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A New Approach to the Theory of Languages and Automata

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Abstract

In this paper appears an approach to the theory of Languages and Automata with the use of tools and methods from the Theory of the Hypercompositional Structures. The set of the words V^* over an alphabet V, with the compositional laws from the Theory of Languages and the hypercompositional laws that are being introduced, becomes a new hypercomposional structure, the hyperringoid. Its action as a set of operators or hyperoperators on a given set is being presented along with the notions of the hypermoduloid and the supermoduloid. The study of this structure gives interesting results for both theories.

It is known that a language whose strings are written using letters from an alphabet V is defined as a subset of the free monoid V^*, which is genrated by V. It is also known that the languages that are accepted from finite automata are the regular sets over V. The regular sets are represented by regular expressions. The definition of the regular expresions over V, requires the introduction of the bisets $\{x, y\}$ from V^*. This leads to the definition of the following hypercomposition in V^*: $x + y = \{x, y\} \, V^*$, endowed with this hyperoperation becomes a join hypergroup [3]. The operation of V^*, i.e. the concatention of the words, is bilaterally distributive with regard to this hyperoperation. So a new hypercompositional structure, the Join Hyperringoid came into begin [3].

Definition 1. A hyperringoid is a non void set Y with an operation "." and a hyperoperation "+", that satisfy the axioms:
i. $(Y, +)$ is a hypergroup
ii. $(Y, .)$ is a semigroup
iii. The operation is bilaterally distributive to the hyperoperation.

If $(Y, +)$ is a join hypergroup, then the hyperringoid is called **Join hyperringoid**. The special join hypergroup which dervies in this way from the theory of languages was named B-hypergroup and the respective hyperringoid, B-hyperringoid.

Another important notion in the theory of languages, is the notion of the empty set of words. Actually the use of the "null word" as well as other reasons,

e.g. the necessity of the use of the symbol $< SOS >$ (Start Of String) in the electronical realization of the automaton, which acts as a an annihilator (since it leads from any state to the start state), has led to the introduction of a non scalar neutral element in the join hypergroup. Thus appeared the fortified join hypergroup [3]. This new hypercompositonal structure $(H, +)$ satisfies the axioms:

$FJ1$ There exists a unique neutral element, denoted by 0, such that $x \in x + 0$, for every $x \in H$ and $0+0 = 0$.

$FJ2$ For every $x \in H \backslash \{0\}$, there exists one and only one element$x' \in H \backslash \{0\}$ (denoted by $-x$), such that $0 \in x + x'$.

Especially for the case of the languages, the fortified join hypergroup, which corresponds to them and which motivated the development of these new structures is the dilated B-hypergroup. In this hypergroup every element is selfopposite. More precisely the hypercompositon of this structure is being defined in the following way: $x + y = \{x, y\}$, if $x \neq y$, and $x + x = \{0, x\}$.

The corresponding hyperringoid of all the words, the null word included, is called dilated B-hyperringoid. Generally, if the hypercompositional part of a hyperringoid is a fortified join hypergroup, then it is called fortified join hyperringoid or join hyperring [3].

The hyperringoids which derive from the set of the words V^*, have the property that every one of their elements has a unique factorization into irreducible elements (which are the letters of the alphabet).

Definition 2. A linguistic hyperringoid (resp. dilated linguistic hyperringoid) is a unitary B-hyperringoid (resp. dilated B-hyperringoid) which has a finite prime subset P and which is non commutative for card $P > 1$.

Notice that every B-hyperringoid is not a linguistic one. For example the set of the complex numbers, with the usual multiplication and hypercomposition $a + b = \{a, b\}$. An interesting linquistic hyperringoid appears in the Proposition:

Proposition 1. The set N of the natural numbers is a linguistic and hyperoperation with prime subset the singleton $\{1\}$ and with operation the $xy = x + y$

and hyperoperation the $x + y = \{x, y\}$, for every $x, y \in N$.

This linguistic hyperringoid is used in the counters [6].

