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Elements and Results on Polysymmetrical Hypergroups

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Abstract. *Polysymmetrical Hypergroups are special cases of regular hypergroups i.e. hypergroups which have at least one two-sided identity and each one of their non identities has two-sided inverses. Many types of Polysymmetrical Hypergroups derived from the study of other areas of Algebra (as for example the algebraically closed fields and the linear spaces), or appeared during the study of other mathematical branches (as geometry, or the theory of Languages and Automata) from the point of view of the hypercompositional structures theory. This paper presents and analyzes all the different kinds of Polysymmetrical Hypergroups that have appeared up to now.*
Keywords: *hypergroups, polysimetrical hypergroups*

1. Introduction

1934 was the year that Frederic Marty defined the hypergroup [9]. This happened in connection with his thesis on meromorphic functions, which was written under the direction of Paul Montel. Unfortunately F. Marty died young, during the Second World War, when his airplane was shot down over the Baltic Sea, while he was going on a mission to Finland. In the duration of his short life (1911-1940), F. Marty studied properties and applications of the hypergroups in two more communications ([10, 11]). This new structure though, very soon whipped up the interest of other mathematicians such as M. Krasner, J. Kuntzmann, H. Wall, O. Ore, M. Dresher etc. So, several papers appeared referring to its study and its applications [1, 5, 7, 8, 30]. For the self-sufficiency of this paper, it is mentioned that the axioms which endow the pair (H,·), where H is a nonempty set and "·" is a hypercomposition in H (i.e. a function from H×H to the powerset $\mathfrak{P}(H)$ of H), with the hypergroup structure are:

i. a(bc) = (ab)c for every a,b,c ∈ H (associativity)

ii. aH = Ha = H for every a ∈ H (reproductivity)

In a hypergroup, the result of the hypercomposition is always a nonempty set. Indeed, let $ab=\varnothing$. Then $H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$, which is absurd.

Soon, from the very first stage of this theory, the hypergroup was enriched with new further more axioms. So, not only F. Marty, but also M. Krasner, Kuntzmann, H. Wall, M. Dresher and O. Ore observed that there exist hypergroups with *identity* (or *unit* or *neutral*) *elements* and *inverses*. An element $e \in H$ such that $a \in ea$ for all $a \in H$ is called *left identity*. A *right identity* is defined analogously. An *identity* is an element e that is both left and right identity. Next a′ is a *left inverse* of $a \in H$ if $e \in a'a$, where e is an identity. The *right inverse* is defined in a similar manner and a *two-sided inverse* a′ has the property $e \in a'a \cap aa'$ where e is a two-sided identity. Thus any hypergroup (H,·) satisfying the conditions:

iii. in H there exists two-sided identities

iv. every element of H has two-sided inverses

was named *regular hypergroup*. A hypergroup is said to be *completely regular* if it contains a single identity e with respect to which, every element x has a unique inverse x^{-1}, such that $e \in x^{-1}x \cap xx^{-1}$.

Another important property that holds in certain hypergroups is the reversibility: Suppose that h is as subhypergroup of H (i.e h is a subset of H that satisfies the axioms of a hypegroup). Then h is left reversible in H if any relation $x_2 \in a_1x_1$, where $a_1 \in h$ implies the existence of an $a_2 \in h$ such that $x_1 \in a_2x_2$. Similar is the definition of right reversibility and h is called reversible in H if it is both left and right reversible.

In F. Marty's paper [9] one can also find the definition of the two *induced hypercompositions* (the left and the right division) that derive from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|b} = \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b|} = \{y \in H \mid a \in by\}$$

For the sake of simplicity of the notation, these hypercompositions are also denoted by a:b or a/b and a..b or b\a respectively. When "·" is commutative, a/b = b\a. Using the induced hypercompositions, the closed subhypergroup can be defined as follows: a subhypergroup h of a hypergroup H is closed if $a/b \subseteq h$ and $b\backslash a \subseteq h$, for each $a,b \in h$ [3, 12, 13].

