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On the Borderline of Fields and Hyperfields

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Abstract: The hyperfield came into being due to a mathematical necessity that appeared during the study of the valuation theory of the fields by M. Krasner, who also defined the hyperring, which is related to the hyperfield in the same way as the ring is related to the field. The fields and the hyperfields, as well as the rings and the hyperrings, border on each other, and it is natural that problems and open questions arise in their boundary areas. This paper presents such occasions, and more specifically, it introduces a new class of non-finite hyperfields and hyperrings that is not isomorphic to the existing ones; it also classifies finite hyperfields as quotient hyperfields or non-quotient hyperfields, and it gives answers to the question that was raised from the isomorphic problems of the hyperfields: when can the subtraction of a field F's multiplicative subgroup G from itself generate F? Furthermore, it presents a construction of a new class of hyperfields, and with regard to the problem of the isomorphism of its members to the quotient hyperfields, it raises a new question in field theory: when can the subtraction of a field F's multiplicative subgroup G from itself give all the elements of the field F, except the ones of its multiplicative subgroup G?

Keywords: fields; hyperfields; rings; hyperrings; multiplicative subgroups; hypergroups; canonical hypergroups

MSC: 12-11; 12K99; 12E20; 16Y20; 20N20

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1. Introduction

The hypergroup is the very first hypercompositional structure that appeared in Algebra. It was introduced in 1934 by F. Marty while he was studying problems in non-commutative algebra, such as cosets determined by non-invariant subgroups. Unfortunately, Marty was killed in 1940, at the age of 29, during World War II, while he was serving in the French Air Force as a lieutenant and hence his mathematical heritage on hypergroups was only three papers [1–3]. Nevertheless, his ideas did not remain in France only. They spread quickly throughout Europe and across the pond. Already, by the end of the 1930s and in the 1940s, both in Europe and in the USA, important mathematicians such as M. Krasner [4–8], J. Kuntzmann [8–10], H. Wall [11], O. Ore [12–14], M. Dresher [13], E. J. Eaton [14,15], L. W. Griffiths [16], W. Prenowitz [17–19], and A.P. Dietzman [20], studied the general form of the hypergroup as well as other, special forms of this algebraic structure, resulting to its enrichment with additional axioms. The basic concept behind the hypergroup is the hypercomposition. A *hypercomposition* or *hyperoperation* over a non-empty set E is a mapping from the cartesian product $E\times E$ into the power set $P(E)$ of E. A *hypergroup* is a non-empty set E enriched with a hypercomposition "$\cdot$", which satisfies the following two axioms:

(i) The axiom of *associativity*:

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c, \quad \text{for all} \quad a,b,c \in E$$

(ii) The axiom of *reproductivity*:

$$a{\cdot}E = E{\cdot}a = E, \quad \text{for all} \quad a \in E$$

Papers [21,22] present in detail that the group is defined with exactly the same axioms. Namely, a *group* is a non-empty set E that is enriched with a composition (i.e., a mapping from the cartesian product $E{\times}E$ into the set E) that satisfies the axioms (i) and (ii).

If "·" is an internal composition on a set E and A, B are subsets of H, then $A{\cdot}B$ signifies the set $\{a \cdot b | (a,b) \in A \times B\}$, while if "·" is a hypercomposition then $A{\cdot}B$ is the union $\underset{(a,b)\in A\times B}{\cup} a \cdot b$. Ab and aB have the same meaning as $A\{b\}$ and $\{a\}B$ respectively. In general, the singleton $\{a\}$ is identified with its member a.

Theorem 1. *If either $A = \varnothing$ or $B = \varnothing$, then $AB = \varnothing$ and vice versa.*

Proof. The proof will be given with the use of symbolic logic. So, it must be proved that:

$$A{\times}B = \varnothing \Leftrightarrow (A = \varnothing) \vee (B = \varnothing)$$

or equivalently that:

$$A{\times}\varnothing = \varnothing = \varnothing{\times}B$$

To this end, we have the following equivalent statements:

$$\begin{aligned} & A{\times}B \neq \varnothing \Leftrightarrow \\ & \Leftrightarrow \exists\, (s,t) \in A{\times}B && \text{(definition of the Empty Set)} \\ & \Leftrightarrow \exists\, s{\in}A \wedge \exists\, t{\in}B && \text{(definition of the Cartesian Product)} \\ & \Leftrightarrow A \neq \varnothing \wedge B \neq \varnothing && \text{(definition of the Empty Set)} \\ & \Leftrightarrow \neg\,(A = \varnothing \vee B = \varnothing) && \text{(De Morgan's Laws)} \end{aligned}$$

Hence, by the law of contraposition:

$$(A = \varnothing) \vee (B = \varnothing) \Leftrightarrow A{\times}B = \varnothing \ \square$$

Theorem 2. *Refs. [21,22] The result of the hypercomposition of any two elements in a hypergroup H is always non-void.*

Proof. Suppose that $ab = \varnothing$, for some a, $b \in H$. By the reproductive axiom, $aH = H$ and $bH = H$. Hence:

$$H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$$

which is absurd. □

The second hypercompositional structure that appeared in Algebra was the *hyper-field*. It was introduced by M. Krasner in 1956 for the purpose of defining a certain approximation of a complete valued field by a sequence of such fields [23]. Its construction is as follows:

Let K be a valued field and let $|{\cdot\cdot}|$ be its valuation. Let ρ be a real number such that $0{\leq}\rho{<}1$ and let π_ρ be the equivalence relation in K, which is defined as follows:

$$a \equiv 0 \Leftrightarrow 0 \equiv a\,, \text{ if } a = 0$$
$$b \equiv a \Leftrightarrow \left| \tfrac{b}{a} - 1 \right| \leq \rho \Leftrightarrow |\, b - a \,| \leq \rho |\, a \,|\,, \text{ if } a \neq 0$$

The classes mod π_ρ are circles $C_\xi = C(\xi, \rho|\xi|)$ of center $\xi{\in}K$ and radius $\rho|\xi|$. It turns out that the element-wise (pointwise) multiplication of any two classes (i.e., each element of one class with all elements of the other) is a class, while their element-wise sum is a union of classes. Certain properties apply in the set K/π_ρ of these equivalence classes. These properties were the defining axioms of the hyperfield. So, a *hyperfield* is an algebraic structure $(H,+,\cdot)$ where H is a non-empty set, "$\cdot$" is an internal composition on H, and "+" is a hypercomposition on H, which satisfies the axioms:

I. *Multiplicative axiom*
$H = H^* \cup \{0\}$, where $(H^*,\cdot)$ is a multiplicative group and 0 is a bilaterally absorbing element of H, i.e., $0x = x0 = 0$, for all $x \in H$

II. *Additive axioms*

i. associativity:
$a\cdot(b\cdot c) = (a\cdot b)\cdot c$, for all $a,b,c \in H$

ii. commutativity:
$a\cdot b = b\cdot a$, for all $a,b \in H$

iii. for every $a \in H$ there exists one and only one $a' \in H$ such that $0 \in a+a'$. a' is written $-a$ and called the opposite of a; moreover, instead of $a+(-b)$ we write $a-b$.

iv. reversibility:
if $a \in b+c$, then $c \in a-b$

III. *Distributive axiom*

$$a\cdot(b+c) = a\cdot b+a\cdot c,\ (b+c)\cdot a = b\cdot a+c\cdot a, \text{ for all } a,b,c \in H$$

By virtue of axioms II.iii and II.iv it holds that $a+0=a$ for all $a \in H$. Indeed, $0 \in a-a$; therefore, $a \in a+0$. Next, if for any $x \in H$, it is true that $x \in a+0$, then $0 \in a-x$, consequently, $x=a$.

If the multiplicative axiom ***I*** is replaced by the axiom:

I′. H^* is a multiplicative semigroup having a bilaterally absorbing element 0,

then, a more general structure is obtained which is called *hyperring* [24].

It is easy to see that a non-empty set H enriched with the additive axioms II is a hypergroup. This special hypergroup was named *canonical hypergroup* by Jean Mittas, who studied it in depth and presented his research results through a multitude of papers, e.g., [25–28].

Apparently, fields and rings satisfy the above axioms, and hence, they are also called *trivial hyperfields* and *trivial hyperrings,* respectively. It is worth mentioning, though, that several algebraic properties which are valid for the rings and the fields are not transferred in the hyperrings and hyperfields. The following proposition is such an example.

Proposition 1. *Let P be a hyperring. Then,*

$$(a+b)(c+d) \subseteq ac+ac+ad+bd$$

for any a,b,c,d ∈ P.

