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AN AUTOMATON DURING ITS OPERATION

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(Dedicated to the memory of GEORGE GH. MASSOUROS)

Abstract

In this paper a hypergroup is being presented which describes an automaton during its operation. Using it and through a certain procedure which is being developed here, among others, all the states at which the automaton can possibly be found, at any given time t, are being determined.

1. Introduction.

The attached hypergroup of an automaton, a notion which derives from the attempt of expressing and solving problems of the Theory of Automata and Languages with use of the Theory of the Hypercompositional Structures, has been introduced in [2]. There two kinds of attached hypergroups appear, the *attached order hypergroups* which generally are several types of canonical hypergroups and the *attached grade hypergroup*, which is either join polysymmetrical hypergroup or fortified join hypergroup, when the automaton is minimized. Apart from other applications [3],[4], these attached hypergroups have been used for the minimization of the automaton. Moreover, in [3], another kind of attached hypergroup has been introduced, the *attached hypergroups of the paths*. But all these attached hypergroups deal with the set of states of an automaton and describe its structure. However the operation of an automaton involves the factor of time. Therefore in the following we shall search for a hypergroup which will somehow describe the automaton during its operation.

2. The attached hypergroup of the operational paths of an automaton.

An automaton is a mathematical model for a machine that accepts a particular set of words over some alphabet A [5]. Let's consider an automaton (A, S, s_0, δ, F), the one of figure 1 for instance:

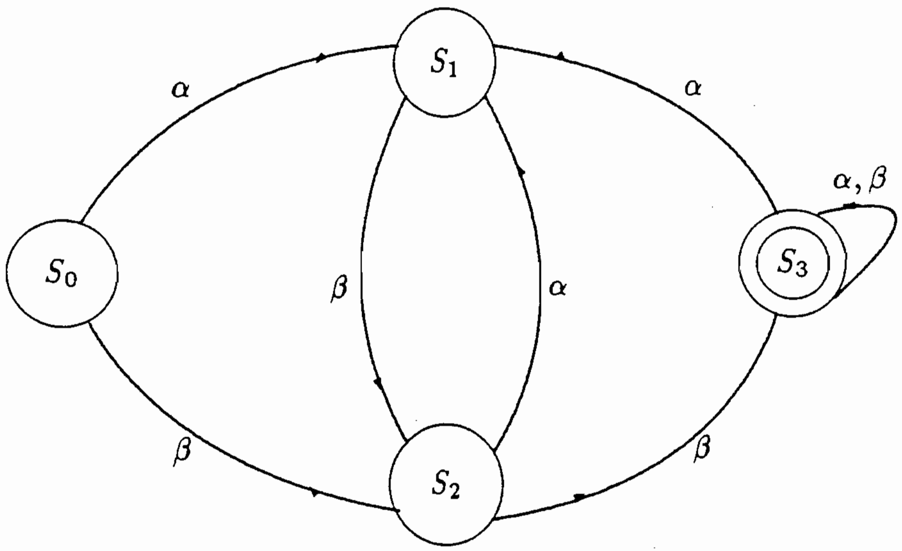

Figure 1.

Definition 2.1. We say that the state s_j is *successive* to s_i if there exists $\chi \in A$ such that $\delta(s_i, \chi) = s_j$.

Obviously the fact that s_j is successive to s_i does not imply that s_i is successive to s_j as well (without excluding it though). Thus the following diagram presents the successive states of the above automaton.

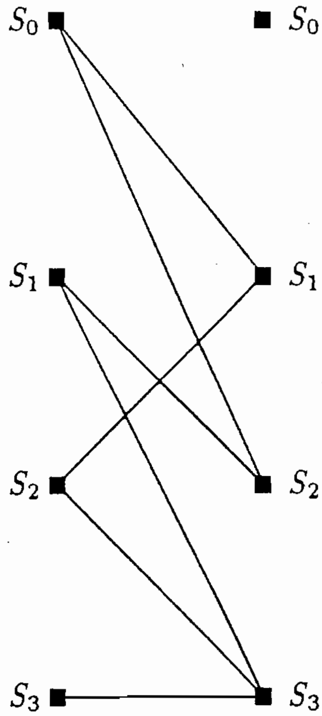

Figure 2.