Hypercompositional structures have not been introduced only in the set of the words. Hypergroup have also been used in the automata in order to describe their structure and operation. More precisely the set of the states of an automaton, endowed with properly defined hypercomposition becomes a hypergroup. Such hypergroups, that are called attached hypergroup to the automaton, are (a) the attached order hypergroup, (b) the attached grade hypergroup, (c) the attached hypergroup of the paths and (d) the attached hypergroup of the operation. Especially the attached order hypergroup and the attached grade hypergroup lead to the creation of the minimum automaton that accepts the same language with the initial one [2], [5].

Beyond the hypercompositonal structures that are being "attached" to the set of the words over an alphabet, or to the set of the states of an automaton, there have also been developed hypercompositional structures that derive from the operation of the words on the states of the automaton [4]. Let's see some defintions and next let's present a series of Theorem which, when applied to the special case of the B-hyperringoids they give the Theorem of Nerod-Myhill.

Definition 3. Let M be an arbitrary set and $(Y, +, .)$ a hyperringoid. Y is a set of operators over M, if there exists an external operation from $M \times Y$ to M, that satisfies the axiom:

$$(s\alpha)\beta = s(\alpha\beta) \quad s \in M \text{ and } \alpha, \beta \in Y$$

If there exists an external hyperoperation from $M \times Y$ to $P(M)$ satisfying the above axiom, then Y is a set of hyperoperators. In the first case we say that Y acts through an operation, while in the second one, through a hyperoperation.

Definition. 4. If M is a hypergroup and Y is a hyperringoid of operators over M such that
i. $(s + t)\lambda = s\lambda + t\lambda$,
ii. $s(\lambda + k) \subseteq s\lambda + sk$,
iii. $s(\lambda k) = (s\lambda)k$
for every $k, \lambda \in Y$ and $s, t \in M$, then M is called (right) hypermoduloid over Y. If Y is a set of hyperoperators, then M is called supermoduloid. If $(Y, +, .)$ is a fortified hyperringoid and $(M, +)$ a fortified join hypergroup, then M is called join hypermodule, resp. join supermodule, if in addition to the above, the following axiom holds:
iv. $s0 = 0$, resp $|iv\xi \quad 0 \in s0$.

Next let M be an arbitrary set with operators or hyperoperators from a hyperringoid Y and let s be an element of M. To every element α of Y we map the $s\alpha$.·If Y is a set of operators, then this mapping (that from now on will be denoted by φs) is a function from

Y to M, while if Y is a set of hyperoperators, then ϕs is a function from Y to $P(M)$, that is:

$$\varphi s : Y \to M \quad \text{such that} \quad \varphi s(\alpha) = s\alpha$$

or

$$\varphi s : Y \to P(M) \quad \text{such that} \quad \varphi s(\alpha) = s\alpha$$

So the definition:

Definition 5. A subset L of Y is called (s, F)-acceptable from M, or simply acceptable when there is no fear of a mix up, if there exists $s \in M$ and $F \subseteq M$ in the case of external operation, or $F \subseteq P(M)$ in the case of external hyperoperation, such that: $\varphi s^{-1}(F) = L$

In the following also appears an important, relation, the homomorphic one [1] Let H, H' be two hypergroupoids and let $R \subseteq H \times H'$. R will be called homomorphic if (I) For every $(a1, b1) \in R$ and $(a2, b2) \in R$ holds:

$$\forall x \in a1 + a2)(\exists y \in b1 + b2)[(x, y) \in R]$$

and

$$\forall y' \in b1 + b2)(\exists x' \in a1 + ab2)[(x, y) \in R]$$

Furthermore if Y and Y' are two hyperringoids, then $R(\subseteq Y \times Y')$ will be called homomorphic if appart from (I) the following is also valid:
(II) For every $(a1, b1) \in R$ and $(a2, b2) \in R$ holds: $(a1a2, b1b2) \in R$.
Next we observe that if an equivalence relation R over a hyperringoid Y satisfies the property:
(II') $(x, y) \in R$ and $w \in Y \longrightarrow (xw, yw) \in R$ and $(wx, wy) \in R$ then R satisfies (II) and conversely.