Proof.

$$(a+b)(c+d) = \bigcup_{x \in a+b} x(c+d) = \bigcup_{x \in a+b} (xc+xd) \subseteq \bigcup_{z \in a+b} zc + \bigcup_{w \in a+b} wc =$$
$$= (a+b)c + (a+b)d = ac+ac+ad+bd$$

(see also [29]) □

Another example is the polynomials over a hyperring P. As in the case of rings, a polynomial p over a hyperring P is defined as an ordered set $(a_0, a_1, \ldots)$ where all the $a_i's$ after a certain one (say after a_n) are zero. The elements a_i are the coefficients of p and n is the degree of p. If $p = (a_i)$ and $q = (b_j)$ then

$$p+q = \{(c_i) | c_i \in a_i + b_i\} \quad \text{and} \quad pq = \left\{ (c_i) | c_i \in \sum_{j+k\,=\,i} a_j b_k \right\}$$

The set of the polynomials over P is not a hyperring since its multiplicative part is not a semigroup, but it is a semihypergroup. This algebraic structure was named *superring* by J. Mittas [30,31]. In [32], R. Ameri, M. Eyvazi, and S. Hoskova-Mayerova proved that the distributive axiom is not valid for the multiplication of the polynomials over a hyperring. More precisely, it is indicated that the weak distributive axiom holds, i.e.,

$$r \cdot (p+q) \subseteq r \cdot p + r \cdot q,\ (p+q) \cdot r \subseteq p \cdot r + q \cdot r$$

Moreover, as it is proved in [33] (Theorem 16), the direct sum of hypermodules is not a hypermodule but a weak hypermodule in the sense that it satisfies the weak distributive axiom. Unfortunately, there are numerous published papers that contain incorrect results as they are based on the erroneous assumption that the direct sum of hypermodules is a hypermodule or that the distributivity holds for the multiplication of polynomials over a hyperring.

Krasner named the hyperfields, which he used for the approximation of the complete valued field, *residual hyperfields*. Next, while working on the question of how rich the class of the hyperrings and hyperfields is, he was led to the construction of a more general class of hyperrings and hyperfields, i.e., the class of the *quotient hyperfields* and the *quotient hyperrings* [24].

Note on the notation: In the following pages, in addition to the typical algebraic notations, we are using Krasner's notation for the complement and the difference [34]. So, we denote by $A \cdot\cdot B$ the set of elements that are in the set A but not in the set B. If K is a field or a hyperfield, then K^* denotes the set $K \cdot\cdot \{0\}$.

2. The Quotient Hyperfield/Hyperring

The construction of the quotient hyperfield or hyperring is based on a field or ring, respectively. Let F be a field and G a subgroup of F's multiplicative group F^*. Then, the multiplicative classes modulo G in F form a partition of F. Krasner observed that the product of two such classes, considered as subsets of F, is also a class modulo G, while their sum is a union of such classes. Next, he proved that the set F/G of the classes of this partition becomes a hyperfield if the multiplication and the addition are defined as follows:

$$xG \cdot yG = xyG$$
$$xG \dagger yG = \{(xp + yq)G \,|\, p, q \in G\}$$

for all $xG,\ yG \in F/G$.

Moreover, Krasner proved that if R is a ring and G is a normal subgroup of its multiplicative group, then the above construction gives a hyperring [24].

From the proof that R/G is a hyperring, it derives that the definition of the addition in R/G as well as the proof of the additive axioms do not require the normality of G. On the other hand, the definition of the multiplication and the proof of the multiplicative and distributive axioms require only that the equality:

$$xG \cdot yG = \{xg_1 y g_2 \mid g_1, g_2 \in G\} = \{xyg \mid g \in G\} = xyG$$

holds. But the validity of this equality is equivalent to the normality of G only when G is a subgroup of a group and not when G is a subgroup of a semigroup, which is the case when R is a ring. This was proved by Ch. Massouros [35] via an example, which is generalized below.

Example 1. *Let R_o be a unitary ring such that $2 \neq 0$. Let us consider the cartesian product $R = {R_o}^n$. R is enriched with the following addition and multiplication:*

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$
$$(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (a_1(b_1 + \ldots + b_n), \ldots, a_n(b_1 + \ldots + b_n))$$

It is well known that $(R,+)$ is a group. Next, observe that the multiplication is not commutative. Indeed:

$$(a_1,\ldots,a_n)(b_1,\ldots,b_n) = (a_1(b_1+\ldots+b_n),\ldots,a_n(b_1+\ldots+b_n))$$

while:

$$(b_1,\ldots,b_n)(a_1,\ldots,a_n) = (b_1(a_1+\ldots+a_n),\ldots,b_n(a_1+\ldots+a_n))$$

On the contrary, the multiplication is associative:

$$\begin{aligned}
&(a_1,\ldots,a_n)[(b_1,\ldots,b_n)(c_1,\ldots,c_n)] = \\
&= (a_1,\ldots,a_n)(b_1(c_1+\ldots+c_n),\ldots,b_n(c_1+\ldots+c_n)) = \\
&= \begin{pmatrix} a_1(b_1(c_1+\ldots+c_n)+\ldots+b_n(c_1+\ldots+c_n)),\ldots \\ \ldots,a_n(b_1(c_1+\ldots+c_n)+\ldots+b_n(c_1+\ldots+c_n)) \end{pmatrix} = \\
&= (a_1(b_1+\ldots+b_n)(c_1+\ldots+c_n),\ldots,a_n(b_1+\ldots+b_n)(c_1+\ldots+c_n)) = \\
&= (a_1(b_1+\ldots+b_n),\ldots,a_n(b_1+\ldots+b_n))(c_1,\ldots,c_n) = \\
&= [(a_1,\ldots,a_n)(b_1,\ldots,b_n)](c_1,\ldots,c_n)
\end{aligned}$$

and distributive:

$$\begin{aligned}
&(a_1,\ldots,a_n)[(b_1,\ldots,b_n)+(c_1,\ldots,c_n)] = \\
&= (a_1,\ldots,a_n)(b_1+c_1,\ldots,b_n+c_n) = \\
&= (a_1(b_1+c_1+\ldots b_n+c_n),\ldots,a_n(b_1+c_1+\ldots b_n+c_n)) = \\
&= (a_1(b_1+\ldots+b_n)+a_1(c_1+\ldots+c_n),\ldots,a_n(b_1+\ldots+b_n)+a_n(c_1+\ldots+c_n)) = \\
&= (a_1(b_1+\ldots+b_n),\ldots,a_n(b_1+\ldots+b_n))+(a_1(c_1+\ldots+c_n),\ldots,a_n(c_1+\ldots+c_n)) = \\
&= (a_1,\ldots,a_n)(b_1,\ldots,b_n)+(a_1,\ldots,a_n)(c_1,\ldots,c_n)
\end{aligned}$$

Thus $(R,+,\cdot)$ is a ring. A non-zero element $(a_1,\ldots,a_n)$ of R is idempotent if $a_1+\ldots+a_n = 1$. Indeed:

$$(a_1,\ldots,a_n)^2 = (a_1(a_1+\ldots+a_n),\ldots,a_n(a_1+\ldots+a_n)) = (a_1\cdot 1,\ldots,a_n\cdot 1) = (a_1,\ldots,a_n)$$

Thus, the elements $e_1 = (1,\ldots,0),\ldots,e_n = (0,\ldots,1)$ are idempotent. Moreover, the opposite of the $e_i = (0,\ldots,1,\ldots,0)$, $i = 1,\ldots,n$ is $-e_i = (0,\ldots,-1,\ldots,0)$, which is different from the e_i because $2e_i = (0,\ldots,2,\ldots,0) \neq (0,\ldots,0) = 0$. Since $(-e_i)^2 = {e_i}^2 = e_i$, the 2-element sets $G_i = \{-e_i,\ e_i\}$, $i = 1,\ldots,n$ are multiplicative subgroups of R. Next, if $a = (a_1,\ldots,a_n)$ is an element in R, then:

$$\begin{aligned}
aG_i &= (a_1,\ldots,a_n)\{-e_i,e_i\} = \\
&= \{(a_1,\ldots,a_n)(0,\ldots,-1,\ldots,0),\ (a_1,\ldots,a_n)(0,\ldots,-1,\ldots,0)\} = \\
&= \{(-a_1,\ldots,-a_n),\ (a_1,\ldots,a_n)\} = \{-a,a\}
\end{aligned}$$

while

$$\begin{aligned}
G_ia &= \{-e_i,e_i\}(a_1,\ldots,a_n) = \\
&= \{(0,\ldots,-1,\ldots,0)(a_1,\ldots,a_n),\ (0,\ldots,-1,\ldots,0)(a_1,\ldots,a_n)\} = \\
&= \{(0,\ldots,-a_1-\ldots-a_n,\ldots,0),\ (0,\ldots,a_1+\ldots+a_n,\ldots,0)\}
\end{aligned}$$

Consequently, the multiplicative subgroups G_i, $i = 1,\ldots,n$ are not normal. Nevertheless, they satisfy the condition:

$$(aG_i)(bG_i) = abG_i$$

Indeed,

$$(aG_i)(bG_i) = \{-a,a\}\{-b,b\} = \{(-a)(-b),\ (-a)b,\ a(-b),\ ab\} = \{-ab,\ ab\} = abG_i$$

Therefore, the quotients R/G_i, $i = 1,\ldots,n$ are hyperrings. Observe that G_i is a right neutral element for multiplication in R/G_i, but it is not a left one as well. In contrast, when the quotient *hyperring is constructed via a normal subgroup G of the ring's multiplicative semigroup, then G is a bilateral neutral element for the multiplication in the quotient hyperring.*

The aforementioned hyperrings, although they are not quotient hyperrings of a ring by a normal subgroup of its multiplicative semigroup, they are still embeddable in such quotient hyperrings [35,36].

