (0 + 0) \neq \emptyset$ is valid. □

3 Structure of the additive hypergroup of a join hyperring. The characteristic of a join hyperring

The additive hypergroup of a join hyperring is a fortified join hypergroup. Certain significant properties of the FJH are [17, 4]:

i. *the sum of two a-elements is a subset of A and it always contains the two addends,*
ii. *the sum of two non opposite c-elements consists of c-elements, while the sum of two opposite c-elements contains all the a-elements,*
iii. *the sum of an a-element with a non zero c-element is the c-element.*

The structure of the additive hypergroup imposes significant properties on the multiplicative semi-group of a join hyperrings. Thus [20]:

i. $C^2 \subseteq C$ *and* $CA = AC = \{0\}$.
ii. *In a join hyperrring which contains a c-element, the product of two a-elements equals to zero.*
iii. *The equalities*

$$x(-y) = (-x)y = -xy,$$
$$(-x)(-y) = xy,$$
$$w(x - y) = wx - wy, \ (x - y)w = xw - yw$$

hold if -x, -y, x, y, w are not divisors of zero.

iv. *Every join hyperring which has no divisors of zero is normal.*

If Y is any FJH, a multiplication $xy = 0$, for all $x, y \in Y$ can be defined. It is clear that this composition is associative and distributive with respect to hypercomposition and thus a join hyperring is obtained. A join hyperring of this type is called zero join hyperring. The existence of such join hyperrings shows that there is nothing that one can say in general about the structure of the FJH of a join hyperring. However, simple restrictions on the multiplicative semi-group of a join hyperring impose strong restrictions on the hypergroup. For example suppose that Y has an identity 1. If c is a canonical element in Y, then $c = 0+c = c(0+1)$, thus $0 + 1$ must be equal to 1 and therefore 1 has to be canonical. On the other hand if a is an attractive element in Y, then $\{0, a\} = 0 + a = a(0+1)$, thus $0+1$ must be equal to the set $\{0, 1\}$ which means that 1 has to be attractive. Hence:

Proposition 3.1. *If a join hyperring Y is unitary, then either Y is a join hyperring which consists only of attractive elements or Y is a (Krasner's) hyperring.*

Next one can easily see that

Proposition 3.2. *A join hyperring Y with a unitary element $1 \neq 0$, is a division join hyperring if and only if it has no proper left (right) ideals.*

In any join hyperring Y expressions like $x + x$ are abbreviated by $2x$ and generally we put:

$$n \cdot x = \begin{cases} x + x + \cdots + x & \text{(n times)} \quad \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ (-x) + (-x) + \cdots + (-x) & \text{(-n times)} \quad \text{if } n < 0. \end{cases}$$

It is easily seen that $(mn)x = m(nx), m, n \in Z$. On the contrary, if $n < 0$ and $y = -x$, the rule $n(x + y) = nx + ny$ is not always valid. This rule is true if Y is normal. Also when Y is normal it holds:

$$m \cdot x + n \cdot x = \begin{cases} (m+n) \cdot x & \text{if } mn > 0, \\ (m+n) \cdot x + \min\{|m|, |n|\} \cdot (x - x) & \text{if } mn < 0. \end{cases}$$

For this reason, the join hyperrings that are used in the following text are normal.

Let us consider the additive order of x in the normal join hyperrings, i.e. the order of x in the additive hypergroup of Y. The symbol $\omega(x)$ is introduced which is named the additive order of x. Two cases can appear such that one revokes the other:

I. $0 \notin$ m·x + n·(x-x), for any (m,n)∈Z×N, with m $\neq$ 0. Then the order of x is defined to be the infinity and ω(x)=$+\infty$ is written.

II. There exist $(m, n) \in Z \times N$ with $m \neq 0$ such that $0 \in m \cdot x + n \cdot (x - x)$. Let p be the minimum positive integer, such that there exists $n \in N$ for which

$0 \in p \cdot x + n \cdot (x - x)$. Let $m = kp$, $(k \in Z)$ and $q(k)$ the minimum non negative integer for which $0 \in kp \cdot x + q(k) \cdot (x - x)$. A function $q\colon Z \to N$ is defined such that it corresponds k to $q(k)$. Then the order of x is the pair $\omega(x) = (p, q)$. The number p is called the principal order of x and the function q is called the associated order of x.

