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LANGUAGES - AUTOMATA AND HYPERCOMPOSITIONAL STRUCTURES

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ABSTRACT

In this paper, an association is begun between the theory of Languages and Automata theory on the one hand and the Hypercompositional Structures on the other. For this purpose, not only Hypercompositional Structures are used but certain other new ones are introduced, such as the hyperringoid, the fortified join and the fortified join polysymmetrical hypergroup. Moreover the notion of the order of an automaton's state is introduced along with several equivalence relations in the set S of the states of an automaton. Finally certain hypercompositional structures that are defined in S allow us to introduce in it the notions of valuation and hypervaluation.

1. INTRODUCTION

This is an introductory paper in which an association is begun between the theory of automata and languages on one hand and on the other, hypercompositional structures, that are defined through them. Moreover relevant applications are presented.

If A is an alphabet, A* the set of words that are defined from A - closure of A - and λ the empty word, then the set A* under the operation of concatenation of words i.e. $x.y = xy$ (1), is a monoid, with neutral element λ, since for every $x \in A^*$, by definition, $\lambda a = a\lambda = a$ (2)

Length $l(x)$ of a word $x \in A^*$ is the number of letters in x [thus $l(\lambda) = 0$] and so $l(xy) = l(x) + l(y)$ for every $x,y \in A^*$.

The definition of regular expressions (which are fundamental in the theory of languages) over an alphabet A, presumes the introduction of the sets $\{x,y\}$ from A*. This suggests the definition of a hypercomposition in A*: $(x + y) = \{x,y\}$ (3).

An **Automaton**, as is well known, is a collection of five objects things (S, A, S_o, F, t), where S is a finite set of states, A is an alphabet of input letters, S_o and F are the sets of starting and final states, respectively, and t is a transition function with domain $S \times A$ and range S, if the automaton is d e t e r m i n i s t i c , otherwise range $P(S)$, if it is n o n d e t e r m i n i s t i c. [1], [4], [13].

2. LANGUAGES AND HYPERCOMPOSITIONAL STRUCTURES

From the above we can see that if we give A^* the composition (1) and the hypercomposition (3), then we get the hypercompositional structures $(A^*,+)$ and $(A^*,+,.)$. It is almost obvious that the first structure is a commutative hypergroup. Moreover we observe that the induced hypercomposition $x : y$ [6], [7] is

$$x : y = \{ \omega \in A^* \mid x \in \omega + y \} = \begin{cases} x, & \text{if } x \neq y \\ A^*, & \text{if } x = y \end{cases}$$

Also, considering all the possible cases, it can be proved that for $a,b,c,d \in A^*$ we have:

$$(a : b) \cap (c : d) \neq \emptyset \implies (a + d) \cap (b + c) \neq \emptyset$$

So we have the Proposition:

Proposition 2.1. The structure $(A^*,+)$ is a join hypergroup [3], [7]

The hypergroup $(A^*,+)$, as well as every hypergroup $(H,+)$ of this type, has the following properties

PROPERTIES (2.1)

i. It lacks scalar elements (i.e. elements $s \in H$ such that for every $x \in H$ the sum $s + x$ is a singleton)

ii. Each one of its elements x is neutral and absorbing (with broad meaning), that is for every $y \in H$, $y \in x+y$ and $x \in x+y$

iii. Each one of its subsets is a sub-hypergroup of it.
Indeed, let $B \subseteq H$ and $x \in B$. Then, $x + B = \bigcup_{a \in B}(x+a) = \bigcup_{a \in B}\{x,a\} = B$

iv. None of its sub-hypergroups is closed [5]. Because if B is a sub-hypergroup of H and x an element which does not belong to B, then $(x + B) \cap B = (\{x\} \cup B) \cap B = B \neq \emptyset$
Also it can be proved that

v. It doesn't have inversible [5] subhypergroups.

REMARK 2.1.

The join hypergroup $(H,+)$ with $x + y = \{x, y\}$ has, by definition, the above property ii. Yet there exist join hypergroups $(H,+)$ with elements $x \in H$ such that $x \notin x + y$ for every $y \in H$, with $x \neq y$, $y \neq 0$ (in the case that there exists 0 in H), as it is shown in the example:

EXAMPLE 2.1.

If H is a totally ordered and dense set, then H with hypercomposition:

$$x+y = \begin{cases} x, & \text{for } x = y \\ ]\min\{x,y\}, \max\{x,y\}[, & \text{for } x \neq y \end{cases}$$

is a join hypergroup, which obviously satisfies the above property.