A large number of papers has been published on the hyperfields and hyperrings, starting from the pioneer work of J. Mittas [37–44] and continuing with a plenitude of researchers, such as Ch. Massouros [29,35,45–51], A. Nakassis [36], G. Massouros [50–54], R. Rota [55,56], S. Jančic-Rašović [57–59], I. Cristea [58–64], H. Bordbar [59–61], M. Kankaraš [62], V. Vahedi et al. [63–65], M. Jafarpour et al. [63–66], A. Connes and C. Consani [67,68], O. Viro [69,70], R. Ameri, M. Eyvazi and S. Hoskova-Mayerova [32,71], M. Baker et al. [72–74], J. Jun [75], O. Lorscheid [76], Z. Liu [77], H. Shojaei and D. Fasino [78], K. Das et al. [79], K. Roberto et al. [80–82], P. Corsini [83], B. Davvaz, V. Leoreanu-Fotea [84], C. Yatras [85–87], S. Atamewoue Tsafack, S. Wen, B.O. Onasanya, et al. [88], A. Linz, and P. Touchard [89], S. Creech [90], T. Gunn [91], etc. In the recent years, several hyperfields which belong to the class of quotient hyperfields have appeared, a fact that is not mentioned or even noticed, while, sometimes, an unsuccessful terminology is used for them. More specifically:

(a) In the papers [67,68] by A. Connes and C. Consani and afterward in many subsequent papers (e.g. [69,72,75,76]), the name «*Krasner's hyperfield*» is used for the hyperfield, which is constructed over the set {0, 1} via the hypercomposition:

$$0+0=0,\ 0+1=1+0=1,\ 1+1=\{0,1\}$$

Oleg Viro, in his paper [69], justifiably states about this hyperfield: «*To the best of my knowledge, K did not appear in Krasner's papers*». His remark is absolutely correct. Actually, the above is a special case of a quotient hyperfield, and in this sense, it belongs to a special class of Krasner hyperfields. Indeed, for a field F and its multiplicative subgroup F^*, the quotient hyperfield $F/F^* = \{0, F^*\}$ is isomorphic to the hyperfield considered by A. Connes and C. Consani. More precisely, in the case of hyperfields with cardinality 2, the following theorem holds:

Theorem 3. *The two-element non-trivial hyperfield is isomorphic to a quotient hyperfield.*

(b) In the papers [67,68] by A. Connes and C. Consani, a hyperfield is considered over the set {−1, 0, 1} with the following hypercomposition:

$$0+0=0,\ 0+1=1+0=1,\ 1+1=1,\ -1-1=-1,\ 1-1=-1+1=\{-1,0,1\}$$

This hyperfield is now called «*sign hyperfield*» by some authors. Nevertheless, this hyperfield is a quotient hyperfield as well. Indeed, let F be an ordered field and let F^+ be its positive cone. Then the quotient hyperfield $F/F^+ = \{-F^+, 0, F^+\}$ is isomorphic to the sign hyperfield.

(c) The «*phase hyperfield*» that appeared recently in the bibliography (see, e.g., [69,72]) is just the quotient hyperfield $\mathbb{C}/\mathbb{R}^+$, where $\mathbb{C}$ is the field of complex numbers and $\mathbb{R}^+$ is the set of positive real numbers. The elements of this hyperfield are the rays of the complex field with origin at the point (0,0). The sum of two elements $z\mathbb{R}^+$, $w\mathbb{R}^+$ of $\mathbb{C}/\mathbb{R}^+$ is the set $\{(zp+wq)\mathbb{R}^+ \mid p,q \in \mathbb{R}^+\}$. When $z\mathbb{R}^+ \neq w\mathbb{R}^+$, this sum consists of all the interior rays $x\mathbb{R}^+$ of the convex angle which is created from $z\mathbb{R}^+$ and $w\mathbb{R}^+$, while if $w\mathbb{R}^+ = -z\mathbb{R}^+$ then, the sum of the two opposite rays $z\mathbb{R}^+$, $-z\mathbb{R}^+$ is the set $\{\, 0,\ -z\mathbb{R}^+,\ z\mathbb{R}^+\}$. This hyperfield is presented in detail in [46].

Note on the notation: In the following theorems, new hyperfields are constructed via other hyperfields or fields. To avoid any confusion between the new and the old hypercomposition we use + as the sign for the initial addition and symbols such as $\dot{+}$, $\hat{+}$, $\tilde{+}$, etc., to denote the new one.

Theorem 4. *Let* $(F, +, \cdot)$ *be a field. If we define the hypercomposition* $\dotplus$ *on F as follows:*

$$\begin{aligned} &x \dotplus y = \{\, x, y, x + y \,\}, && \text{if } y \neq -x \text{ and } x, y \neq 0, \\ &x \dotplus (-x) = F, && \text{for all } x \in F^{*}, \\ &x \dotplus 0 = 0 \dotplus x = x, && \text{for all } x \in F, \end{aligned}$$

then $(F, \dotplus, \cdot)$ *is a hyperfield isomorphic to a quotient hyperfield.*

Proof. From the verification of the axioms, it follows that $(F, \dotplus, \cdot)$ is a hyperfield (see also [46]). Next, since $(F,+,\cdot)$ is a field, the polynomial ring $F[x]$ is an integral domain, and so the field $F(x)$ of the rational functions over F can be defined. We can then assume that in all rational functions, the coefficient of the highest power of the denominator's polynomial is 1 since, if this is not the case, we can make it via the appropriate division. Now, let G be the set

$$G = \{\, \pi(x) \in F(x) \mid a_m = 1 \,\}$$

where a_m is the coefficient of the numerator's highest power. G is a multiplicative subgroup of the multiplicative group of $F(x)$. Therefore, we can consider the quotient hyperfield $(F(x)/G, \dotplus, \cdot)$. The function $\varphi : F \to F(x)/G$, with $\varphi(a) = aG$, for each $a \in F$, is one-to-one, since if a, b are distinct elements in F, then

$$aG = \{\pi(x) \in F(x) \text{ with } a_m = a\,\} \quad \text{and} \quad bG = \{\pi(x) \in F(x) \text{ with } a_m = b\,\}$$

are distinct elements of $F(x)/G$. Moreover, φ is a surjection since every element aG of $F(x)/G$ is the image of the corresponding element a of F. Next, let

$$\pi_1(x) = \frac{\sum_{i=1}^{k} a_i t^{a_i}}{\sum_{j=1}^{l} b_j t^{b_j}},\ a_k = 1,\ b_l = 1 \quad \text{and} \quad \pi_2(x) = \frac{\sum_{i=1}^{n} a'_i t^{a_i}}{\sum_{j=1}^{l} b_j t^{b_j}},\ a'_n = 1,\ b_l = 1$$

be two elements in G. We assume that $\pi_1(x)$ and $\pi_2(x)$ have the same denominator because if they are rational expressions with unlike denominators, we can convert them into rational expressions with common denominators. Let us consider the sum:

$$aG \dotplus bG = \{\, [a\pi_1(x) + b\pi_2(x)]G \mid \pi_1(x), \pi_2(x) \in G \,\} \text{ with } bG \neq -aG$$

Then:

(i) If $k > n$, then the coefficient of the highest power of the polynomial $a\pi_1(x) + b\pi_2(x)$ is a, thus $a\pi_1(x) + b\pi_2(x) \in aG$, and therefore $aG \in aG \dotplus bG$. On the other hand, the coefficient of the highest power of the polynomial $b\pi_1(x) + a\pi_2(x)$ is b, thus $b\pi_1(x) + a\pi_2(x) \in bG$ and therefore $bG \in aG \dotplus bG$.

(ii) If $k = n$, then the coefficient of the highest power of the polynomial $a\pi_1(x) + b\pi_2(x)$ is $a + b$, thus $a\pi_1(x) + b\pi_2(x) \in (a + b)G$, and therefore $(a + b)G \in aG \dotplus bG$.

Consequently, φ is an isomorphism, and thus the Theorem. □

It needs to be clarified here that the definition of the hypercomposition for the non-opposite elements, in combination with the axioms of the hyperfield, allows no different way for the definition of the hypercomposition of two opposite elements. More precisely, we have the following two Propositions (for their proofs see [46]):

Proposition 2. *In a hyperfield K, with card*$K > 3$*, the sum* $x+y$ *of any two elements* $x,y \neq 0$ *contains these two elements if and only if the difference* $x-x$ *equals K for all* $x \neq 0$.

Proposition 3. *In a hyperfield K, with cardK > 3, the sum x+y of any two non-opposite elements x,y≠0 does not contain the participating elements if and only if the difference x−x equals to* {−*x*,0,*x*}*, for all x≠*0.

The hypercomposition that appears in Proposition 2 is called *closed* (or *containing*; sometimes it is also called *extensive* [92]), while the hypercomposition that appears in Proposition 3 is called *open* [93]. In particular, a hypercomposition in a hypergroupoid $(E,+)$ is called *right closed* if $a\in b+a$ for all $a,b\in E$, *left closed* if $a\in a+b$ for all $a,b\in E$, and *closed* if $\{a,b\}\subseteq a+b$ for all $a,b\in E$. A hypercomposition is called *right open* if $a\notin b+a$ for all $a,b\in E$ with $b\neq a$ while it is called *left open* if $a\notin a+b$ for all $a,b\in E$ with $b\neq a$. A hypercomposition is called *open* if it is both right and left open. Right closed hypercompositions are left open, and left closed compositions are right open. If the commutativity is valid, then the right/left closed and the closed (resp. the right/left open and the open) hypercompositions coincide.

The following Theorem presents the construction of a hyperfield that is equipped with a closed hypercomposition, and therefore, the definition of the sum of two opposite elements in it is restricted by the provisions of Proposition 2.