Thus if x is an a-element, then $0 \in x + (x - x)$ and therefore $\omega(x) = (1, q)$ with $q(k) = 1$ for every $k \in Z..\{0\}$. Moreover, if x is a selfopposite c-element, then $0 \in 2 \cdot x + 0 \cdot (x - x)$, if $x \notin x - x$ and $0 \in x + (x - x)$, if $x \in x - x$ and thus $\omega(x) = (2, q)$ with $q(k) = 0$ in the first case and $\omega(x) = (1, q)$ with $q(k) = 1$ in the second case (for every $k \in Z$).

Definition 3.3. The **characteristic** $\chi(x)$ of an element $x \in Y$ is the principal order of x in the additive hypegroup of Y, if $\omega(x) \neq +\infty$ and the 0, if $\omega(x) = +\infty$.

Proposition 3.4. *If $x \in Y$ divides $y \in Y$, then $\chi(x)$ divides $\chi(y)$.*

Proof. The proposition is obvious when $\chi(x) = 0$. Next suppose that $y = ax$. Then $0 \in \chi(x)x + n(x - x)$, $n \in N$ which implies that $0 \in \chi(x)ax + n(ax - ax)$. Thus $0 \in \chi(x)y + n(y - y)$ from which it derives that $\chi(x)$ is multiple of $\chi(y)$. □

Definition 3.5. The **characteristic** $\chi(Y)$ of the join hyperring Y is the least common multiple of $\chi(x)$, x in Y. If no common multiple exists, then $\chi(Y)$ is defined to be 0.

If $\chi(\Upsilon) \neq 0$, then $0 \in \chi(Y)x + \Omega(x)$, for all $x \in Y$ and if n is an integer such that $0 \in nx + \Omega(x)$, then n is multiple of $\chi(\Upsilon)$. If $\chi(Y) = 0$, then no integer n satisfies the relation $0 \in nx + \Omega(x)$ for all x in Y.

Proposition 3.6. *If $x \in Y$ is not a zero divisor then $\chi(Y) = \chi(x)$.*

Proof. The proposition is obvious when $\chi(x) = 0$. Next suppose that x is not a zero divisor e.g. from the left and that $\chi(x) \neq 0$. Then $0 \in \chi(x)x + \Omega(x)$ and $0 \in \chi(x)xy + \Omega(x)y$, for all y in Y. Next note that $\Omega(x)y = \left[\bigcup_{n \in N} n(x - x)\right]y = \bigcup_{n \in N} n(xy - xy) = x\left[\bigcup_{n \in N} n(y - y)\right] = x\Omega(y)$, thus $0 \in x\left[\chi(x)y + \Omega(y)\right]$, from which it derives that $0 \in \chi(x)y + \Omega(y)$ for all y in Y. Hence the principal order of all y in Y is finite and $\chi(y)$ divides $\chi(x)$. Therefore $\chi(x)$ is the least common multiple of $\chi(y)$, y in Y and so $\chi(Y) = \chi(x)$. □

Proposition 3.7. *Let Y be a unitary join hyperring, then*

i. $\chi(Y) = \chi(1)$.

ii. *If Y is a proper join hyperring, then $\chi(Y) = 1$.*

The proof derives from Propositions 3.4. and 3.1.

4 Homomorphisms of Join Hyperrings

According to the terminology that M. Krasner introduced [8], if Y and Y' are two JH, then a homomorphism from Y to Y' is a mapping $f\colon Y \to P(Y')$ such that

$$f(x+y) \subseteq f(x) + f(y) \text{ and } f(xy) = f(x)f(y) \text{ for all } x, y \text{ in } Y$$

A homomorphism is named good or normal if f is a mapping from Y to Y' such that: $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x, y in Y. This paragraph deals with the good homomorhisms. As usual [8], the kernel of f, denoted by $kerf$, is the subset $f^{-1}(f(0))$ of Y. The homomorphic image $f(Y)$ of Y, is denoted by Imf.

Proposition 4.1.

(i) *if x belongs to $C \cap kerf$ then $-x$ belongs also to $kerf$,*

(ii) *if $C \neq \emptyset$, then the set of the attractive elements of Y is a subset of $kerf$,*

(iii) *if $f(0) = 0'$, then $f(A) \subseteq A'$ and $f(C) \subseteq C'$.*

Proof.

(i) If x is a canonical element, then $-x$ is canonical as well [17, 4]. Since $f(0) \in f(x-x) = f(x) + f(-x) = f(0) + f(-x) = f[0 + (-x)] = f(-x)$, it derives that $f(-x) = f(0)$, hence $-x \in kerf$.

(ii) Since $A \subseteq x - x$, for all $x \in C$ [17, 4], it derives that $f(A)$ is a subset of $f(x-x)$. Because of (i) it holds $f(x-x) = f(0)$. Therefore $A \subseteq kerf$.