W. Prenowitz, in order to study Geometries using tools and methods of the hypergroup theory, introduced the hypergroups, which are known as *join spaces* [29]. Now, the term *join space* or *join hypergroup* signifies a commutative hypergroup that satisfies the transposition property:

$$(a/b) \cap (c/d) \neq \varnothing, \text{ implies } \; ad \cap bc \neq \varnothing$$

J. Jantosciak, in his paper [3] removes the commutativity from the axioms of the join hypergroup introducing thus the *transposition hypergroup*. More precisely, a transposition hypergroup is a hypergroup which satisfies the axiom:

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing$$

2. M-Polysymmetrical Hypergroups

Special cases of regular hypergroups are the polysymmetrical hypergroups. The term "polysymmetrical" was used by J. Mittas, who introduced them and studied them in connection with the algebraically closed fields.

The characteristic feature of the polysymetrical hypergroups is a more or less strong axiom of reversibility that permits to shift elements of the relation $z \in xy$ from the one part to the other, using the symmetric elements of x,y,z.

At first, in 1970, Jean Mittas [23], in connection with his work on algebraically closed fields, defined a commutative hypergroup $(H,\cdot)$ having the following axioms:

M_1i. in H there exists one two-sided identity e

M_1ii. every element x of H has two-sided inverses x' such that $xx'=x'x=e$
the set $S(x)=\{x' \in H \mid x'x=e\}$ is called the symmetric set of x

M_1iii. if $z \in xy$ and $x' \in S(x)$, $y' \in S(y)$, $z' \in S(z)$, then $z' \in y'x'$

Next in 1983 in a joint paper by S. Ioulidis and J. Mittas [2], a hypergroup $(H,\cdot)$ with the following axioms was introduced:

M_2i. in H there exists one two-sided identity e such that $ex=xe=x$, for every $x \in H$ (*scalar identity*)

M_2ii. every element x of H has two-sided inverses x' such that $e \in xx' \cap x'x$. The set of the inverses x' of an element x is denote by S(x) and S(x) is called the symmetric set of x

M_2iii. for every $x, z \in H$ and for every $x', x'' \in S(x)$, there exist $y_1 \in x'z$ and $y_2 \in zx''$ such that $z \in xy_1 \cap y_2x$.

(see also [26]). J. Mittas named the hypergroups that satisfy the axioms M_1i-M_1iii, first kind polysymmetrical hypergroups and the hypergroups that satisfy the axioms M_2i-M_2iii, second kind polysymmetrical hypergroups. Later on C. Yatras who studied in details the commutative polysymmetrical hypergroups of the first kind [31], [32] named them M-polysymmetrical hypergroups, using the letter "M" after the name of J. Mittas, who introduced them.

Finally in 2005, in his paper [28], J. Mittas defined the generalized M-polysymmetrical hypergroup. This hypergroup satisfies the axioms:

GMi. in H there exists at least one two-sided identity e (i.e. $x \in xe=ex$ for every $x \in H$). the set of identities is denoted by U

GMii. for each element x of H there exists at least one two-sided inverse x' with respect to each element of U, i.e. $(\forall x \in H)(\exists\, x' \in H)(\forall e \in U)[e \in xx'=x'x]$

GMiii. $x x' = x'x = U$ for each $x \in H$, $x' \in S(x)$

GMiv. for each $x, y \in H$ it holds: $xy \cap U \neq \varnothing \Rightarrow xy = U$

GMv. if $z \in xy$ and $x' \in S(x)$, $y' \in S(y)$, $z' \in S(z)$, then $z' \in y'x'$

The following example shows the close connection between the generalized M-polysymmetrical hypergroup and the first kind M-polysymmetrical hypergroup.

Example 2.1. Let K be the set of the points of a conical surface of revolution around the axis Oz of the Oxyz system, having vertex O. In K the following hypercomposition is introduced

$$(x_1,y_1,z_1)+(x_2,y_2,z_2) = \{ (x,y,z) \in K, \ z = z_1+z_2 \}$$

i.e. the result of the hypercomposition of any two elements (x_1,y_1,z_1), (x_2,y_2,z_2) is the set of the points of a circle of the conical surface having $z=z_1+z_2$. K, endowed with this hypercomposition, is a M-polysymmetrical hypergroup of the first kind. The neutral element of this hypergroup is the conical surface's vertex (0,0,0), while the symmetric set S(x) of any element $x \in K$ is the set of the points of the circle which is symmetric to the circle defined by x with center of symmetry the point (0,0,0). Now if we consider the union of the points of the conical surface with the points of the plane xOy, and we endow it with the same hypercomposition, we end up to a generalized M-polysymmetrical hypergroup. In this hypergroup, the set of the neutral elements is the set of the points of the plain xOy.