Theorem 5. *Ref. [46] Let* $(H, +, \cdot)$ *be a hyperfield. If we define a new hypercomposition «$\dotplus$» on H as follows:*

$$\begin{aligned} &x \dotplus y = \{x, y\} \cup (x + y), && \text{for all } x, y \in H^*, \text{with } y \neq -x,\\ &x \dotplus (-x) = H, && \text{for all } x \in H^*,\\ &x \dotplus 0 = 0 \dotplus x = x, && \text{for all } x \in H, \end{aligned}$$

then, $(H, \dotplus, \cdot)$ *is a hyperfield and when* $(H, +, \cdot)$ *is a quotient hyperfield, then* $(H, \dotplus, \cdot)$ *is also a quotient hyperfield.*

The proof of this theorem can be found in [46].

The hyperfield, which is constructed by the above Theorems 4 and 5, will be termed *augmented hyperfield* because the composition or the hypercomposition is augmented to contain the two addends. The augmented hyperfield of a field or a hyperfield *F* is denoted by [*F*]. The augmented hyperfield's distinctive feature is that it always provides the information (the elements) that produced the result. As shown in the following sections, different hyperfields can have the same augmented hyperfield.

Theorems 4 and 5 ensure that the augmented hyperfield of a field or a quotient hyperfield is a quotient hyperfield, but it is not known yet whether all the members of a family of hyperfields whose augmented hyperfield is a quotient hyperfield are quotient hyperfields.

In the following construction Theorems, Proposition 2 is used to define the sum of two opposite elements:

Theorem 6. *Ref [46] Let G be a non-unitary multiplicative group and let* $(H^*,\cdot)$ *be its direct product with the multiplicative group* {−1,1}*. Consider the set* $H = H^*\cup\{0\}$*, where* 0 *is a bilaterally absorbing element in H, i.e.,* 0*w*=*w*0=0*, for all w∈H. The following hypercomposition is introduced on H:*

$$\begin{aligned} (x,i) \hat{+} (y,j) &= \{(x,i),(y,j)\}, && if\ (y,j) \neq (x,-i),\\ (x,i) \hat{+} (x,-i) &= H, && for\ all\ (x,i) \in H^*,\\ (x,i) \hat{+} 0 = 0 \hat{+} (x,i) &= (x,i) \quad and \quad 0 \hat{+} 0 = 0 && for\ all\ (x,i) \in H^*. \end{aligned}$$

Then, $(H,\hat{+},\cdot)$ *is a hyperfield.*

Theorem 7. *Ref. [46] Let* $(G, \cdot)$ *be a non-unitary multiplicative group and* 0 *a bilaterally absorbing element. If we define a hypercomposition* $\hat{+}$ *on* $H = G \cup \{0\}$ *as follows:*

$$\begin{array}{lll} x \hat{+} y &= \{x, y\}, & for\ all\ x, y \in G,\ with\ y \neq x, \\ x \hat{+} x &= H, & for\ all\ x, y \in G, \\ x \hat{+} 0 &= 0 \hat{+} x = x, & for\ all\ x \in H, \end{array}$$

then, the triplet $(H, \hat{+}, \cdot)$ *becomes a hyperfield.*

In [46], it is proved that the above Theorem constructs a family of hyperfields, which contains quotient hyperfields, but it is not known yet whether this family contains non-quotient hyperfields as well.

Theorem 8. *Let Q be a multiplicative group that has more than two elements and let* 0 *be a multiplicatively bilaterally absorbing element. If we define a hypercomposition* $\tilde{+}$ *on* $H = Q \cup \{0\}$ *as follows:*

$$\begin{array}{lll} x \tilde{+} y &= Q, & for\ all\ x, y \in Q,\ with\ y \neq x, \\ x \tilde{+} x &= H \cdot\cdot \{x\}, & for\ all\ x \in Q, \\ x \tilde{+} 0 &= 0 \tilde{+} x = x, & for\ all\ x \in H, \end{array}$$

then, the triplet $H(Q) = (Q \cup \{0\}, \tilde{+}, \cdot)$ *is a hyperfield.*

The following example proves the existence of quotient hyperfields which are constructed according to the above Theorem.

Example 2. (i) *Consider the field* $\mathbb{Z}_{41}$*. This field's multiplicative subgroup of order* 4

$$G = \{1, 4, 10, 16, 18, 23, 25, 31, 37, 40\}$$

has the property $G–G=G+G=\mathbb{Z}_{41} \cdot\cdot G$ *and* $xG+yG=\mathbb{Z}_{41} \cdot\cdot \{0\}$ *when* $x \neq y$ *with* $x, y \in \left\{ 3^k \,\middle|\, k = 0, 1, 2, 3 \right\}$*. Therefore, the quotient hyperfield*

$$\mathbb{Z}_{41}/G = \left\{0,\ G,\ 3G,\ 3^2G,\ 3^3G\right\}$$

is of the type of hyperfields of Theorem 8.

(ii) *Consider the field* $\mathbb{Z}_{71}$*. Its multiplicative subgroup of order* 5 *is*

$$G = \{1, 20, 23, 26, 30, 32, 34, 37, 39, 41, 45, 48, 51, 70\}$$

and it has the property $G–G=G+G=\mathbb{Z}_{71} \cdot\cdot G$ *and* $xG+yG=\mathbb{Z}_{71} \cdot\cdot \{0\}$ *when* $x \neq y$ *with* $x, y \in \left\{ 2^k \,\middle|\, k = 0, 1, 2, 3, 4 \right\}$*. Therefore, the quotient hyperfield*

$$\mathbb{Z}_{71}/G = \left\{0,\ G,\ 2G,\ 2^2G,\ 2^3G,\ 2^4G\right\}$$

is of the type of hyperfields of Theorem 8.

(iii) *Consider the field* $\mathbb{Z}_{101}$*. This field's multiplicative subgroup of order* 5

$$G = \{1, 6, 10, 14, 17, 32, 36, 39, 41, 44, 57, 60, 62, 65, 69, 84, 87, 91, 95, 100\}$$

has the property $G–G=G+G=\mathbb{Z}_{101} \cdot\cdot G$ *and* $xG+yG=\mathbb{Z}_{101} \cdot\cdot \{0\}$ *when* $x \neq y$ *with* $x, y \in \left\{ 2^k \,\middle|\, k = 0, 1, 2, 3, 4 \right\}$*. Therefore, the quotient hyperfield*

$$\mathbb{Z}_{101}/G = \left\{0,\ G,\ 2G,\ 2^2G,\ 2^3G,\ 2^4G\right\}$$

is of the type of hyperfields of Theorem 8.

The hyperfields of Theorems 6 and 7 are called *b-hyperfields* due to the binary result of the hypercomposition, which consists of the two addends when they are different elements. Moreover, the hyperfields of Theorems 4, 5, 6, and 7 were termed *monogenic* (*monogène*) because they are generated by just a single element of the hyperfield [46]. Additionally, the hyperfield which is constructed by Theorem 8 is monogenic (monogène) because $H=x\tilde{+}x\tilde{+}x\tilde{+}x$. The *monogenic* (*monogène*) canonical hypergroup was introduced and studied in depth by J. Mittas [26]. The set of the canonical subhypergroups of a canonical hypergroup H is a complete lattice, thus for a given subset X of H there always exists the least (in the sense of inclusion) canonical subhypergroup $\overline{X}$ of H which contains X. Now, if X is the singleton $\{x\}$, then the canonical subhypergroup that is generated from it, is called *monogenic* (*monogène*). If $H = \overline{\{x\}}$, then H itself is called *monogenic* (*monogène*). The study of the monogenic (monogène) hypergroups led to the definition of the *order* of a canonical hypergroup's elements [26] and sequentially to the order of the elements of a hyperfield [41]. Since:

$$mx + nx = \begin{cases} (m+n)x, & \text{if } mn > 0 \\ (m+n)x \ + \ \min\{\ |m|,\ |n|\ \}\ (x-x), & \text{if } mn < 0 \end{cases}$$

for the monogenic (monogène) hypergroup it holds:

$$\overline{\{x\}} = mx + n(x-x), \quad m,n \in \mathbb{Z}$$

and as it is true that $-(x-x)=x-x$, we can assume that $(m,n)\in\mathbb{Z}\times\mathbb{N}$ instead of $\mathbb{Z}\times\mathbb{Z}$. Thus, two mutually exclusive cases can appear:

(I) For every $(m,n)\in\mathbb{Z}\times\mathbb{N}$, with $m\neq 0$, $0\notin mx+n(x-x)$, in which case x, as well as $\overline{\{x\}}$ are said to be of *infinite order* denoted by $\omega(x)=+\infty$.

Proposition 4. *Ref.* [26] $\omega(x)=+\infty$ *if and only if* $m'x\cap m''x=\varnothing$, *for every* $m',m''\in\mathbb{Z}$ *with* $m'\neq m''$.

(II) There exists $(m,n)\in\mathbb{Z}\times\mathbb{N}$, with $m\neq 0$, such that $0\in mx+n(x-x)$. In the following, p will denote the minimum positive integer for which there exists $n\in\mathbb{N}$, such that $0\in px+n(x-x)$.

