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SOLVING EQUATIONS AND SYSTEMS IN THE ENVIRONMENT OF A HYPERRINGOID

by

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ABSTRACT This paper deals with the manipulation of first ordered equations and systems of equations within the structure of the hyperringoid.

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1. INTRODUCTION

In 1966 M. Krasner introduced a hypercompositional structure, which he named hyperring [2]. A **hyperring** is a triplet $(H,+,\cdot)$, where $(H,+)$ is a canonical hypergroup, $(H,\cdot)$ is a semigroup, the neutral element of the canonical hypergroup is bilaterally absorbing with regard to the multiplication and also the multiplication distributes over the hyperoperation from both sides.

In 1990, during the 4th AHA, we have presented with J. Mittas [5] a hypercompositional structure, which is a generalization of Krasner's hyperring. This new structure has proved to be a very suitable tool for the study of the theory of Automata and Languages from the standpoint of the theory of the Hypercompositional Structures. We have called it **hyperringoid** and its definition is as follows:

Definition 1.1. A non void set Y endowed with a composition "·" and a hypercomposition "+" is called a **hyperringoid** if

I. (Y,+) is a hypergroup

II. (Y,·) is a semigroup

III. The composition distributes over the hypercomposition from both sides.

According to the properties of their additive part we distinguish several types of hyperringoids. Therefore, if the hypergroup is a join one we have the **Join hyperringoid**, if it is a Fortified Join hypergroup we have the **Fortified Join hyperringoid** or the **Join hyperring**. Moreover if $x+y = \{x, y\}$ (*B*-hyperoperation) then we have the **B-hyperringoid**, which is a Join hyperringoid. The above hyperoperation has derived in a natural way from the theory of the Languages [5], [6]. It is worth mentioning though that this hypercomposition can be found in a paper by L. Konguetsof, written as early as the 60's [1].

Furthermore if the B-hyperoperation has a neutral element with regard to which every element is selfopposite, i.e. if

$$x+y = \begin{cases} \{x, y\} & \text{if } x \neq y \\ \{0, x\} & \text{if } x = y \end{cases}$$

then we have the **Dilated B-hyperringoid**, which is a Join hyperring. The properties of the hyperringoid have been studied in [7]. Here appears the subject regarding the equation and system solving in the hyperringoids.

2. Solutions of Equations in a Hyperringoid

In an additive group, as it is known from the relevant theory, there always exists a unique solution for the equation $\gamma = x+\beta$. Also, in the case of the hypergroups the relation $\gamma \in x+\beta$ is being satisfied by all the elements x of the set $\gamma:\beta$, which, as it is known [3], [4], is non empty. In a hyperringoid though (like in the case of the hyperring) there may not

exist elements x which satisfy the relation $\gamma \in \alpha x+\beta$, in an analogous manner as in the case of the rings where there does not always exist a solution in the equation $\gamma = \alpha x+\beta$. But in a hyperringoid there appear equations for which it is possible to exist a solution. Such equations will be dealt with in the following, where we assume that the hyperringoid Y is unitary and that when $(Y,+)$ has a neutral element 0, then $\alpha\beta \neq 0$ if $\alpha, \beta \neq 0$ and also the coefficient of the unknown set X is not zero. Moreover α^0 as well as A^0, (where $\alpha \in Y$ and $A \subseteq Y$) will denote the unit element of Y.

Proposition 2.1. *The existence of the set* $\sum_{n=0}^{\infty}\alpha^n\beta$ *provides a solution of the equation*

$$\alpha X+\beta = X \tag{1}$$

where $\alpha, \beta \in Y$ *and* $X \subseteq Y$.

P r o o f. Substituting in the first part of the equation we have:

$$\alpha \sum_{n=0}^{\infty}\alpha^n\beta + \beta = \left(\alpha\sum_{n=0}^{\infty}\alpha^n + 1\right)\beta = \left(\sum_{n=1}^{\infty}\alpha^n + 1\right)\beta = \sum_{n=0}^{\infty}\alpha^n\beta$$

Now if we have the subset B of Y instead of the element β, then the set $\left(\sum_{n=0}^{\infty}\alpha^n\right)B$ satisfies the relation $X \subseteq \alpha X+B$. Indeed, in the case of sets, the distributivity is $(A+B)\Gamma \subseteq A\Gamma + B\Gamma$ and so we have:

$$\alpha\left(\sum_{n=0}^{\infty}a^n\right)B+B \supseteq \left(\alpha\sum_{n=0}^{\infty}a^n+1\right)B = \left(\sum_{n=1}^{\infty}a^n+1\right)B = \left(\sum_{n=0}^{\infty}a^n\right)B$$

Therefore the next Proposition is valid:

Proposition 2.2. *The existence of* $\left(\sum_{n=0}^{\infty}a^n\right)B$ *provides a set which verifies the relation:*

$$X \subseteq \alpha X + B \tag{2}$$

where $\alpha \in Y$ *and* $B, X \subseteq Y$.