Definition 2.2. We say that the state s_j is *connected with* s_i if there exists a word $\chi \in A^*$ such that $\delta^*(s_i, \chi) = s_j$.

The automaton always starts its operation from the start state s_0 and while reading one letter at every moment (clock pulse) it moves to the next successive state. It is possible though, during its operation, to pass from the same state at different moments. Thus for instance the automaton of figure 1 can be found on state s_1 with the word α, with the word $\beta\alpha$, with the word $\alpha\beta\alpha$, etc. This means that it can be found on state s_1 during the first moment of its operation, during the second moment, the third etc.

Therefore if we wish to describe the operation of an automaton, we can consider the cartesian product $S \times N$ where S the set of states of the automaton and N the set of the non negative integers. We will use the notation s_i^k to represent the ordered pair (s_i, k) which shows the state s_i at the moment k. The graph of these ordered pairs can be drawn in the same manner as the following one, which refers to the automaton of figure 1. In this graph, where the horizontal axis, which is the set N, represents the clock pulses, and the vertical axix represents the states, we have connected all the successive states up to the tenth clock pulse of the automaton's operation.

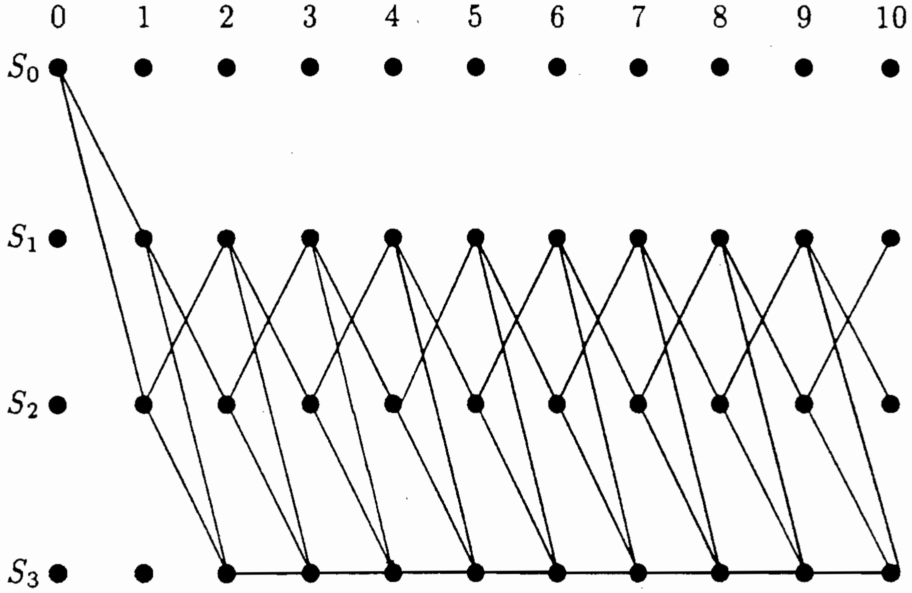

Figure 3.

From the graph we see that in fact we are interested in the states on which the automaton can possibly be found. Hence we introduce the definition:

Definition 2.3. An element s_i^t of the cartesian product $S \times N$ is called *activated*, if, after t clock pulses, the automaton can be found on the state s_i.

When we want to especially emphasize the moment of the activation, then we speak for the *t-activated* element.

Next let's denote with A the set of the activated elements. In this set we generalize the notions of the functions δ and δ^* in the following way:

$\delta_A(s_i^t, x) = s_j^{t+1}$, where $s_j = \delta(s_i, x)$ and
$\delta_A^*(s_i^t, w) = s_j^{t+|w|}$, where $s_j = \delta^*(s_i, w)$ and
($|w|$is the length of the word w(that is the number of w's letters [2]).

(Using simpler notations the above can be written as: $\delta_A(s_i^t, x) = \delta(s_i, x)^{t+1}$ and $\delta_A^*(s_i^t, x) = \delta^*(s_i, x)^{t+|x|}$.)

Definition 2.4. The elements s_j^r and s_i^t will be called *successive* if the state s_j is successive to the state s_i and $r = t + 1$. Also the element s_j^r is called *connected* with the s_i^t if the state s_j is connected with the state s_i and $t < r$.