(iii) If x is an attractive element, then $f(x)+0' = f(x+0) = f\{x, 0\} = \{f(x), 0'\}$. If x is a canonical element, then $f(x) + 0' = f(x+0) = f(x)$. □

Proposition 4.2.

(i) *$kerf$ is a hyperindealoid of Y.*

(ii) *Imf is a subhyperringoid of Y' which generally does not contains the element $0'$ of Y', but $f(0)$ is neutral element in Imf.*

(iii) *if T is a subhyperringoid of Y, then $f(T)$ is a subhyperringoid of Y'.*

Proof.

(i) If $x \in kerf$, then $f(x + kerf) = f(0)$. Thus $x + kerf \subseteq kerf$. Next let $y \in kerf$ and suppose that x is a canonical element. Then $-x \in kerf$, so $y \in y + 0 \subseteq y + (x - x) = (y - x) + x \subseteq x + kerf$. Now suppose that x is an attractive element. Then $y \in x + y$, if y is an attractive element, or $y = x + y$ if y is a canonical element. Thus $kerf \subseteq x + kerf$ and therefore $x + kerf = kerf$. Also if $x \in kerf$, and $y \in Y$, then $f(xy) = f(x)f(y) = f(0)f(y) = f(0)$, hence $xy \in kerf$.

(ii) If x', y' are two elements of Y' then there exist $x, y \in Y$, such that $f(x) = x'$, $f(y) = y'$. Thus $x'y' = f(x)f(y) = f(xy) \in Imf$. Also $x' + Imf = f(x) + f(Y) = \bigcup_{z\in Y} f(x+z) = f(Y)$. Thus Imf is a subhyperringoid of Y'. Yet $f(0) \in f(x-x) = f(x) + f(-x)$ is valid and since $x \in x + 0$, it holds that $f(x) \in f(x) + f(0)$. Hence (ii).

(iii) The proof is similar to (ii). □

Proposition 4.3.

(i) *The quotient $Y/kerf$ becomes a hyperringoid if a hypercomposition and a composition is defined as follows:*

$$(x + kerf) + (y + kerf) = \{w + kerf : w \in x + y\}$$
$$\textit{and } (x + kerf)(y + kerf) = xy + kerf.$$

(ii) *$Y/kerf$ is isomorphic to Imf.*

(iii) *$f = e \circ \pi$, where π is the natural epimorphism from Y to $Y/kerf$, i.e. $\pi(x) = x + kerf$ and e is the isomorphism from $Y/kerf$ to Imf, with $e(x + kerf) = f(x)$.*

Proposition 4.4. *If f is an epimorphism, then $f(0) = 0'$.*

Proof. Since f is an epimorphism then for the $-f(0)$ there exists an element x of Y such that $f(x) = -f(0)$. Consequently it holds: $0' \in -f(0) + f(0) \Rightarrow 0' \in f(x) + f(0) \Rightarrow 0' \in f(x + 0) \Rightarrow 0' \in f(\{x, 0\}) \Rightarrow 0' \in \{f(x), f(0)\}$. So either $f(0) = 0'$ or $f(x) = 0'$ from where $-f(0) = 0'$, and thus $f(0) = 0'$. □

Proposition 4.5. *If f is a monomorhpism, then $kerf = \{0\}$.*

Proof. Let $x \in kerf$, then $f(x) = f(0)$, hence $x = 0$ and $kerf = \{0\}$. □

As it is analyzed in [8], the fact that x is an a-element and $x \in kerf$, does not imply that $-x$ belongs to $kerf$ as well. Thus the complete homomorphism was defined, which is a homomorphism that satisfies the implication: $x \in kerf \Rightarrow -x \in kerf$.

Proposition 4.6. *If f is an epimorphism, then it is complete.*

Proof. Since f is an epimorphism, $f(0) = 0'$. Suppose next that f is not complete. Then there exists an element $x \in Y$ such that $f(x) = 0'$ and $f(-x) \neq 0'$. Then for $-f(-x)$ there exists $y \in Y$, which does not belong to $kerf$, such that $-f(-x) = f(y)$. Thus $0 \in f(-x) + f(y) = f(-x + y)$ and therefore $(-x + y) \cap kerf \neq \emptyset$. Let $w \in (-x + y) \cap kerf$. Since $-x \notin kerf$, the reversibility of $-x$ is valid [17] and so $y \in x + w$, thus $y \in kerf$, which contradicts the assumption for y. Thus $f(-x) = 0$ and therefore f is complete. □