Corollary 2.1. *In a B-hyperringoid the set* $\sum_{n=0}^{\infty} A^n B$ *is a solution of the equation:*

$$AX + B = X \tag{3}$$

where $A, B, X \subseteq Y$.

The solution of the equation (1), which was given in Proposition 2.1 is not unique. We will see it in the next Propositions, but before that it is necessary to prove the following Lemma:

Lemma 2.1. *If every two elements of Y are contained in their hypersum then, every semi-subhyperringoid is a subhyperringoid.*

P r o o f. Let Y' be a semi-subhyperringoid of Y and $\alpha \in Y'$. Then $\alpha + Y' \subseteq Y'$. Moreover:

$$\alpha + Y' = \bigcup_{\beta \in Y'} (\alpha+\beta) \supseteq \bigcup_{\beta \in Y'} \{\alpha, \beta\} = Y'$$

Thus $\alpha + Y' = Y'$ and so the Lemma.

Proposition 2.3. *If the hypersum of every element of Y with itself gives as a result only this element and also if every two elements of Y are contained in their hypersum, then the set* $\{1, \alpha\}^{-}\beta$ *is another solution of (1)*

P r o o f. It is known [7] that the semi-subhyperringoid A^{-} which is generated by a set A consists of the union of finite sums whose addends are products of the type:

$$\prod_{i=1}^{k} \varepsilon_i, \text{ where } \varepsilon_i \in A, i = 1 \ldots k$$

Thus $\{1, \alpha\}^{-}$, which is a subhyperringoid of Y (Lemma 2.1.), is the union of all the sums of the type:

$$\sum_{\alpha} = \alpha^{k1} + \alpha^{k2} + \ldots + \alpha^{kv}$$

where $k_1, \ldots, k_v \in N_0$. So substituting $\{1, \alpha\}^-$ in (1) we have:

$$\alpha(\{1, \alpha\}^-\beta) + \beta = (\alpha\{1, \alpha\}^- + 1)\beta$$

But the set $\alpha\{1, \alpha\}^-$ consists of all the sums $\sum_a$ which do not contain the unity and so, adding the unit element to this sum we get the whole subhyperringoid $\{1, \alpha\}^-$. Consequently $\alpha\{1, \alpha\}^- + 1 = \{\alpha, 1\}^-$ and thus

$$[\alpha\{1, \alpha\}^- + 1]\beta = \{1, \alpha\}^-\beta$$

Also, analogously to Proposition 2.2, we have:

Proposition 2.4. *If the hypersum of every element of Y with itself gives as a result only this element and also if every two elements of Y are contained in their hypersum, then the set $\{1, \alpha\}^-B$ verifies the relation (2).*

Moreover the next Proposition holds:

Proposition 2.5. *If $\{1, \alpha\}^-$ is a subhyperringoid of Y such that its multiplicative part is a subgroup of the multiplicative semigroup of Y, then $\{1, \alpha\}^-\beta$ is a solution of (1), while the $\{1, \alpha\}^-B$ satisfies the relation (2)*

Proposition 2.6. *If every two elements of Y are contained in their hypersum, then the set $\{1, \alpha\}^-\beta$ is also a solution of the equation:*

$$\sum_{i=0}^{n} a^i X + \beta = X$$

where $\alpha, \beta \in Y$ and $X \subseteq Y$.