3. Canonical Polysymmetrical Hypergroups

The additive part of the hyperfield is a special type of a completely regular commutative hypergroup, having a scalar (and thus unique) identity and a unique inverse for each one of its elements. The hyperfield is a hypercompositional structure, analogous to the field, (with the addition being a hypercomposition and the multiplication a composition) which was introduced, in 1956, by M. Krasner, who constructed it as the proper algebraic tool in order to define a certain approximation of a complete valued field, by sequences of such fields [6]. J. Mittas named the additive hypergroup of the hyperfield *canonical hypergroup* and he studied it in depth (e.g. see [24]). A canonical hypergroup is a completely regular commutative hypergroup, having a scalar (and so unique) identity, a unique inverse for each one of its elements, and is also enriched with the axiom:

$$z \in xy \Rightarrow y \in zx' \quad (x' \text{ is the inverse of } x)$$

Equivalently, canonical hypergroup is a join hypergroup that has a scalar identity. Next, in 1985, J. Mittas, in his paper [25], defined the canonical polysymmetrical hypergroup. According to his definition, a *canonical polysymmetrical hypergroup (CPH)* is a second kind commutative polysymmetrical hypergroup which satisfies the axiom:

$$z \in xy \Rightarrow (\exists x' \in S(x)) \ [y \in zx']$$

From this axiom it derives that in a CPH, for any two elements x, y, it holds $S(xy) \subseteq S(x)S(y)$. Also in a CPH the implications are valid:

$$z \in xy \Rightarrow (\exists x' \in S(x)) \ (\forall y' \in S(y)) \ (\exists z' \in S(z))[z' \in x'y']$$

$z \in xy \quad \Rightarrow \quad (\exists z' \in S(z)) \quad (\exists (x',y') \in S(x) \times S(y))$
$(\exists z' \in S(z))[z' \in x'y']$

It is worth mentioning that although the transposition holds in the case of the canonical hypergroups, this does not happen in the case of the canonical polysymmetrical hypergroups. Indeed:

Example 3.1. Let H be a set totally ordered and symmetric around a center, denoted by $0 \in H$. Then H with the commutative hypercomposition

$$x+y = \begin{cases} y, & \text{if } |x|<|y| \text{ for every } x,y \in H^- \cup \{0\} \text{ or } x,y \in \{0\} \cup H^+ \\ [x, y], & \text{if } x \in H^- \text{ and } y \in H^+ \end{cases}$$

becomes a canonical polysymmetrical hypergroup. Suppose now that $x,y,a,b \in H^+$ and $x<y<a<b$. Then $x/y=a/b=H^-$. Thus $x/y \cap a/b \neq \varnothing$. But $x+b=\{b\}$, $y+a=\{a\}$ and so $x+b \cap y+a = \varnothing$, i.e. transposition is not valid.

In the same paper [25], J. Mittas introduced stronger definitions of polysymmetrical hypergroups. Thus he defined the fortified canonical polysymmetrical hypergroup and the extremely fortified canonical polysymmetrical hypergroup.

A *fortified canonical polysymmetrical hypergroup (F-CPH)* is a first kind commutative polysymmetrical hypergroup $(H,\cdot)$, which satisfies the axiom:

$S(xy) \subseteq [x'S(y)] \cap [y'S(x)]$, for each $x,y \in H$, $x' \in S(x)$, $y' \in S(y)$

One can observe that a F-CPH is a CPH. Also in a F-CPH it holds:

$$z \in xy \Rightarrow (\forall (x',z') \in S(x) \times S(z))\ (\exists y' \in S(y))[z' \in x'y']$$

An *extremely fortified canonical polysymmetrical hypergroup (EF-CPH)* is a first kind commutative polysymmetrical hypergroup $(H,\cdot)$, which satisfies the axiom:

$$z \in xy \Rightarrow (\forall (x',y',z') \in S(x) \times S(y) \times S(z))[z' \in x'y']$$