Now, for the hypercompositional structure $(A^*,+,.)$, we observe that the multiplication (concatenation) is distributive with regard to the addition. This property, together with the properties of the structures $(A^*,+)$, $(A^*,.)$ leads to a definition of a new hypercompositional structure, which is more general than that of a hyperring.

Definition 2.1. A non void set H under a composition "." and a hypercomposition "+" such that the following conditions are satisfied:

i) (H,+) is a join hypergroup

ii) (H,.) is a semigroup

iii) the composition is distributive with regard to the hypercomposition

is a **hyperringoid.**

So we have

Proposition 2.2. The hypercompositional structure $(A^*,+,.)$ is a unitary hyper-ringoid [because of (2)].

The hyperringoid $(A^*,+,.)$, as well as the hypergroup $(A^*,+)$ are called **attached** hyperstructures to the alphabet A. The form of the attached hyperringoid of A leads to the following more general consideration:

Proposition 2.3. Let (D,.) be an arbitrary semigroup. Then D under the hypercomposition (3) becomes a hypercompositional structure (D,+,.) which is a hyperringoid.

REMARK 2.2

It is obvious that if the additive hypergroup of a hyperringoid (H,+,.) has a scalar neutral element 0 (consequently unique [7]) and if 0 also is bilaterally absorbing element for the multiplication (i.e. $0x = x0 = 0$ for every $x \in H$) then the hyperringoid is a hyperring.

Now given a hyperringoid $(A^*,+,.)$, the notion of the length of a word leads to the consideration of the binary relation L in A^* such that:

$$x \; L \; y, \quad \text{when } l(x) = l(y)$$

which is obviously an equivalence relation, called length equivalence. If we denote by C_x the class which contains the word x, we have that $C_x+C_y=\bigcup_{x'\in C_x, y'\in C_y}(x'+y')=C_x \cup C_y=\bigcup_{z\in x'+y'}C_z$ i.e. a union of classes C_z with $z \in x'+y'$ for arbitrary $x' \in C_x$ and $y' \in C_y$. So the equivalence relation L is normal, therefore the quotient set A*/L under the hypercomposition

$$C_x + C_y = \{C_x, C_y\} \qquad (4)$$

is a hypergroup. Also the set A* under the hypercomposition

$$x + y = C_x \cup C_y \qquad (5)$$

is a hypergroup as well. On the other hand as for the multiplication, taking into consideration that

$$l(xy) = l(x)+l(y),\ l(xy) = l(x'y') \text{ for every } x' \in C_x,\ y' \in C_y,$$

we have that

$$C_x . C_y = \{ x'y';\ x' \in C_x,\ y' \in C_y \} = C_{xy} \qquad (6)$$

which means that the equivalence relation is normal with regard to the multiplication. Proceeding to the verification of the other axioms we get the propositions:

Proposition 2.4. The set A*/L of the classes mod L under the hypercomposition (4) is a join hypergroup which satisfies the properties 2.1.

Proposition 2.5. The hypercompositional structure (A*/L,+,.) under the hypercomposition (4) and composition (6) becomes a unitary hyperringoid with multiplicatively neutral element the class $C_\lambda = \{ \lambda \}$.

Proposition 2.6. The set A* under the hypercomposition (5) is a join hypergroup.

Proposition 2.7. The hypercompositional structure (A*,+,.) under the hypercomposition (5) and composition the concatination, becomes a unitary hyperringoid with multiplicatively neutral element λ.

With regard to the Remark (2.2) we observe that it is possible for the hyperringoid to contain a nonscalar additively neutral element which is also absorbing with regard to the multiplication. Such an example results from proposition (2.3), if the semigroup (D,.) has a bilaterally absorbing element, denoted by 0, which obviously is a nonscalar neutral element for the addition as well (because $0 + x = \{0, x\}$, for every $x \in D$). But in this case, the existence of a nonzero opposite element x' is possible for every nonzero x in D, provided that the hypercomposition $x + y$ in D is altered in the following way:

$$x + y = \{x, y\} \text{ if } x \neq y \quad \text{and} \quad x + x = \{0, x\} \qquad (7)$$

But in this case, obviously, every element $x \in D$ is self-opposite and so for $x' = x$ it follows that:

$z \in x+y \Longrightarrow$ either $y \in z+x'$ or $x \in z+y'$, for every $x,y,z \in D$
More generally, if R is an equivalence relation in D for which $C_0 = \{0\}$, and if we define the hypercomposition in D as follows:

$$x + y = C_x \cup C_y \text{ if } C_x \neq C_y \text{ and } x + y = C_0 \cup C_x \text{ if } C_x = C_y \quad (8)$$

then every element $x \in D$ has many symmetricals. Also it can be verified that with the above altered hypercompositions, each of the hypercompositional structures (D,+), (D,+) remains a join hypergroup, and moreover is **partially** r e v e r s i b l e [2], which leads to the introduction of the following notions:

Definition 2.2. If a join hypergroup (H,+) satisfies the axioms:

1) There exists a unique neutral element in H denoted by 0 - the **zero** of H - for which every nonzero element x in H has one and only one nonzero **inversible** - **opposite** or **symmetrical** in H - denoted by $-x$,
2) The hypergroup (H,+) is **partially reversible**, i.e.
$\omega \in x+y \Longrightarrow$ either $y \in \omega-x$ or $x \in \omega-y$, for every $x,y,\omega \in H$

then it is called **fortified join hypergroup.**

Definition 2.3. If a join hypergroup (H,+) satisfies the axioms:

1) There exists a unique neutral element in H denoted by 0 - the **zero** of H - for which every nonzero element x in H has at least one nonzero **inversible** - **opposite** or **symmetrical** - in H denoted by S(x)
2) The hypergroup (H,+) is **partially reversible**, i.e it holds:
$\omega \in x+y \Longrightarrow (\exists x' \in S(x))\,[\,y \in \omega+x'\,]$ or
$(\exists y' \in S(y))\,[\,x \in \omega+y'\,]$

then it is called **fortified polysymmetrical join hypergroup.**

REMARK 2.2
This neutral $0 \in H$ is generally nonscalar. If it is scalar, then obviously the join hypergroup (H,+) becomes a canonical one, or a canonical polysymmetrical one respectively [7], [11].
REMARK 2.3
If in a fortified polysymmetrical join hypergroup H the sets S(x), $x \in H$ form a partition of H, then the set $H/R = \{ S(x) \mid x \in H \}$ under the hypercomposition:

$$S(x) + S(y) = \{ S(w) \mid w \in S(x) \cup S(y) \}$$

becomes a fortified join hypergroup.

Definition 2.4. A hyperringoid (H, +, .) is called **fortified** if its additive join hypergroup is fortified and its zero element is a bilaterally absorbing element for the multplication.

It results that if we adjoin to the set A^* an element 0, considering it, in a way, as a **zero word**, with the properties:

$$0x = x0 = 0, \quad 0 + x = \{0, x\}, \quad x + x = \{0, x\} \tag{9}$$

for every $x \in A = A^* \cup \{0\}$, we have that:

Proposition 2.8. The hypercompositional structure $(A,+,.)$ is a fortified unitary hyperringoid.

In the set A the **order** (ord x) of a word x, $x \neq 0$ is defined to be its length plus 1, that is

$$\text{ord } x = l(x) + 1, \quad \text{for } x \neq 0, \quad \text{and} \quad \text{ord } 0 = 0$$

Then the binary relation O such that: $x \equiv y \ (O)$, if ord x = ord y is obviously an equivalence relation in A that is called **order equivalence** and the restriction of it to A^* coincides, as it is obvious, with the length equivalence in it. It can be easily shown that relation O is normal for both the hypercomposition and the composition in A and thus we have the Proposition:

Proposition 2.9. The hypercompositional structure $(A/O,+,.)$ with hypercomposition and composition analogous to (4) and (6) is a fortified unitary hyperringoid.

REMARK 2.4.

The sets of classes mod O in A and mod L in A^*, obviously, are each isomorphic to N and so each is totally ordered. Therefore it is possible for more hypercompositional structures, to be defined as the next example shows (also see corresponding cases in paragraph 3).

With regard to the fortified join hypergroups and hyperringoids we have the folowing example, (apart from the theory of languages and with non self-opposite elements).