Thus it is obvious that in the case of a deterministic automaton two t-activated elements cannot be connected.

In the set of the activated elements we observe that two elements can be:

(a) not connected (e.g. the s_2^2 and s_2^1 of fig. 3).

(b) connected one with the other by one word (e.g. the s_3^3 and s_1^1 of fig. 3).

(c) connected one with the other by more than one word (e.g. the s_3^4 and s_0^0 of fig. 3).

These remarks lead to the introduction of a hypercomposition "+" on the set A of the activated elements of the cartesian product $S \times N$:

$$s_i^m + s_j^n = \begin{cases} \{\delta_A^*(s_i^m, x) & \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n\}, \text{ if } s_j^n \\ & \text{is connected with } s_i^m, \\ \{s_i^m, s_j^n\}, & \text{if } s_j^n \text{ is not connected with } s_i^m. \end{cases}$$

This hypercomposition is associative. Indeed let the element s_i^m be connected with the s_j^n, s_k^p and let the s_j^n be connected with s_k^p. Also let $m<n<p$. Then

$$\begin{aligned} (s_i^m + s_j^n) + s_k^p &= \{\delta_A^*(s_i^m, x) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n\} + s_k^p = \\ &= \{\delta_A^*(\delta_A^*(s_i^m, x), y) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n, \\ &\qquad y \in \text{Prefix}(q),\ \delta_A^*(\delta_A^*(s_i^m, x), q) = s_k^p\} = \\ &= \{\delta_A^*(s_i^m, v) \mid v \in \text{Prefix}(w),\ \delta_A^*(s_i^m, w) = s_k^p\}. \end{aligned}$$

Moreover,

$$\begin{aligned} s_i^m + (s_j^n + s_k^p) &= s_i^m + \{\delta_A^*(s_j^n, x) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_j^n, r) = s_k^p\} = \\ &= \{\delta_A^*(s_i^m, z) \mid z \in \text{Prefix}(u),\ \delta_A^*(s_i^m, u) = s_k^p \text{ or} \\ &\quad \delta_A^*(s_i^m, u) = \delta_A^*(s_j^n, x), x \in \text{Prefix}(r), \delta_A^*(s_j^n, r) = s_k^p\} = \\ &= \{\delta_A^*(s_i^m, v) \mid v \in \text{Prefix}(w),\ \delta_A^*(s_i^m, w) = s_k^p\}. \end{aligned}$$

Next let the elements s_i^m, s_j^n and s_i^m, s_k^p be connected one with the other, while s_j^n, s_k^p are not connected. Then

$$\begin{aligned}(s_i^m + s_j^n) + s_k^p &= \{\delta_A^*(s_i^m, x) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n\} + s_k^p = \\ &= (s_i^m + s_j^n) + (s_i^m + s_k^p).\end{aligned}$$

Moreover $(s_i^m + s_j^n) + s_k^p = s_i^m + \{s_j^n, s_k^p\} = (s_i^m + s_j^n) + (s_i^m + s_k^p)$.

Now if the elements s_i^m, s_j^n are connected, while the s_k^p is not connected with none of the other two, then

$$(s_i^m + s_j^n) + s_k^p = \{\delta_A^*(s_i^m, x) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n)\} + s_k^p.$$

But the element s_k^p is not connected with none of the $\delta_A^*(s_i^m, x)$ since if it were, then s_i^m would have been connected with s_k^p, which is absurd. Thus we have:

$$(s_i^m + s_j^n) + s_k^p = \{\delta_A^*(s_j^n, x) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n\} \cup \{s_k^p\}.$$

Moreover

$$\begin{aligned}s_i^m + (s_j^n + s_k^p) &= s_i^m + \{s_j^n, s_k^p\} = (s_i^m + s_j^n) + (s_i^m + s_k^p) = \\ &= \{\delta_A^*(s_i^m, x) \mid x \in \text{Prefix}(r),\ \delta_A^*(s_i^m, r) = s_j^n\} \cup \{s_k^p\}.\end{aligned}$$