Proposition 5. *Ref.* [26] *For a given* $m\in\mathbb{Z}$ *there exists* $n\in\mathbb{N}$ *such that* $0\in mx+n(x-x)$, *if and only if m is divided by p.*

For $m=kp$, $k\in\mathbb{Z}^*$, let $q(k)$ be the minimum nonnegative integer such that $0\in kpx+q(k)(x-x)$. Then q is a function from $\mathbb{Z}$ to $\mathbb{N}$. Mittas called the pair $\omega(x)=(p,q)$ *order* of both x and $\overline{\{x\}}$. Also, he named p the *principal order* of x and q the *associative order* of x [26,41]. Therefore, the order of all the elements of the hyperfields which are constructed by the Theorems 4, 5 and 6 is (1,1) because $0\in x+(x-x)$, while the order of the elements of the monogenic (monogène) hyperfield of Theorem 7 is (2,0), since $0\in x+x=2x+0(x-x)$ and of the hyperfield of Theorem 8 is (4,0), since $0\in x+x+x+x=4x+0(x-x)$.

These definitions were later used in other hypercompositional structures, such as the fortified transposition hypergroups [22], the hyperringoids [52], the M-polysymmetrical hyperrings [86] etc.

3. The Non-Quotient Hyperfields/Hyperrings

M. Krasner realized that the existence of non-quotient hyperfields and hyperrings was an essential question for the self-sufficiency of the theory of hyperfields and hyperrings vis-à-vis that of fields and rings, since if all hyperrings and hyperfields could be isomorphically embedded into the quotient hyperrings, then several conclusions of their theory could have been obtained in a direct and straightforward way, through the use of the ring, field and modules theories, instead of developing new techniques and proof methodologies. Therefore, in his paper [24], he raised the relevant question. The answer to this question led to the construction of two classes of hyperfields and hyperrings, which contain elements that are not isomorphic to the quotient ones. The following Theorems 9 and 10 which were proved by Ch. Massouros, refer to hyperfields with closed hypercompositions and they prove the existence of finite and infinite non-quotient hyperfields. The subsequent Theorem 11 was proved by A. Nakassis, it is on hyperfields with open hypercompositions and it reveals another class of finite non-quotient hyperfields. Moreover, Theorem 12 gives a new class of infinite non-quotient hyperfields which do not belong to the previous two classes, and Theorem 13 uncovers a new class of infinite non-quotient hyperrings.

Theorem 9. *Ref. [35] Let* Θ *be a multiplicative group that has more than two elements and let* $(K^*, \cdot)$ *be its direct product with the multiplicative group* $\{-1, 1\}$*. Consider the set* $K = K^* \cup \{0\}$*, where* 0 *is a bilaterally absorbing element in K, i.e.,* $0w = w0 = 0$*, for all* $w \in K$*. The following hypercomposition is introduced on K:*

$$
\begin{aligned}
(x,i) \dotplus (y,j) &= \{(x,i),\ (y,j),\ (x,-i),\ (y,-j)\}, \quad if\ \ (y,j) \neq (x,i),\ (x,-i) \\
(x,i) \dotplus (x,i) &= K \cdot \cdot \{(x,i),\ (x,-i),\ 0\} \\
(x,i) \dotplus (x,-i) &= K \cdot \cdot \{(x,i),\ (x,-i)\} \\
(x,i) \dotplus 0 &= 0 \dotplus (x,i) = (x,i)\ \ and\ \ 0 \dotplus 0 = 0
\end{aligned}
$$

Then, the triplet $K(\Theta) = (K, \dotplus, \cdot)$ *is a hyperfield that does not belong to the class of quotient hyperfields when* Θ *is a periodic group.*

For the proof of the above Theorem, see [35].

Theorem 10. *Refs. [29,47] Let* Θ *be a multiplicative group which has more than two elements and let* 0 *be a multiplicatively bilaterally absorbing element. If we define a hypercomposition* $\dotplus$ *on* $H = \Theta \cup \{0\}$ *as follows:*

$$
\begin{aligned}
x \dotplus y &= \{x, y\}, && for\ all\ x, y \in \Theta, with y \neq x, \\
x \dotplus x &= H \cdot \cdot \{x\}, && for\ all\ x \in \Theta, \\
x \dotplus 0 &= 0 \dotplus x = x, && for\ all\ x \in H,
\end{aligned}
$$

then, the triplet $H(\Theta) = (\Theta \cup \{0\}, \dotplus, \cdot)$ *is a hyperfield which is not isomorphic to a quotient hyperfield when* Θ *is a periodic group.*

For the proof of the above Theorem, see [29,47].

Proposition 6. *Ref. [36] Let* $(T, \cdot)$ *be a multiplicative group of order m, with* $m > 3$*. Additionally, let* $H = T \cup \{0\}$ *where* 0 *is a multiplicatively absorbing element. If H is equipped with the hypercomposition:*

$$
\begin{aligned}
x \dotplus y &= H \cdot \cdot \{0, x, y\} && for\ all\ x, y \in T,\ with\ y \neq x, \\
x \dotplus x &= \{0, x\}, && for\ all\ x \in T, \\
x \dotplus 0 &= 0 \dotplus x = x, && for\ all\ x \in H,
\end{aligned}
$$

then, $H(T) = (T \cup \{0\}, \dotplus, \cdot)$ *is a hyperfield.*

It is worth noting here that the elements of the above hyperfield are self-opposite, and since the hypercomposition is open, Proposition 3 imposes the definition of the sum of the self-opposite elements so that $H(T)$ fulfills the axioms of the hyperfield.

Theorem 11. *Ref. [36] If T is a finite multiplicative group of m, $m>3$ elements and if the hyperfield $H(T)$ is isomorphic to a quotient hyperfield F/Q, then $Q \cup \{0\}$ is a field of $m-1$ elements while F is a field of $(m-1)^2$ elements.*

Obviously, the cardinality of T can be chosen in such a way that $H(T)$ cannot be isomorphic to a quotient hyperfield.

For the proof of Theorem 11, the following important counting lemma was introduced and used by A. Nakassis.

Lemma 1. *Ref. [36] Let H be a hyperfield equipped with a hypercomposition such that the differences $x-x$, $x \in H$ have only 0 in common. If H is isomorphic to a quotient hyperfield F/Q, then the cardinality of the sum of any two non-opposite elements is equal to the cardinality of Q.*

Proof. Suppose that H is a hyperfield equipped with a hypercomposition such that $(x-x) \cap (y-y) = \{0\}$ for all $x, y \in H$ with $x \neq y$. Assume that H is isomorphic to a quotient hyperfield F/Q. Let a', b' with $a' \neq b'$ be two elements in H and let aQ, bQ be their homomorphic images in F/Q. Then $a'+b'$ has the same cardinality with $aQ + bQ = \{(a+bq)Q \mid q \in Q\}$. Next, if $(a+bq)Q = (a+bp)Q$, then

$$a + bq = (a+bp)r \Leftrightarrow a - ar = bq - bpr \Rightarrow (aQ - aQ) \cap (bQ - bQ) \neq \varnothing$$

However, since the equality $(aQ - aQ) \cap (bQ - bQ) = \{0\}$ is valid, it follows that $a - ar = 0$. Therefore $r=1$ and consequently $bq - bp = 0$ or equivalently $q = p$. Hence $card(aQ+bQ) = cardQ$ and so the lemma. □

A direct consequence of Nakassis' lemma is that if a hyperfield H is isomorphic to a quotient hyperfield and the differences $x-x$, $x \in H$ have only 0 in common, then the sums of the non-opposite elements have the same cardinality. This result is very useful to the classification of hyperfields which is presented in Section 5.

In the following, the class of non-quotient hyperrings and hyperfields will be enriched with another family of such structures.

J. Mittas in the first section of [41], constructed the following hyperfield, which is called *tropical hyperfield* nowadays (see, e.g., [69,70,72,75,76]) because it is proved to be a suitable and effective algebraic tool for the study of tropical geometry:

Example 3. *Ref. [41] Let $(E, \cdot)$ be a totally ordered multiplicative semigroup, having a minimum element 0, which is bilaterally absorbing with regard to the multiplication. The following hypercomposition is defined on E:*

$$x \hat{+} y = \begin{cases} \max\{x, y\} & if\ x \neq y \\ \{z \in E \mid z \leq x\} & if\ x = y \end{cases}$$

Then $(E, \hat{+}, \cdot)$ is a hyperring. If $E \cdot \cdot \{0\}$ is a multiplicative group, then $(E, \hat{+}, \cdot)$ is a hyperfield.

A slight modification of the definition of the above hypercomposition, when x is equal to y, gives the following Theorem:

Theorem 12. *Let $(E, \cdot)$ be a totally ordered multiplicative semigroup, having a minimum element 0, which is bilaterally absorbing with regard to the multiplication. The following hypercomposition is defined on E:*

$$x \breve{+} y = \begin{cases} \max\{x, y\} & if\ x \neq y \\ \{z \in E \mid z < x\} & if\ x = y \end{cases}$$

Then $(E, \breve{+}, \cdot)$ is a non-quotient hyperring. If $E \cdot \cdot \{0\}$ is a multiplicative group, then $(E, \breve{+}, \cdot)$ is a non-quotient hyperfield.