P r o o f. Because of Lemma 2.1., Y does not have any semi-subhyperringoids, since all of them are subhyperringoids. Therefore $\{1, \alpha\}^-$ is a subhyperringoid of Y and substituting to the equation we get:

$$\sum_{i=0}^{n} a^i \{1, \alpha\}^-\beta + \beta = \left[\sum_{i=0}^{n} a^i \{1, \alpha\}^- + 1\right]\beta = \left[\sum_{i=1}^{n} a^i \{1, \alpha\}^- + \{1, \alpha\}^- + 1\right]\beta$$

Moreover, since $\alpha^i \in \{1, \alpha\}^-$ it derives that $\alpha^i\{1, \alpha\}^- \subseteq \{1, \alpha\}^-$. Hence

$\sum_{i=1}^{n} a^i \{1, \alpha\}^- \subseteq \{1, \alpha\}^-$. Also $\{1, \alpha\}^-$ is a subhyperringoid and therefore $A + \{1, \alpha\}^- = \{1, \alpha\}^-$ for every $A \subseteq \{1, \alpha\}^-$. Thus:

$$\left[\sum_{i=1}^{n} a^i \{1, \alpha\}^- + \{1, \alpha\}^- + 1\right] \beta = \left[\{1, \alpha\}^- + 1\right] \beta = \{1, \alpha\}^- \beta$$

Corollary 2.2. *If every two elements of Y are contained in their hypersum, then the set $[A \cup \{1\}]^- \beta$ is a solution of the equation:*

$$\sum_{i=0}^{n} A^i X + \beta = X$$

where $\beta \in Y$ and $A, X \subseteq Y$.

Moreover, since $\left(\sum_{i=0}^{n} A^i\right) X \subseteq \sum_{i=0}^{n} A^i X$, we have the

Corollary 2.3. *If every two elements of Y are contained in their hypersum, then the set $[A \cup \{1\}]^- B$ verifies the relation:*

$$X \subseteq \left(\sum_{i=0}^{n} A^i\right) X + B$$

where $A, B, X \subseteq Y$.

For the following it is necessary to define a relation in $\wp(Y)$. We can give though a more general definition, which will apply in every hypergroup and so we proceed with an arbitrary hypergroup H and we introduce in $\wp(H)$ a relation " $\leq$ " defined as follows:

$$A \leq B \Leftrightarrow A+B = B$$

for every $A, B \subseteq H$. We note that if A is a singleton, e.g. $A=\{\alpha\}$, then for the sake of simplicity of the symbolism we write $\alpha \leq B$.

Lemma 2.2. *The relation "$\leq$" is transitive and antisymmetric.*

P r o o f. Let $A \leq B$ and $B \leq \Gamma$. Then $A+B = B$ and $B+\Gamma = \Gamma$. Thus

$$A+\Gamma = A + (B+\Gamma) = (A+B) + \Gamma = B+\Gamma = \Gamma$$

and so $A \leq \Gamma$, i.e. the relation " $\leq$ " is transitive. Next if $A \leq B$ and $B \leq A$ then $A+B = B$ and $B+A = A$, therefore $A=B$ and so the antisymmetric property. It is worth mentioning here that the relation $A \leq B$ does not imply that $A \subseteq B$. Indeed, if (J, +) is the join hypergroup of the Euclidean plane [8], B is the interior of a convex subset K of J and A is the boundary of K, then $A+B = B$ and $A \cap B = \varnothing$.

Next let $\mathcal{O}$ be the family of the subsets of H with the property $A+A = A$. For this family it holds:

Lemma 2.3. *The relation " $\leq$ " is an order relation in $\mathcal{O}$*

The above defined relation " $\leq$ " can also be introduced in a hyperringoid Y, where it holds:

Proposition 2.7. *If $A \leq B$ and $\Gamma \leq \Delta$, then*

i) $A + \Gamma \leq B + \Delta$

and ii) $A\lambda \leq B\lambda$ *for arbitrary* $\lambda \in Y$.

P r o o f. From the relations $A \leq B$ and $\Gamma \leq \Delta$ it derives that $A+B = B$ and $\Gamma+\Delta = \Delta$. Adding these equalities we have:

$$(A + B) + (\Gamma + \Delta) = B + \Delta \Leftrightarrow$$

$$\Leftrightarrow (A + \Gamma) + (B + \Delta) = B + \Delta \Leftrightarrow$$

$$\Leftrightarrow A + \Gamma \leq B + \Delta$$

Moreover, if we multiply both sides of $A+B = B$ with $\lambda \in Y$ we have:

$$(A + B)\lambda = B\lambda \Leftrightarrow A\lambda + B\lambda = B\lambda \Leftrightarrow A\lambda \leq B\lambda$$

Proposition 2.8. *In the family $\mathcal{P}$ of the subsets of Y the equation (1) has minimum (with regard to the order "$\leq$") solution, the set* $\sum_{n=0}^{\infty}\alpha^n\beta$