Finally if the elements s_i^m, s_j^n, s_k^p are not connected, then

$$\begin{aligned}(s_i^m + s_j^n) + s_k^p &= \{s_i^m, s_j^n\} + s_k^p = (s_i^m + s_k^p) \cup (s_j^n + s_k^p) = \\ &= \{s_i^m, s_j^n, s_k^p\}.\end{aligned}$$

Similarly, $s_i^m + (s_j^n + s_k^p) = \{s_i^m, s_j^n, s_k^p\}$. For every element s_i^m from A, we have $s_i^m + A = A + s_i^m = A$ and so we have the proposition:

Proposition 2.1. *The set A of the activated elements of the cartesian product $S \times N$, where S is the set of the states of an automaton, becomes a hypergroup if we introduce in it the above defined hypercomposition.*

This hypercomposition is not commutative and therefore we have the following definitions of the two induced hypercompositions ":" and "..", [1]

$$s_i^m : s_j^n = \begin{cases} \{s_k^p & \mid x \in A^* \text{ such that } \delta_A^*(s_k^p, x) = s_i^m\}, \\ & \text{if } m < n \text{ and } s_i^m, s_j^n \text{ are connected,} \\ \{s_i^m\}, & \text{if } s_i^m, s_j^n \text{ are not connected;} \end{cases}$$

$$s_i^m \cdot\cdot s_j^n = \begin{cases} \{s_k^p & \mid x \in A^* \text{ such that } \delta_A^*(s_j^n, x) = s_k^p\}, \\ & \text{if } m < n \text{ and } s_i^m, s_j^n \text{ are connected}, \\ \{s_i^m\}, & \text{if } s_i^m, s_j^n \text{ are not connected.} \end{cases}$$

Hence the hypergroup $(A, +)$ does not satisfy the join axiom.

3. A method for calculating the results of the hypercomposition and giving information for the operation of the automaton.

In the last part of this work we present a method with which we can find the results of the hypercomposition defined above, that is we can see if s_i^t is a t-activated element and at the same time we can determine all the intermediate states that the automaton passes from until it reaches the s_i^t. The procedure is the following:

Initially the "matrix of the possible states" $\Pi\delta$ is defined as an $n \times n$-matrix with rows and columns the n states of the automaton. Every element $\Pi\delta ij$ of this matrix is calculated according to the rule:

$$\Pi\delta ij = \begin{cases} \text{the state } s_j \text{, if it is possible for the automaton to reach this state} \\ \text{during the next clock pulse, starting from state } s_i, \\ \\ 0 \text{ in any other case.} \end{cases}$$

Next the "matrix of the first clock pulse" $\Pi[1]$ is defined as an $n \times n$-matrix with its columns the elements s_i^1, $i = 1, 2, \ldots, n$ and rows the elements s_i^0, $i = 1, 2, \ldots, n$, that is the elements of the previous clock pulse. Every entry $\Pi 1 ij$ is being defined from the rule:

$$\Pi 1 ij = \begin{cases} s_0^0 s_j^1, \text{ if the automaton can reach state } s_j \text{ at the first clock pulse,} \\ \\ 0 \text{ in any other case.} \end{cases}$$

In matrix $\Pi[1]$ the row corresponding to the element s_0^0 is expected to have entries different from zero, since at this phase the automaton is obviously starting from the start state s_0.

Finally the matrix of every subsequent clock pulse derives from the matrix of the previous clock pulse with the use of $\Pi\delta$. Thus for the i row of the "matrix of the t clock pulse" $\Pi[t]$, [which has n columns (the elements s_i^t, $i = 1, 2, \ldots, n$) and n rows (the elements s_i^{t-1}, $i = 1, 1, \ldots, n$)] we have

- It comes from the i column of the matrix $\Pi[t-1]$ and the i row of $\Pi\delta$.

- It has zero everywhere if the entire i row of $\Pi\delta$ or the entire i column of $\Pi[t-1]$ is zero, otherwise:
- It has zero at all the places where the i row of $\Pi\delta$ has zero and
- In every one of the rest places we write the set of the paths that derives if at the end of each path which appears in the entire column of the matrix $\Pi[t-1]$, (regardless of the row) we add the element that comes from the state which is written at the respective place of the matrix $\Pi\delta$, at the clock pulse t.