From the above definition it derives that in a EF-CPH, for any two non identities x, y, the equality $S(xy)=S(x)S(y)=x'y'$ is valid. This property has interesting consequences:

a) an EF-CPH is a F-CPH
b) if x is a non identity of a EF-CPH H and $y \in H$, $y',y'' \in S(y)$, then $xy'=xy''$ and $yy'=yy''$
c) for each $x \in H$, $x',x'' \in S(x)$ it holds $S(x')=S(x'')$
d) for each $x,y,z \in H$, with z non identity, the implication holds:

$$z \in xy \Rightarrow (\forall x' \in S(x))[y' \in zx']$$

Next in [27], in connection with vector spaces, the notion of the generalized canonical polysymmetrical hypergroup was introduced. A *generalized canonical polysymmetrical hypergroup (G-CPH)* is a commutative hypergroup $(H,\cdot)$ endowed with the following axioms:

i. $(\exists e \in H)(\forall x \in H)[x \in ex]$
ii. $(\forall x \in H)(\ (\exists x' \in H)\ [e \in xx']$
iii. $S(e)=e$
iv. for each $x,y,z \in H$, the implication holds:

$$z \in xy \Rightarrow (\exists x^* \in S(S(z)))(\exists x' \in S(x))[y \in z^* x']$$

The G-CPH appeared as the attached hypergroup of vector spaces. Indeed, let V be a vector space over a field F. Then [27]:

Proposition 3.1. *V, endowed with hypercomposition*

$$x \dagger y = \{\kappa x+\lambda y \ / \ \kappa,\lambda \in F_+^*,\ \kappa+\lambda=1\},\ \text{for every } x,y \in V^*$$

and $\quad x \dagger 0 = 0 \dagger x = x,\ \text{for every } x \in V$

is a G-CPH, but not a join one.

Proposition 3.2. *V, endowed with hypercomposition*

$$x \dagger y = \{\kappa x+\lambda y \ / \ \kappa,\lambda \in F_+^*,\ \kappa+\lambda=1\},\ \text{for every } x,y \in V^*$$

and $\quad x \dagger 0 = 0 \dagger x = \{\lambda x \ / \ \lambda \in F_+^*,\ \lambda \le 1\},\ \text{for every } x \in V$

is a join G-CPH.

Two other G-CPHs derive from the Propositions 3.1. and 3.2., if the hypercompositions are properly modified in such a way that the sum $x \dagger y$ denotes, geometrically, not only the set of points of the segment xy (for x≠y), but also the set of all the points of the line through x,y (x≠y)

Proposition 3.3. *V, endowed with hypercomposition*

$$x \dagger y = \{\kappa x+\lambda y \ / \ \kappa,\lambda \in F^*,\ \kappa+\lambda=1\},\ \text{for every } x,y \in V^*$$

and $\quad x \dagger 0 = 0 \dagger x = x,\ \text{for every } x \in V$

becomes a non join G-CPH.

Proposition 3.4. *V, endowed with hypercomposition*

$$x \dagger y = \{\kappa x+\lambda y \ / \ \kappa,\lambda \in F^*,\ \kappa+\lambda=1\},\ \text{for every } x,y \in V^*$$

and $\quad x \dagger 0 = 0 \dagger x = \{\lambda x \ / \ \lambda \in F^*,\ \lambda \le 1\},\ \text{for every } x \in V$

becomes a join G-CPH.

Also the hypercomposition of the next Proposition gives an interesting and useful result. The figure which presents geometrically this hypercomposition, is the set of the vectors with origin 0, that fill the parallelogram with sides x and y, and end to the opposite sides of x and y.

Proposition 3.5. *V, endowed with hypercomposition*

$$x \dagger y = \{x+\lambda y,\ y+\lambda x \ / \ \lambda \in F_+^*,\ \lambda \le 1\},\ \text{for every } x,y \in V$$

becomes a join G-CPH.

In [12, 13, 27] it is proved that the convex sets of a vector space and its vector subspaces are directly connected to the semi-sub-hypergroups and to the closed sub-hypergroups respectively, of some of the vector space's attached hypergroups. This approach leads to remarkable results and also lets the generalization of already known theorems of the vector spaces, in sets with fewer axioms [13, 14, 15].