EXAMPLE 2.2

Every totally ordered set $(H,\leq)$ symmetrical to a center $0 \in H$ gives rise to a partition

$$H = H^- \cup \{0\} \cup H^+$$

($H^-=\{x\in H \mid x<0\}$, $H^+=\{x\in H \mid x>0\}$) such that $x < 0 < y$ for every $x \in H^-$, $y \in H^+$, $x \leq y \Longrightarrow -y \leq -x$ for every $x, y \in H$, where $-x$ is the symmetrical of x with regard to 0. Particularly every symmetrical subset of an abelian totally ordered group, becomes a fortified join hypergroup, under the hypercomposition:

$$x + y = \{x, y\} \quad \text{if} \quad |x| \neq |y|$$

$$x + x = x, \quad x - x = [-|x|, |x|]$$

where $|x| = x$, if $x \in H^+$, $-x$, if $x \in H^-$ and 0, if $x = 0$

In the special case that the set H is an ordered field e.g. the set Q, then, under the above hypercomposition $x + y$ and composition the multiplication in H the structure $(H,+,.)$ is fortified hyperringoid and more precicely a **fortified hyperfieldoid** with the obvious meaning of the term.

Detailed study of these structures, both independently as well as in relation to the theory of languages is already being carried out by G. Massouros and it is the subject of other forthcoming articles of his.

3. AUTOMATA AND HYPERCOMPOSITIONAL STRUCTURES

Let (S, A, S_o, F, t) be an automaton, (deterministic or not). In order to define the **order** of a state we suppose that there exists a **conventional** start state q_o', which is connected to every start state $q_o \in S$ (even if there exists only one such state) with the empty word λ.

Definition 3.1. We define the order of a state $q \in S$, denoted with ord q, to be the natural number $l+1$, where l is the minimum length of the word which leads from the conventional start state q_o' to q.

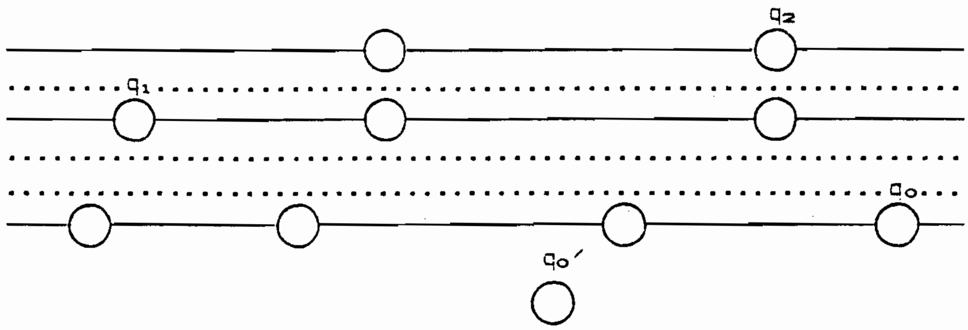
By definition ord $q_o' = 0$.

Using this notion we define in the set of the states S the following **order equivalence**. For every $q_1, q_2 \in S$, $q_1 O q_2$ if ord q_1 = ord q_2

Yet we observe that, as in the corresponding case of the sets A^* and A, the set of the classes is isomorphic to a subset of N (with possible extention of the definition of the automaton for infinite number of states, to N itself). Therefore it is possible to make S a hypercompositional structure, having in mind analogous cases of such structures starting from totally ordered sets, e.g. using the, by definition, commutative hypercomposition:

$$q_1 + q_2 = \begin{cases} q_2, & \text{if ord } q_1 < \text{ord } q_2 \\ \bigcup_{\text{ord } q \leq \text{ord } q_1} C_q, & \text{if ord } q_1 = \text{ord } q_2 \end{cases}$$

which makes it a polysymmetrical canonical hypergroup [11] with neutral element q_o' and with opposites of an $q \in S$ the states of the class C_q (and so q itself).



From the several cases of hypercompositional structures of S we mention the following three ones, where the structure (S,+) with hypercomposition, by definition, commutative, is canonical hypergroup (with self opposite elements) [8].

1st

$$q_1 \overset{\bullet}{+} q_2 = \begin{cases} q_2, & \text{if ord } q_1 < \text{ord } q_2 \\ \bigcup_{\text{ord } q < \text{ord } q_1} C_q, & \text{if ord } q_1 = \text{ord } q_2 \ (\neq 0) \end{cases}$$

and $q_0' + q_0' = q_0'$.

(This results from the above if in the definition of the hypercomposition and for the case ord q_1 = ord q_2 we replace the $\leq$ with $<$). It can be proved that the set S itself is, in this case, totally ordered and the hypergroup (S,+) is s u p e r i o r l y c a n o n i c a l [10].