As it seems, from the procedure of its construction, at every position of $\Pi[t]$ there may be more than one entries (paths). Also the position at which every set of the paths that derives as described above will be written is determined from the respective position of the state of $\Pi\delta$, which will consist the last element of the path.

Additionally we remark that:

1. The corresponding element s_i^t to the column of matrix $\Pi[t]$ with zero entries everywhere is a non activated element.
2. The corresponding element s_j^t to the row of matrix $\Pi[t]$ with zero entries everywhere is a non activated element.
3. The corresponding to the zero entry elements of matrix $\Pi[t]$ are not connected elements.

Applying the above to the automaton we have described in fig. 1 we have:

$\Pi\delta =$

	s_0	s_1	s_2	s_3
s_0	0	s_1	s_2	0
s_1	0	0	s_2	s_3
s_2	0	s_1	0	s_3
s_3	0	0	0	s_3

Also the first 4 matrices are:

$\Pi[1] =$

	s_0^1	s_1^1	s_2^1	s_3^1
s_0^0	0	$s_0^0 s_1^1$	$s_0^0 s_2^1$	0
s_1^0	0	0	0	0
s_2^0	0	0	0	0
s_3^0	0	0	0	0

$\Pi[2] =$

	s_0^2	s_1^2	s_2^2	s_3^2
s_0^1	0	0	0	0
s_1^1	0	0	$s_0^0 s_1^1 s_2^2$	$s_0^0 s_1^1 s_3^2$
s_2^1	0	$s_0^0 s_2^1 s_1^2$	0	$s_0^0 s_2^1 s_3^2$
s_3^1	0	0	0	0

$\Pi[3] =$

	s_0^3	s_1^3	s_2^3	s_3^3
s_0^2	0	0	0	0
s_1^2	0	0	$s_0^0 s_2^1 s_1^2 s_2^3$	$s_0^0 s_2^1 s_1^2 s_3^3$
s_2^2	0	$s_0^0 s_1^1 s_2^2 s_1^3$	0	$s_0^0 s_1^1 s_2^2 s_3^3$
s_3^2	0	0	0	$s_0^0 s_1^1 s_3^2 s_3^3$ $s_0^0 s_2^1 s_3^2 s_3^3$

$\Pi[4] =$

	s_0^4	s_1^4	s_2^4	s_3^4
s_0^3	0	0	0	0
s_1^3	0	0	$s_0^0 s_1^1 s_2^2 s_1^3 s_2^4$	$s_0^0 s_1^1 s_2^2 s_1^3 s_3^4$
s_2^3	0	$s_0^0 s_2^1 s_1^2 s_2^3 s_1^4$	0	$s_0^0 s_2^1 s_1^2 s_2^3 s_3^4$
s_3^3	0	0	0	$s_0^0 s_2^1 s_1^2 s_3^3 s_3^4$ $s_0^0 s_1^1 s_2^2 s_3^3 s_3^4$ $s_0^0 s_1^1 s_3^2 s_3^3 s_3^4$ $s_0^0 s_2^1 s_3^2 s_3^3 s_3^4$

Interpreting the results of matrix $\Pi[4]$ for instance, we see that, at the fourth moment of its operation, the automaton of the example can be found:

- on state s_1 having followed the path $s_0 s_2 s_1 s_2 s_1$,
- on state s_2 having followed the path $s_0 s_1 s_2 s_1 s_2$,
- on state s_3 having followed one of the paths: $s_0 s_1 s_2 s_1 s_3$, or $s_0 s_2 s_1 s_2 s_3$, or $s_0 s_2 s_1 s_3 s_3$, or $s_0 s_1 s_2 s_3 s_3$, or $s_0 s_1 s_3 s_3 s_3$, or $s_0 s_2 s_3 s_3 s_3$.

So the 4-activated elements form the set $\{s_1^4, s_2^4, s_3^4\}$, and the result of the hypercomposition, say $s_0^0 + s_1^4$, is: $s_0^0 + s_1^4 = \{s_0^0, s_2^1, s_1^2, s_2^3, s_1^4\}$, while $s_1^2 + s_1^4 =$ $= \{s_1^2, s_2^3, s_1^4\}$.

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