It is possible though that an equivalence relation satisfies only one of the two conditions of (II'). In this case it is called right (or resp. left) congruence relation as to the operation.

Proposition 2. If R is a homomorphic equivalence relation in a hyperringoid y as to the hyperoperation and a right congruence as to the multiplication, then the quotient set y/R becomes a (right) hypermoduloid over y.

Thus, according to whether $rk(R)$ is finite or not, the deriving hypermoduloid will respectively be finite or not. So if Y is a linguistic hyperringoid and if $rk(R) < +\infty$ then Y/R is a finite hypermoduloid. The elements of such a hypermoduloid can be considered as the states of an automaton, which is completely defined only when one of the classes of Y/R is defined as the start state and if a set of these classes is considered to be the set of the final states. Therefore it is possible, for the hypermoduloid deriving from certain equivalence relations, to lead to an automaton.

Theorem 1. Let L be a subset of a hyperringoid y. Then the relation RL defined from:

$$x RL y \quad \Leftrightarrow \quad (\forall a, b \in y)[xa \in L$$

$\Leftrightarrow \quad ya \in L(i)$ and $bx \in L$

$\Leftrightarrow \quad by \in L(ii)]$

is an equivalence relation in y, satisfying the property:

if $(a1, b1) \in RL$ and $(a2, b2) \in RL$, then
$$(a1a2, b1b2) \in RL.$$

If it satisfies only the (i) (symb. RL') or only the (ii) (symb. $'RL$) then it is a right, resp. a left congruence. Finally, if y is a B-hyperringoid, then RL is a homomorphic relation.

Proposition 3. If y is a B-hyperringoid, then the quotient y/RL is a B-hyperringoid.

Since the linguistic hyperringoid is a B-hyperringoid we have:

Corollary. 1. If L is a language of the linguistic hyperringoid, then the relation RL defined above is a homomorphic equivalence relation in it, while RL' and $'RL$ are a right and a left congruence relation as to the concatenation.
Moreover:

Theorem 2. If there exists a homomorphic equivalence relation R in y for which L is a union of classes, then $rk(RL) \leq rk(R)$ and therefore is $rk(R) < \infty$ then $rk(RL) < \infty$. Similar inequalities hold, for RL' and $'RL$ if R is a right or a left congruence as to the multiplication.

Corollary 2. If the language L consists of a union of classes of a right congruence equivalence relation R, for which $rk(R) < \infty$ holds, then we have $rh(RL') < \infty$. Respective results hold for the left congruence of the homomorphic equivalence relation.
We also have the Theorem:

Theorem 3. Let R be a homomorphic equivalence relation in y as to the hyperoperation and a right congruence as to the multiplication and let L be a subset of y which is a union of classes modulo R. Then there exists hypermoduloid M as to which L is acceptable, and $rh(RL') \leq rk(R)$.

Proposition 4. If L is a subset of a B-hyperringoid, then there exists a hypermoduloid M as to which L is acceptable.

Proposition 5. If $rk(RL') < \infty$, then L is acceptable from a finite hypermoduloid M.

Corollary 3. If for the language L we have $rk(RL') < \infty$, then there exists an automaton A which accepts L as its language.

Theorem 4. If L is an acceptable from M subset of y, then there exists an equivalence relation R in y as to which L is a union of classes. If M is finite, then $rk(R) < \infty$.

Corollary 4. If the language L is recognized by an automaton, then there exists an equivalence relation R as to which L is a union of classes and $rk(R) < \infty$.

From the above Theorems 1-4, Propositions 1-5 and Corollaries 1-4 derives the known in the theory of languages Theorem of Nerode [8]. Also Myhill [7] had come into similar conclusions with Nerode, sortly before him and so this Theorem is often refered to as the Theorem of Myhill-Nerode.

The Theorem of Myhill-Nerode. If L is a language over an alphabet V, then the following are equivalent:
(i) L is a language of an automaton.
(ii) there exists a right congruence equivalence relation R in V^* for which L is a union of certain equivalence classes and $rk(R) < \infty$
(iii) $rk(RL') < \infty$.

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