P r o o f. Let Ψ be a solution of (1). Then

$$\Psi + \beta = (\alpha\Psi + \beta) + \beta = \alpha\Psi + (\beta + \beta) = \alpha\Psi+\beta = \Psi$$

thus $\beta \leq \Psi$ (i). Moreover

$$\Psi + \alpha\Psi = (\beta + \alpha\Psi) + \alpha\Psi = \beta + (\alpha\Psi + \alpha\Psi) = \beta + \alpha\Psi = \Psi$$

thus $\alpha\Psi \leq \Psi$ (ii)

Multiplying now both sides of (i) with α we get $\alpha\beta \leq \alpha\Psi$, which, due to (ii) becomes $\alpha\beta \leq \Psi$. Repeating this procedure n times we get $\alpha^n\beta \leq \Psi$ and adding these inequalities for all $n \in N_0$ we have:

$$\sum_{n=0}^{\infty}\alpha^n\beta \leq \Psi$$

Corollary 2.4. *In a B-hyperringoid the equation (3) has minimum (with regard to the order "$\leq$") solution, the set* $\sum_{n=0}^{\infty}A^nB$

3. Systems of Equations in a Hyperringoid

Let's consider the system:

$$\Psi = \alpha\Psi + 1$$
$$X = \Psi B \qquad \text{(S1)}$$

where $\alpha \in Y$ and $X, \Psi, B \subseteq Y$.

Then, according to Proposition 2.1, the existence of the set $\Psi = \sum_{n=0}^{\infty}\alpha^n$ provides with one solution the first equation. Substituting this to the second equation we have $X = \left(\sum_{n=0}^{\infty}\alpha^n\right)B$. Moreover if Y is such that the hypersum of every one of its elements with itself gives as a result only

this element and also the hypersum of every two of its elements contains them, or if $\{1, \alpha\}^{\sim}$ is a subhyperringoid of Y such that its multiplicative part is a subgroup of the multiplicative semigroup of Y, then (Prop. 2.3) the subhyperringoid $\{1, \alpha\}^{\sim}$ is also a solution of the system's first equation and so $\Psi = \{1, \alpha\}^{\sim}$ and $X = \{1, \alpha\}^{\sim}B$. Thus:

Proposition 3.1. *The existence of the sets* $X = \left[\sum_{n=0}^{\infty}\alpha^n\right]B$, *and* $\Psi = \sum_{n=0}^{\infty}\alpha^n$ *provides the system of the equations (S1) with a solution, which is also the minimum solution when the coefficients belong to the family* $\mathcal{O}$. *Moreover, if* Y *has the property that the hypersum of every one of its elements with itself gives as a result only this element and also the hypersum of every two of its elements contains the two addends, or if* $\{1, \alpha\}^{\sim}$ *is a subhyperringoid of* Y *such that its multiplicative part is a subgroup of the multiplicative semigroup of* Y, *then*

$$(X, \Psi) = \left(\{1, \alpha\}^{\sim}B, \{1, \alpha\}^{\sim}\right)$$

is also a solution of (S1).

Corollary 3.1. *In a B-hyperringoid the system*

$$\Psi = A\Psi + 1$$
$$X = \Psi B$$

has minimum solution the

$$(X, \Psi) = \left(\sum_{n=0}^{\infty}A^nB, \sum_{n=0}^{\infty}A^n\right)$$

Proposition 3.1. *In a B-hyperringoid Y there exists a solution for the system of the equations:*

$$X_1 = A_{11}X_1 + A_{12}X_2 + \ldots + A_{1n}X_n + B_1$$
$$X_2 = A_{21}X_1 + A_{22}X_2 + \ldots + A_{2n}X_n + B_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

.....

.....

$$X_n = A_{n1}X_1 + A_{n2}X_2 + \ldots + A_{nn}X_n + B_n$$

where the unknowns X_i, *the coefficients* A_{ij} *and the* B_i *for* $i, j = 1,\ldots,n$ *are subsets of* Y.

P r o o f. The proof of this proposition is achieved through the Corollary 2.1. Indeed, from the n equations, the last one can be considered as one equation with X_n the unknown set, its coefficient A the set A_{nn} and $B = A_{n1}X_1 + A_{n2}X_2 + \ldots + A_{n(n-1)}X_{n-1} + B_n$. The solution of this equation can be substituted to the rest n-1 equations and thus there will derive a new system with n-1 equations and n-1 unknown sets. Repeating the same procedure for each one of the rest unknown sets we can reach the solution of the initial system.

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