Proof. The verification of the axioms of the hyperring and the hyperfield proves that $(E, \breve{+}, \cdot)$ is such a structure. Next suppose that $(E, \breve{+}, \cdot)$ is isomorphic to a quotient hyperring $(R/Q, +, \cdot)$. As $x \notin x + x$, for all $x \in E$ and because $2 = 1 + 1 \in Q + Q$, it follows that $2 \notin Q$. Hence $2Q$ is a class different from Q which belongs to $Q + Q$, therefore $2Q < Q$ and so $2Q + Q = Q$. Next:

$3 = 2 + 1 \in 2Q + Q = Q$

$4 = 3 + 1 \in Q + Q$, thus $4 \notin Q$

$4 = 2 + 2 \in 2Q + 2Q$, thus $4 \notin 2Q$

Consequently $4Q$ is a new class different from Q and $2Q$ and furthermore, since it belongs to $Q + Q$, it holds that $4Q < Q$. Therefore:

$4Q + Q = Q$

$5 = 4 + 1 \in 4Q + Q = Q$

$6 = 2 \cdot 3 \in 2Q \cdot Q = 2Q$

Hence, for 7, we have:

on the one hand $7 = 6 + 1 \in 2Q + Q = Q$

while, on the other hand, $7 = 4 + 3 \in Q + Q$, subsequently $7 \notin Q$.

This is a contradiction and therefore $(E, \breve{+}, \cdot)$ does not belong to the class of quotient hyperrings or hyperfields. □

Note that Theorem 12′s hypercomposition is neither open nor closed. Also, note that the above Theorem enriches the class of non-quotient hyperrings with many new members in addition to the ones it is constructing. Indeed, [35] gives a method of constructing non-quotient hyperrings when at least one non-quotient hyperfield is known. In particular, the following Theorem is valid:

Theorem 13. *Ref. [35] The direct sum of the hyperrings S_i, $i \in I$ is not isomorphic to a sub-hyperring of a quotient hyperring if at least one of the S_i is not a quotient hyperfield.*

Thus, for example, if $\mathbb{R}$ is the field of the real numbers and $\breve{\mathbb{R}}_+$ the hyperfield of Theorem 12 which is constructed over the set of the non-negative real numbers, then $\mathbb{R} \oplus \breve{\mathbb{R}}_+$ is a non-quotient hyperring.

Another class of non-quotient hyperrings was constructed by Nakassis in [36]. Nakassis' hyperrings are endowed with open hypercompositions.

4. Problems in the Theory of Fields that arose from a Question in the Theory of Hyperfields

The constructions of specific monogenic (monogène) hyperfields in the early 1980s, led directly to the hitherto open question of whether these constructions can produce non-quotient hyperfields as well [35,49,94]. It should be noted that to date they have given several hyperfields all of which are quotient [46,47,49]. Theorem 4 gives a family of such monogenic quotient hyperfields. If $x - x = H$, $x \in H^*$ is valid in a monogenic (monogène) hyperfield H which is isomorphic to a quotient hyperfield F/G, then $G - G = F$. Hence, the problem of the isomorphism of monogenic hyperfields to quotient hyperfields, simultaneously brought into being the following problem in the theory of fields:

When can a subgroup G of the multiplicative group of a field F generate F via the subtraction of G from itself?

The answer to this question for subgroups of finite fields of index 2 and 3 was given in [49]. The following Theorem presents the results of papers [47,49,95,96] collectively:

Theorem 14. *Refs. [33,48] Let F be a finite field and G be a subgroup of its multiplicative group of index n and order m. Then, G–G=F, if and only if:*

$$\begin{aligned}
&n = 2 \quad and \quad m > 2, \\
&n = 3 \quad and \quad m > 5, \\
&n = 4, \ -1 \in G \quad and \quad m > 11, \\
&n = 4, \ -1 \notin G \quad and \quad m > 3, \\
&n = 5, \ charF = 2 \quad and \quad m > 8, \\
&n = 5, \ charF = 3 \quad and \quad m > 9, \\
&n = 5, \ charF \neq 2, 3 \quad and \quad m > 23
\end{aligned}$$

Remark 1. From the above Theorem, it becomes apparent that the validity of the equality *G*–*G*=*F* depends on the cardinality of *G*. However, this does not mean that any subset *S* of the field *F* with the same cardinality as *G* has the property *S*–*S*=*F*. For example, if $F=\mathbb{Z}_{19}$, then its multiplicative subgroup of index 3, *G*={1,7,8,11,13,17} satisfies the equality *G*–*G*=*F*, while its subset *S*={1,6,8,11,13,17}, which has the same cardinality as *G*, does not. It must also be noted that *G*'s cosets have the same property as *G*.

Working with the subgroups of index 6, in light of the above Theorem, we have the following Proposition:

Proposition 7. *If G is a subgroup of index 6 of the multiplicative group of a finite field F such that* $G-G=F$ *and* $-1\notin G$*, then G has more than* 10 *elements.*

Proof. $-G$ and G have the same number of elements and $-G\cap G=\emptyset$. Moreover, $(-G)(-G)=G$. Consequently $W=-G\cup G$ is a subgroup of index 3 of the multiplicative group of *F*. Thus, by Theorem 14, *cardW*>5 and therefore *cardG*>10. □

Proposition 7 provides a very accurate result. Indeed, the field with the fewest elements which has a multiplicative subgroup of index 6 that satisfies the assumptions of the above Proposition is $\mathbb{Z}_{67}$ and this field's multiplicative subgroup of index 6 is *G* = {1,9,14,15,22,24,25,40,59,62,64}. As shown in Cayley Table 1, $G–G=\mathbb{Z}_{67}$ is valid.

Table 1. The Cayley table of the subtraction *G*–*G*.

	1	9	14	15	22	24	25	40	59	62	64
1	0	8	13	14	21	23	24	39	58	61	63
9	59	0	5	6	13	15	16	31	50	53	55
14	54	62	0	1	8	10	11	26	45	48	50
15	53	61	66	0	7	9	10	25	44	47	49
22	46	54	59	60	0	2	3	18	37	40	42
24	44	52	57	58	65	0	1	16	35	38	40
25	43	51	56	57	64	66	0	15	34	37	39
40	28	36	41	42	49	51	52	0	19	22	24
59	9	17	22	23	30	32	33	48	0	3	5
62	6	14	19	20	27	29	30	45	64	0	2
64	4	12	17	18	25	27	28	43	62	65	0

Lemma 1. *Fields of characteristic* 2 *have no multiplicative subgroups of index* 6.

Proof. The multiplicative subgroup of a field of characteristic 2 has $2^k - 1$ elements. Therefore, it is not divisible by 6, because it has an odd number of elements. □

Lemma 2. *Fields of characteristic* 3 *have no multiplicative subgroups of index* 6.

Proof. The multiplicative subgroup of a field of characteristic 3 has $3^k - 1$ elements, which is a non-multiple of number 3 and hence non-divisible by 6. □

Taking into consideration Proposition 7, Lemmas 1, 2 and applying techniques that are similar to the ones developed in [47,49,95,96], we have the Theorem:

Theorem 15. *Let F be a finite field and G be a subgroup of its multiplicative group of index* 6 *and order m. Then, G–G=F, if and only if:*

$$\begin{array}{l} -1 \notin G, \text{ and } m \geq 11, \\ -1 \in G, \ charF = 11 \ \text{ and } \ m \geq 20, \\ -1 \in G, \ charF = 13 \ \text{ and } \ m \geq 28, \\ -1 \in G, \ charF \neq 11, \ 13 \ \text{ and } \ m \geq 30. \end{array}$$

The conclusions of the above Theorem are sharp. The examples that follow are indicative of this fact.

Example 4. *The field* $GF[11^2]$ *consists of all the linear polynomials with coefficients in the field of residues modulo* 11. *In* $\mathrm{GF}[11^2]$, *the polynomial* $x^2 + 1$ *is irreducible. Thus, in the multiplication the polynomials are combined according to the ordinary rules, setting* $x^2 = -1 = 10$, *and working modulo* 11. *GF* $[11^2]$ *has the following multiplicative subgroup of index* 6,

$$G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, x, 2x, 3x, 4x, 5x, 6x, 7x, 8x, 9x, 10x\}$$

which has 20 *elements. It can be verified that G–G=GF*$[11^2]$.

Example 5. *The field GF*$[13^2]$ *consists of all the linear polynomials with coefficients in the field of residues modulo* 13. *The addition and the multiplication are defined in the usual way, replacing* x^2 *by* 11, *since the polynomial* $x^2 + 2$ *is irreducible. GF*$[13^2]$ *has the following multiplicative subgroup of index* 6,

$$G = \left\{ \begin{array}{l} 1, \ 5, \ 8, \ 12, \\ 5x+1, \ 8x+1, \ 2x+2, \ 11x+2, \ 3x+3, \ 10x+3, \ 5x+4, \ 8x+4, \ x+5, \ 12x+5, \\ x+6, \ 12x+6, \ x+7, \ 12x+7, \ x+8, \ 12x+8, \ 5x+9, \ 8x+9, \\ 3x+10, \ 10x+10, \ 2x+11, \ 11x+11, \ 5x+12, \ 8x+12 \end{array} \right\}$$

which has 28 *elements. It can be verified that G–G=GF*$[13^2]$.

Example 6. *The field* $\mathbb{Z}_{181}$*of residues modulo* 181 *has the following multiplicative subgroup of index* 6,

$$G = \left\{ \begin{array}{l} 1, \ 5, \ 25, \ 27, \ 29, \ 36, \ 42, \ 46, \ 48, \ 49, \ 56, \ 59, \ 64, \ 67, \ 82, \ 99, \\ 114, \ 117, \ 122, \ 125, \ 132, \ 133, \ 135, \ 139, \ 145, \ 152, \ 154, \ 156, \ 176, \ 180 \end{array} \right\}$$

which has 30 *elements. It can be verified that* $G - G = \mathbb{Z}_{181}$.