4. Transposition Polysymmetrical Hypergroups

A *Transposition Polysymmetrical Hypergroup* (TPH) is a transposition hypergroup $(H,\cdot)$ that contains an identity e which satisfies the axioms:

TP1. $ee=e$

TP2. $x \in xe = ex$

TP3. for every $x \in H-\{e\}$ there exists at least one element $x' \in H-\{e\}$, symmetric of x, such that $e \in x x'$, and furthermore x' satisfies $e \in x'x$

A commutative transposition polysymmetrical hypergroup is called *Join Polysymmetrical Hypergroup* (JPH). The JPH appeared as an attached hypergroup of an automaton, during the study of automata theory with tools from the theory of hypercompositional structures. Indeed, based on the notion of the grade of a state s_i of an automaton, which is the set of the elements of the language over an alphabet A that lead the automaton from this state (s_i) to one of its final states [17], there can be introduced an equivalence relation in the set of the states, named *grade equivalence*, as follows:

$$s_i \, R \, s_j \text{ when the grade of } s_i \text{ is equal to the grade of } s_j$$

Considering that the automaton has only one final state s_F (real or conventional), the set of its states can be equipped with the following hypercomposition:

$$s_i + s_j = \begin{cases} C_{si}^R \cup C_{sj}^R & \text{if } C_{si}^R \neq C_{sj}^R \text{ and } s_i, s_j \neq s_F \\ C_{si}^R \cup \{s_F\} & \text{if } C_{si}^R = C_{sj}^R \end{cases}$$

where C_{si}^R is the equivalence class of s_i. With this hypercomposition, the set of the states of the automaton becomes a join polysymmetrical hypergroup [18, 19]. Next, since the attached hypergroup of the grade of an automaton is a JPH, one can construct a fortified join hypergroup (FJH) [21] in the set of this JPH's equivalence classes. This last FJH, used as the attached hypergroup of the grade, leads to the construction of a new equivalent automaton, the states of which are the equivalence classes of the states of the initial automaton. Apparently, this new automaton has fewer states and accepts the same language with the one it started with [17, 18]. Moreover, by eliminating those states for which the order can not be defined, there derives the automaton which has the minimum number of states and which accepts exactly the same language with the initial one [17, 18, 20].

Another example of a JPH is the following:

Example 4.1. Let K be a field and G a subgroup of its multiplicative group. Also let a hypercomposition $\dotplus$ be defined in K as follows:

$$x \dotplus y = \{ z \in K \mid z = xp+yq, \; p,q \in G \}$$

Then $(K, \dotplus)$ is a join polysymmetrical hypergroup having the 0 of K as its neutral element. The symmetric set of an element x of K is the set $S(x) = \{xp \mid p \in G\}$.

Keeping in mind the general definition of the completely regular hypergroups, the (partial) reversibility in the case of the TPH is:

For every x,y,w ∈ H there exists either x′∈ S(x) or y′∈ S(y) such that: w ∈ xy implies either y ∈ x′w or x ∈ w y′

Obviously if w ∈ xy implies that y ∈ x′w, x′∈ S(x) and x ∈ wy′, y′∈ S(y), then the reversibility is complete

In the TPH, there exist three possible cases:

- The TPH in which the complete reversibility holds
- The TPH in which the partial reversibility holds
- The TPH in which the reversibility is not valid

Thus there exist three kinds of transposition polysymmetrical hypergroups. In [22] one can find examples of each one of them.

From the different examples of TPH it derives that the identity e is possible either to belong or not to belong in the result of the hypercomposition ex. More precisely it holds that ex⊆{e}∪S(S(x)) for each element x of a TPH (H,·). Thus in TPH there exist two kinds of elements, the attractive elements and the non attractive elements. An element x of a TPH, is called *attractive* if e∈xe, while a nonidentity element x is called *non attractive* if e∉xe [21, 22]. If x is an attractive element of a TPH, then S(x) consists of attractive elements, while if x is a non attractive element, then S(x) consists of non attractive elements. Also if x is an attractive element of a TPH, then all the elements of ex are attractive while if x is a non attractive element, then ex contains only non attractive elements. Moreover the result of the hypercomposition of two attractive elements contains only attractive elements, while the result of the hypercomposition of an attractive element with a non attractive element, consists of non attractive elements. It is proved that the set A of the attractive elements of a TPH is a closed subhypergroup of H and further more that A is the minimum (in the sense of inclusion) closed subhypergroup of H. In the study of TPH the symmetric subhypergroups play a significant part. A subhypergroup h of a transposition polysymmetrical hypergroup is called symmetric, if x∈h implies that S(x)⊆ h. It is proved that the intersection of two symmetric subhypergroups is a symmetric subhypergroup. Thus the set of the symmetric subhypergroups of a TPH is a complete lattice.