Proposition 4.7. *If f is an epimorphism and $kerf = \{0\}$, then it is an isomorphism.*

Proof. Since f is an epimorphism, it is complete and $f(0) = 0'$. Thus, $0' \in f(x - x) = f(x) + f(-x)$. Since $kerf = \{0\}$, for $x \neq 0$ it holds that $f(x), f(-x) \neq 0$ and therefore $f(-x) = -f(x)$. Next suppose that $f(x) = f(y)$, then $0' \in f(x) - f(y) = f(x) + f(-y) = f(x - y)$ and since $kerf = \{0\}$ it derives that $0 \in x - y$, hence $x = y$. □

Proposition 4.8. *If $C \cap kerf \neq \emptyset$, then f is complete.*

Proof. According to Proposition 4.1 (i), if the kernel of a normal homomorphism contains a c-element then it will also contain its opposite as well as all the a-elements. Therefore for every element of $kerf$, its opposite will be in $kerf$ as well. □

Proposition 4.9. *Let f be a complete and good homomorphism for which $f(0) = 0'$. Then*

(i) $f(-x) = -f(x)$.

(ii) *Imf is a symmetric subhyperring of Y'.*

(iii) *$kerf$ is a symmetric hyperideal of Y.*

Proof.

(i) Since $f(0) = 0'$ it derives that $0'$ belongs to Imf. Now let $f(x)$ be an arbitrary element of Imf. Then it holds $0' = f(0) \in f(x - x) = f(x) + f(-x)$ and since f is a complete homomorphism, if $f(x) \neq 0$ then $f(-x) \neq 0$ as well. Thus it derives that $f(-x) = -f(x)$.

(ii) According to Proposition 4.2 (ii), Imf is a subhyperringoid of Y'. Next since $f(-x) = -f(x)$ for all $x \in Y$ it derives that for every element of Imf its inverse belongs to Imf as well and therefore Imf is a symmetric subhyperring of Y'.

(iii) According to Proposition 4.2 (i), $kerf$ is a hyperidealoid of Y. In [8] it has been proved that the set $[kerf] = -f^{-1}(f(0)) \cup f^{-1}(f(0))$ is a symmetric subhypergroup. Since f is complete $-f^{-1}(f(0)) = f^{-1}(f(0)) = kerf$. Thus $kerf$ is a symmetric hyperideal of Y. □

Proposition 4.10. *Let f be a complete and good homomorphism, for which $f(0) = 0'$. Then*

(i) *if T be a symmetric subhyperring of Y, $f(T)$ is a symmetric subhyperring of Y'.*

(ii) *if I is a symmetric hyperideal of $Y, f(I)$ is a symmetric hyperideal in Imf.*

(iii) *if I' is a symmetric hyperideal in Imf, $f^{-1}(I')$ is a symmetric hyperideal in Y.*

(iv) *if I is a maximal symmetric hyperideal in $Y, f(I)$ is maximal in Imf.*

Proof.

(i) According to Proposition 2.2 (iii), $f(T)$ is a subhyperringoid of Y'. Next since $f(-x) = -f(x)$ for all $x \in T$ it derives that for every element of $f(T)$ its inverse belongs to $f(T)$ as well and therefore $f(T)$ is a symmetric subhyperring of Y'.

(ii) Because of (i), $f(I)$ is a symmetric subhyperring of Imf. Next if $f(x)$ is an element of $f(I)$ and $f(y)$ an element of Imf, then $f(x)f(y) = f(xy) \in f(I)$.

(iii) Let x, y be elements in $f^{-1}(I')$. Then $f(x), f(y)$ belong in I', thus $f(x) + f(y) \subseteq I'$ or $f(x+y) \subseteq I'$, so $x + y \subseteq f^{-1}(I')$. Next if $x \in f^{-1}(I')$, then $f(x) \in I'$ and since I' is symmetric, $-f(x) \in I'$. But according to Proposition 4.9 (i) it holds that $-f(x) = f(-x)$. Hence $-x \in f^{-1}(I')$. Furthermore, if $x \in f^{-1}(I')$ and $y \in Y$, it holds: $f(xy) = f(x)f(y) \in I'Y \subseteq I'$. Therefore $xy \in f^{-1}(I')$.

(iv) Suppose that there exists an hyperideal J in Imf, such that $f(I) \subseteq J \subseteq Imf$. Then because of (iii) $f^{-1}(J)$ is a hyperideal and furthermore $I \subseteq f^{-1}(J) \subseteq Y$, which contradicts the assumption that I is maximal.

□

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