Similar conclusions to those of Theorem 14 for the multiplicative subgroups of index 3 have been published in [97] without however mentioning the mathematical necessity that led to this problem. The papers [98–100] also deal with this problem without proving though the clear and accurate results that are given by Theorems 14 and 15. On the other hand, in [98–100], the following Theorem is proved:

Theorem 16. *Refs.* [98–100] *If G is a subgroup of finite index in the multiplicative group of an infinite field F, then G−G=F.*

The above Theorem leads to an extension of Theorems 9 and 10. Indeed, since all finite groups are periodic, while there also exists infinite periodic groups, Theorems 9 and 10 generate finite and infinite non-quotient hyperfields. However, according to Theorem 16, if a hyperfield H is the quotient of an infinite field with a multiplicative subgroup of finite index, then $x-x=H$ for all $x\in H$. Thus, the following Theorem holds:

Theorem 17. *There do not exist finite quotient hyperfields with the hypercompositions which are defined in Theorems 9 and 10.*

Furthermore, Theorem 8 sets a new question in the theory of fields:

When can a subgroup G of the multiplicative group of a field F generate F··G via its subtraction from itself?

Example 2 presents three finite fields which have a multiplicative subgroup G possessing the above property, while the sum of any two of its cosets gives all the non-zero elements of the field F. It is worth mentioning that the rather old paper [101] investigates conditions under which the sum of two cosets of a multiplicative subgroup G of a finite field has a nonempty intersection with at least 3 cosets of G.

5. Classification of Finite Hyperfields into Quotient and Non-Quotient Hyperfields

The enumeration of certain finite hyperfields has been conducted in several papers [66,71,73,77]. Paper [66] deals with hyperfields of order less than or equal to 4, [73,77] deals with hyperfields of order less than or equal to 5, and [71] deals with hyperfields of order less than or equal to 6. In [71], R. Ameri, M. Eyvazi, and S. Hoskova-Mayerova make a thorough check of the isomorphism of these hyperfields to the quotient hyperfields using conclusions from the papers [46–48,95–97]. This section addresses the isomorphism problems with the use of the techniques which were developed from the above study, while it covers some of the gaps that appear in [71].

5.1. Hyperfields of Order 2

According to Theorem 3 there is one two-element non-trivial hyperfield, which is isomorphic to the quotient hyperfield F/F^*, where F is any field with $cardF>2$ and F^* is its multiplicative group. Hence, there exist two hyperfields of order 2, the above and $\mathbb{Z}_2$

5.2. Hyperfields of Order 3

Hyperfields of order 3 have two non-zero elements. There are five isomorphism classes of these hyperfields [66,71,73,77]. The trivial hyperfield $\mathbb{Z}_3$ is the first of them. Next, there are three hyperfields of order 3, which derive as quotients of a finite field F by an index 2 multiplicative subgroup G of its multiplicative group. According to Theorem 14, the following three cases can be valid for the subgroup G:

i. $G – G \neq F$, which applies only when $F = \mathbb{Z}_5$ and $G = \{1,4\}$

ii. $-1 \notin G$ (i.e., $\{-1, 1\} \nsubseteq G$) and $G – G = F$, which applies when

$$cardF = 2(cardG) + 1 = 2(2k + 1) + 1 = 4k + 3$$

iii. $-1\in G$ (i.e., $\{-1, 1\} \subseteq G$) and $G – G = G + G = F$, which applies when

$$cardF = 2(cardG) + 1 = 2(2k) + 1 = 4k + 1,\ k > 2$$

Therefore, there exist the corresponding three isomorphism classes of quotient hyperfields of order 3 constructed from finite fields:

i. $\mathbb{Z}_5/\{1,4\}$
ii. $GF[p^q]/G$, p^q=3(mod4)
iii. $GF[p^q]/G$, p^q=1(mod4)

The above classification can also derive as follows:

The first two classes are the field $\mathbb{Z}_3$ and its augmented hyperfield. The Cayley tables of their additive parts are shown in the following Table 2:

Table 2. The Cayley tables of the additive group of $\mathbb{Z}_3$ and of the additive canonical hypergroup of its augmented hyperfield $[\mathbb{Z}_3]$.

$\mathbb{Z}_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$[\mathbb{Z}_3]$	0	1	2
0	0	1	2
1	1	$\{1,2\}$	$\{0,1,2\}$
2	2	$\{0,1,2\}$	$\{1,2\}$

By Theorem 4, the augmented hyperfield of $\mathbb{Z}_3$ is a quotient hyperfield. Observe that $[\mathbb{Z}_3]$ is isomorphic to the quotient hyperfield $\mathbb{Z}_7/\{1,2,4\}$. More generally, the augmented hyperfield of $\mathbb{Z}_3$ is isomorphic to the quotient hyperfield $GF[p^q]$/G, p^q = 3(mod4), G being an index 2 multiplicative subgroup of the field's multiplicative group.

The next two classes are the quotient hyperfield $\mathbb{Z}_5/\{1,4\}$ and its augmented hyperfield $[\mathbb{Z}_5/\{1,4\}]$. Denoting by 1 the group G={1,4} and by a its coset 2G={2,3}, we have the following Cayley tables (Table 3) for the additive canonical hypergroups of $\mathbb{Z}_5/\{1,4\}$ and of its augmented hyperfield:

Table 3. The Cayley tables of the additive canonical hypergroups of the hyperfield $\mathbb{Z}_5/\{1,4\}$ and its augmented hyperfield $[\mathbb{Z}_5/\{1,4\}]$.

$\mathbb{Z}_5/\{1,4\}$	0	1	a
0	0	1	a
1	1	$\{0,a\}$	$\{1,a\}$
a	a	$\{1,a\}$	$\{0,1\}$

$[\mathbb{Z}_5/\{1,4\}]$	0	1	a
0	0	1	a
1	1	$\{0,1,a\}$	$\{1,a\}$
a	a	$\{1,a\}$	$\{0,1,a\}$

According to Theorem 5, the augmented hyperfield of a quotient hyperfield is a quotient hyperfield. Therefore, $[\mathbb{Z}_5/\{1,4\}]$ is a quotient hyperfield which is isomorphic to $\mathbb{Z}_{13}/\{1,3,4,9,10,12\}$. More generally, $[\mathbb{Z}_5/\{1,4\}]$ is isomorphic to the quotient hyperfield $GF[p^q]$/G, p^q = 1(mod4), G being an index 2 multiplicative subgroup of the field's multiplicative group.

The fifth and final class of the order 3 hyperfields is the quotient of an infinite field, and in particular, it is the quotient of an ordered field F by its positive cone F^+. This is the so-called «*sign hyperfield*» and the Cayley table of its canonical hypergroup is shown in Table 4:

Table 4. The Cayley table of the canonical hypergroup of the hyperfield F/F^+.

	0	−1	1
0	0	−1	1
−1	−1	−1	{−1,0,1}
1	1	{−1,0,1}	1

The above conclusions are summed up in the following Theorem:

Theorem 18. *All the hyperfields of order* 3 *are quotient hyperfields which are classified into* 5 *isomorphism classes having the following representatives:*

i. $\mathbb{Z}_3$ *and its augmented hyperfield* $[\mathbb{Z}_3]$.
ii. $\mathbb{Z}_5/\{1,4\}$ *and its augmented hyperfield* $[\mathbb{Z}_5/\{1,4\}]$.
iii. *The quotient hyperfield of an ordered field F by its positive cone* F^+.

Hence, the next Theorem holds:

Theorem 19. *All the hyperfields of order* 2 *and* 3 *are quotient hyperfields.*

5.3. Hyperfields of Order 4

There are 7 isomorphism classes of hyperfields of order 4, as they have been enumerated in [66,71,73,77]. These consist of the Galois field $GF[2^2]$, 4 classes of quotient hyperfields, and 2 classes of non-quotient hyperfields.

Note on the notation: In the subsequent paragraphs, we denote the quotient hyperfields by QHF_i^j and the non-quotient hyperfields by $NQHF_i^j$. The subscript i denotes the order of the hyperfield, while the superscript j lists the classes.

5.3.i. Quotient Hyperfields of Order 4

The first two classes are the field $GF[2^2]$ and its augmented hyperfield. Recall that, according to Theorem 4, the augmented hyperfield of $GF[2^2]$ is a quotient hyperfield. The Cayley tables of their additive parts are presented in Table 5:

Table 5. The Cayley tables of the additive group of $GF[2^2]$ and of the additive canonical hypergroup of its augmented hyperfield $[GF[2^2]]$, which is also denoted by QFH_4^1.

$GF[2^2]$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

Table 5. *Cont.*

QFH_4^1	0	1	x	x+1
0	0	1	x	x+1
1	1	$\{0,1,x,x+1\}$	$\{1,x,x+1\}$	$\{1,x,x+1\}$
x	x	$\{1,x,x+1\}$	$\{0,1,x,x+1\}$	$\{1,x,x+1\}$
x+1	x+1	$\{1,x,x+1\}$	$\{1,x,x+1\}$	$\{0,1,x,x+1\}$

Regarding their multiplicative part, the four elements are combined according to the usual rules, working modulo 2 and writing x^2 as x+1 since x^2+x+1 is the irreducible polynomial of degree 2. Therefore, Table 6 is the Cayley table of the multiplicative group of the field $GF[2^2]$ and its augmented hyperfield:

Table 6. The Cayley table of the multiplicative group of the field $GF[2^2]$ and of its augmented hyperfield $[GF[2^2]]$.