An element e of a hypergroup H is a *scalar identity* if ex=xe=x for each x in H. If a scalar identity exists in H, then it is unique. An element e of a hypergroup H is a *strong identity* if x∈ex=xe⊆ {x,e} for all x∈H. A strong identity need not be unique [4]. The set E of the strong identities is a central subhypergroup of H [4]. It is proved that if a transposition polysymmetrical hypergroup has a strong identity, then it is unique. In [16] one can find many examples of TPH with strong identity. From these examples, there derive the following interesting remarks:

i) the non existence of e in ab does not necessarily imply that e does not also belong to S(a)S(b), or to S(b)S(a).
ii) the non void intersection S(a)∩S(b) does not imply that S(a) is equal to S(b) i.e. in TPHs the sets S(x), do not define a partition in general.

The following propositions present some properties of the transposition polysymmetrical hypergroups with strong identity (for their proofs see [16])

Proposition 4.1. *Let $x \neq e$ be an attractive element of H, then*

i. $e/x = eS(x) = \{e\} \cup S(x) = S(x)e = x\backslash e$

ii. $x/e = e\backslash x = x$

Proposition 4.2. *If x, y are attractive elements of H, then*

i. $\{x, y\} \subseteq xy$

ii. $x \in x/y$ *and* $x \in y\backslash x$

iii. $x/x = x\backslash x = A$, *where A is the set of the attractive elements*

Proposition 4.3. *If* $a, b \in A$, $e \notin ab$ *and* $e \notin S(b)S(a)$, *then* $S(ab) = S(b)S(a)$

Proposition 4.4. *Let* $x, y, z \in A-\{e\}$ *and* $z \in xy$:

i. *if* $S(x) \cap S(z) = \varnothing$, *then* $y \in x'z$ *for every* $x' \in S(x)$

ii. *if* $S(y) \cap S(z) = \varnothing$, *then* $x \in zy'$ *for every* $y' \in S(y)$

If x is attractive and $x' \in S(x)$, then $e \in xx'$ implies that $x \in ex$ and $x' \in ex'$. Also $e \in ex$ implies that $e \in ex'$, while $x \notin ee$. Thus, from the above observations and from Proposition 4.4, it is clear that in the transposition polysymmetrical hypergroups with strong identity the property of reversibility is valid under conditions.

Next, using the symmetric subhypergroups, cosets can be defined. Indeed if it is assumed that H has a strong identity and consists only of attractive elements, x is an element of H and h a symmetric subhypergroup of H, then $x_{\overleftarrow{h}}$ (the left coset of h determined by x) and dually, $x_{\overrightarrow{h}}$ (the right coset of h determined by x), are given by

$$x_{\overleftarrow{h}} = \begin{cases} h, & \text{if } x \in h \\ x/h, & \text{if } x \notin h \end{cases} \quad \text{and} \quad x_{\overrightarrow{h}} = \begin{cases} h, & \text{if } x \in h \\ x\backslash h, & \text{if } x \notin h \end{cases}$$

Recalling that in any hypergroup, the equality $(B\backslash A)/C = B\backslash(A/C)$ is valid, we have that x_h, i.e. the double coset of h determined by x, is given by

$$x_h = \begin{cases} h, & \text{if } x \in h \\ h\backslash(x/h) = (h\backslash x)/h, & \text{if } x \notin h \end{cases}$$

It is proved that if h is a symmetric subhypergroup of H and x is an element of H which does not belong to h, then the equalities $(x/h)h = xh$ and $h(h\backslash x) = hx$ are valid. Thus, each one of the families $H{:}\overleftarrow{h} = \{x_{\overleftarrow{h}} \mid x \in H\}$, $H{:}\overrightarrow{h} = \{x_{\overrightarrow{h}} \mid x \in H\}$ and $H{:}h = \{x_h \mid x \in H\}$ of the left, right and double cosets are partitions of H. A thorough and detailed study of these cosets as well as the homomorphisms of TPH appears in [16].

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