2nd

$$q_1 + q_2 = \begin{cases} q_2, & \text{if ord } q_1 < \text{ord } q_2 \\ \bigcup_{\text{ord } q < \text{ord } q_1} C_q, & \text{when ord } q_1 = \text{ord } q_2 \text{ and } q_0' \neq q_1 \neq q_2 \neq q_0' \\ \bigcup_{\text{ord } q \leq \text{ord } q_1} C_q, & \text{when } q_1 = q_2 \end{cases}$$

With this hypercomposition (S,+) becomes a s t r o n g l y c a n o n i c a l h y p e r g r o u p [10].

3rd

$$q_1 + q_2 = \begin{cases} C_{q_2}, & \text{if } 0 \neq \text{ord } q_1 < \text{ord } q_2 \\ \bigcup_{0 \neq \text{ord } q \leq \text{ord } q_1} C_q, & \text{when ord } q_1 = \text{ord } q_2 \text{ and } q_0' \neq q_1 \neq q_2 \neq q_0' \\ \bigcup_{\text{ord } q \leq \text{ord } q_1} C_q & \text{when } q_0' \neq q_1 = q_2 \end{cases}$$

and $q_0' + q = q$ for every $q \in S$.

The structure (S,+) is now a canonical hypergroup [8].

It results that the canonical hypergroup is in the first case valuated and in the second case feebly valuated with valuation $|q|$ equal to the order of the state q in both cases:

$$|q| = \text{ord } q, \text{ for every } q \in S$$

The hyperdistance d : S x S -----> R_+ is $d(q_1,q_2) = \max\{|q_1|,|q_2|\}$, if $q_1 \neq q_2$ and $d(q, q) = 0$ and the sums $q_1 + q_2$ are circles of the hypermetrical space (S,d), i.e. $q_1 + q_2 = C(q, p\max\{|q_1|, |q_2|\})$ with an arbitrary $q \in q_1 + q_2$ and the "proportionality coeficient" p is a semireal number $p = 1^-$, for every $q_1, q_2 \in S$ for the first case, and $p = 1^-$, for $|q_1| \neq |q_2|$ and $p = 1$ for $|q_1| = |q_2|$ in the third case

[10].

Finally we introduce the following new notion, the g r a d e (grad q, $q \in S$) of a deterministic automaton's (S, A, S_o, F, t) state q:

Definition 3.2. We call grade of a state $q \in S$ and we denote it with grad q the set

$$\text{grad } q = \{ x \in A^* \mid xq \in F \} \quad (10)$$

where xq is the value $t^*(q,x)$ of the extended state transition function $t^* : A^* \times S \dashrightarrow S$ which is defined recursively as follows:

$(\forall q \in S)\ (\forall a \in A) \qquad t^*(a,q) = t(a,q)$

$(\forall q \in S) \qquad t^*(\lambda.q) = q$

$(\forall q \in S)\ (\forall x \in A^*)\ (\forall a \in A) \qquad t^*(ax,q) = t^*(t(a,q),x)$

($xq \in A^*$ in (10) obviously denotes a path in the corresponding pictorial representation which leads from state q to the final state, or to the conventional final one).

Now in the set of states S of an automaton we define the relation R as follows:

$$q_1 R q_2 \quad \text{when} \quad \text{grad } q_1 = \text{grad } q_2$$

Obviously this relation is an equivalence relation in S, called **grade equivalence.** Using the classes of R, S becomes a join hypergroup, if we consider in it the hypercomposition:

$$q_1 + q_2 = C_{q_1}{}^R \cup C_{q_2}{}^R$$

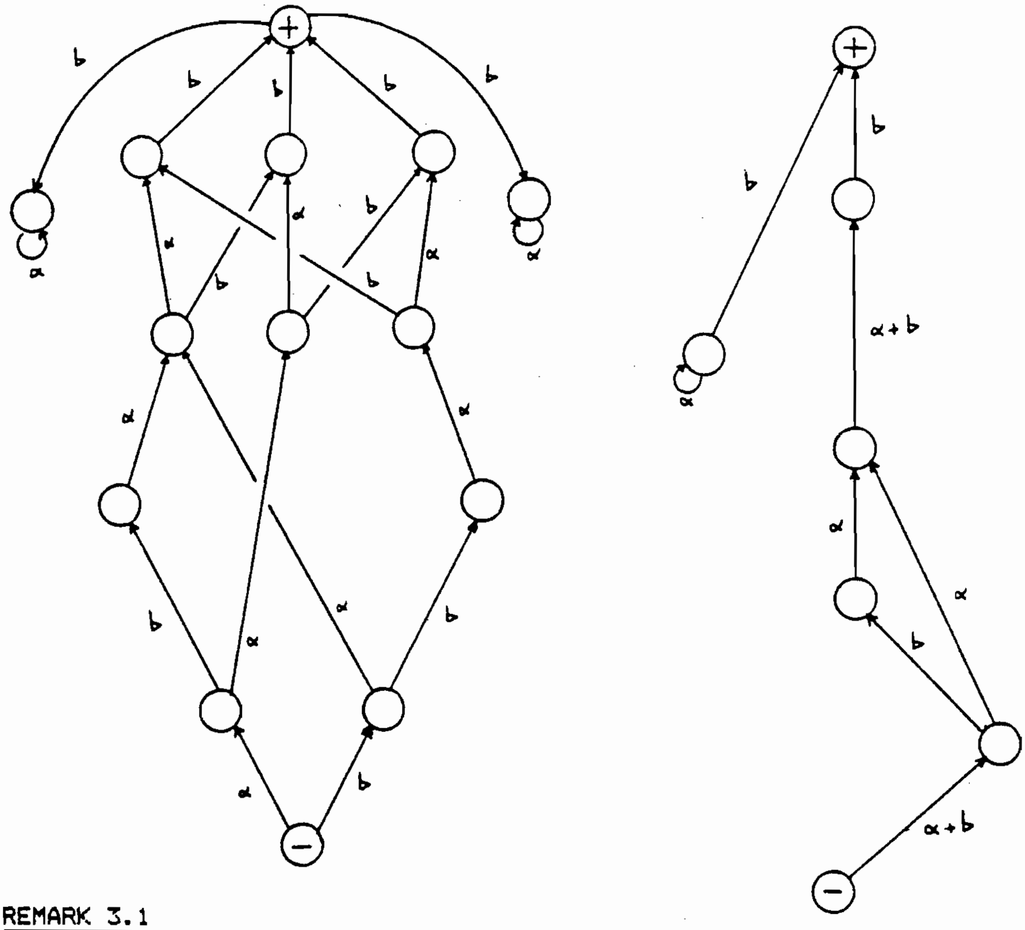
Thus the structure (S,+), according to what has been mentioned in paragraph 2, is a join hypergroup. Now let us suppose that the automaton (S,A,S_o,F,t) we are talking about, has only one final state, the state q_T (otherwise we endow it with a conventional one). Next we define in S the hypercomposition "+" in the following way:

$$q_1 + q_2 = \begin{cases} C_{q_1}{}^R \cup C_{q_2}{}^R, & \text{if } C_{q_1}{}^R \neq C_{q_2}{}^R \text{ and } q_1, q_2 \neq q_T \\ C_{q_1}{}^R \cup \{q_T\}, & \text{if } C_{q_1}{}^R = C_{q_2}{}^R \end{cases}$$

Then the hypergroup (S,+) becomes a polysymmetrical fortified join one. We will name this hypergroup **attached grade hypergroup** of the automaton.

The notion of the grade is directly relevant to the creation of the minimum automaton which accepts the same language with the initial one. Therefore if in an automaton exist two states of the same grade, it makes no difference to the process of reaching the final state, whether we are on one or on the other. So if the attached grade hypergroup is polysymmetrical, then, based on it, and according to what we have mentioned in Remark 2.3, we can construct a fortified join one and the automaton which has this new hypergroup as attached hypergroup has less states from the original one, but it accepts exactly the same language with it.

Example



REMARK 3.1

In a deterministic automaton, A* is obviously a r a n g e o f o p e r a t o r s for the set of states S, and the function t*:A*xS ---> S defines an external composition (from the left) in it. In the case of a non deterministic automaton, t*, properly defined, defines an external hypercomposition in S, and A* becomes a r a n g e o f h y p e r o p e r a t o r s [12]. Using proper type of hypergroup of the S and proper alphabet we can construct hypergroup with operators or hyperoperators respectivly. Especially when S is a proper canonical hypergroup and A* a hyperringoid (or when A is a fortified hyperringoid), then the hypergroup with the operator has a form analogus to the hypermodule (hypemoduloid). This remains a hitherto open question.

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