	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

We keep using Theorem 14 to examine the next two classes. So, according to Theorem 14, for the fields F with cardinality less than or equal to 16, it holds G—$G \neq F$, when G is a multiplicative subgroup of index 3. These fields are $\mathbb{Z}_7$, $\mathbb{Z}_{13}$, and $GF[2^4]$. $GF[2^4]$ is the field of all the polynomials of degree ≤ 3, with coefficients in $\mathbb{Z}_2$.

The multiplicative subgroup of index 3 in the field $\mathbb{Z}_7$ is G={1,6}, and 2G, 2^2G are its cosets. Denoting by $1, a, a^2$ the group G and its two cosets, respectively, we have the following Cayley table (Table 7) for the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_7/\{1,6\}$.

Table 7. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_7/\{1,6\}$.

QHF_4^2	0	1	a	a^2
0	0	1	a	a^2
1	1	$\{0,a\}$	$\{1,a^2\}$	$\{a,a^2\}$
a	a	$\{1,a^2\}$	$\{0,a^2\}$	$\{1,a\}$
a^2	a^2	$\{a,a^2\}$	$\{1,a\}$	$\{0,1\}$

The multiplicative subgroup of index 3 in the field $\mathbb{Z}_{13}$ is G={1,5,8,12} and 2G, 2^2G are its cosets. Denoting by $1, a, a^2$ the group G and its two cosets, respectively, we have the following Cayley Table 8 for the additive canonical hypergroups of the quotient hyperfield $\mathbb{Z}_{13}/\{1,5,8,12\}$.

Table 8. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_{13}/\{1,5,8,12\}$.

QHF_4^3	0	1	a	a^2
0	0	1	a	a^2
1	1	$\{0,a,a^2\}$	$\{1,a,a^2\}$	$\{1,a,a^2\}$
a	a	$\{1,a,a^2\}$	$\{0,1,a^2\}$	$\{1,a,a^2\}$
a^2	a^2	$\{1,a,a^2\}$	$\{1,a,a^2\}$	$\{0,1,a\}$

In the field $GF[2^4]$ of all polynomials of degree ≤ 3 with coefficients in $\mathbb{Z}_2$, the addition and the multiplication of the polynomials are defined in the usual way, by replacing x^4 with x+1, since x^4+x+1 is the irreducible polynomial of degree 4. The multiplicative subgroup of index 3 in the field $GF[2^4]$ is

$$G = \left\{1,\ x^3+x^2,\ x^3+x^2+x+1,\ x^3,\ x^3+x\ \right\}$$

and xG, x^2G are its cosets. Observe that the quotient hyperfield

$$GF\left[2^4\right]\Big/\left\{1,\ x^3+x^2,\ x^3+x^2+x+1,\ x^3,\ x^3+x\right\}$$

is isomorphic to $\mathbb{Z}_{13}/\{1,5,8,12\}$.

Notice that QHF_4^1 is the augmented hyperfield of both QHF_4^2 and QHF_4^3. Moreover, according to Theorem 14, the hyperfield QHF_4^1 is isomorphic to the quotient hyperfield of a finite field F by a subgroup of its multiplicative group of index 3, when *card* $F > 3{\cdot}5 + 1$. The hyperfield $\mathbb{Z}_{19}/\{1,7,8,11,12,18\}$ is a representative of this class of quotient hyperfields.

All the above classes of quotient hyperfields derive from the quotient of finite fields with their multiplicative subgroups, but the last one derives from an infinite field. The Cayley table of the canonical hypergroup of this hyperfield appears in Table 9:

Table 9. The Cayley table of the canonical hypergroup of the quotient hyperfield of an infinite field by a multiplicative subgroup of index 3.

QHF_4^4	0	1	a	a^2
0	0	1	a	a^2
1	1	$\{0,1,a,a^2\}$	$\{1,a\}$	$\{1,a^2\}$
a	a	$\{1,a\}$	$\{0,1,a,a^2\}$	$\{a,a^2\}$
a^2	a^2	$\{1,a^2\}$	$\{a,a^2\}$	$\{0,1,a,a^2\}$

Observe that the hyperfield QHF_4^4 is a monogenic b-hyperfield. In [46], it is proved that there exist monogenic b-hyperfields, which are quotient hyperfields. The above monogenic b-hyperfield is such a hyperfield. Indeed, as it is shown in [97], the multiplicative subgroup $G=v^{-1}(3\mathbb{Z})=\{p^{3k}v \mid k\in\mathbb{Z}$ and v is a p-adic unit$\}$ of the field $\mathbb{Q}_p$ of the p-adic numbers with p-adic valuation v, is of index 3 and $G\subseteq G+aG$, while $a^2G\not\subseteq G+aG$. Therefore, because of Proposition 2, for the quotient hyperfield $\mathbb{Q}_p/G$ it holds that xG—$xG=\mathbb{Q}_p$, $x=1,a,a^2$, and so QHF_4^4 is a quotient hyperfield.

Remark 2. In [71], it is inaccurately stated that the hyperfield QHF_4^4 is isomorphic to $GF[2^4] \Big/ \{1,\ x^3,\ x^3+x,\ x^3+x^2,\ x^3+x^2+x+1\ \}$. This is not true because, as it is shown above, this is isomorphic to QHF_4^3.

5.3.ii. Non-Quotient Hyperfields of Order 4

The non-quotient hyperfields of order 4 are presented next. Since the multiplicative group of the hyperfields of order 4 has 3 elements, Theorem 10 can be applied to construct a non-quotient hyperfield. The Cayley table of the canonical hypergroup of this hyperfield is presented in Table 10:

Table 10. The Cayley table of the additive canonical hypergroup of the non-quotient b-hyperfield with 4 elements.

$NQHF_4^1$	0	1	a	a^2
0	0	1	a	a^2
1	1	$\{0,a,a^2\}$	$\{1,a\}$	$\{1,a^2\}$
a	a	$\{1,a\}$	$\{0,1,a^2\}$	$\{a,a^2\}$
a^2	a^2	$\{1,a^2\}$	$\{a,a^2\}$	$\{0,1,a\}$

Table 11 shows the additive canonical hypergroup of the seventh hyperfield of order 4:

Table 11. The Cayley table of the additive canonical hypergroup of the non-quotient hyperfield $NQHF_4^2$.

$NQHF_4^2$	0	1	a	a^2
0	0	1	a	a^2
1	1	$\{0,1,a\}$	$\{1,a^2\}$	$\{a,a^2\}$
a	a	$\{1,a^2\}$	$\{0,a,a^2\}$	$\{1,a\}$
a^2	a^2	$\{a,a^2\}$	$\{1,a\}$	$\{0,1,a^2\}$

$NQHF_4^2$ is a non-quotient hyperfield. Indeed, having analyzed above all the cases of quotient hyperfields that derive from finite fields, we conclude that if $NQHF_4^2$ belongs to the quotient hyperfields it must originate from a quotient of an infinite field F by some multiplicative subgroup G of index 3. But in this case, G is a subgroup of finite index in the multiplicative group of the infinite field F. Therefore, by Theorem 16, the equality $G-G=F$ must hold. However, this is not true in $NQHF_4^2$. Consequently, $NQHF_4^2$ is not a quotient hyperfield.

5.4. Hyperfields of Order 5

Since the multiplicative group of finite fields is cyclic, the multiplicative group of the quotient hyperfields resulting from finite fields is cyclic as well. Therefore,

Proposition 8. *Finite hyperfields whose multiplicative part is a non-cyclic group cannot be derived from quotients of finite fields.*

Thus, the finite hyperfields whose multiplicative part is a non-cyclic group derive only from quotients of infinite fields. On the other hand, because of Theorem 14, if G is a subgroup of finite index in the multiplicative group of an infinite field F, then $G-G = F$, and therefore, if H is a finite hyperfield isomorphic to a quotient hyperfield of an infinite field F by a subgroup G of its multiplicative group, then $x-x = H$ must hold for all $x \in H^*$. Consequently, the next Theorem holds:

Theorem 20. *If the multiplicative group of a finite hyperfield H is not cyclic and x-$x \neq H$, $x \in H^*$, then H is not isomorphic to a quotient hyperfield.*

There exist two groups of order 4, both of which are Abelian. One is the cyclic group C_4 ($\cong \mathbb{Z}/4\mathbb{Z}$), and the other is F. Klein's Vierergruppe V ($\cong C_2 \times C_2$), which is not cyclic. Moreover, it is known that the multiplicative group of the finite fields is cyclic. However, this is not valid for non-trivial hyperfields. Papers [29,35,46] show how to construct hyperfields from any abelian multiplicative group. Therefore, hyperfields can be constructed from the Vierergruppe as well, and thus, the smallest hyperfield with a non-cyclic multiplicative group has 5 elements.

5.4.1. Hyperfields with the Vierergruppe as Their Multiplicative Group

In [71], it has been shown that there exist 11 hyperfields whose multiplicative group is the Vierergruppe. Recall that the Cayley table of the Vierergruppe is the following Table 12:

Table 12. The Cayley table of the Vierergruppe.

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

As the Vierergruppe is not a cyclic group, the next Corollary follows from the above Theorem 20:

Corollary 1. *If the multiplicative part of a hyperfield H is the Vierergruppe and if $x-x \neq H$, $x \in H^*$, then H is a non-quotient hyperfield.*

By Corollary 1, among the 11 hyperfields whose multiplicative part is the Vierergruppe, the following 4, which are shown in Table 13, are non-quotient hyperfields.

Table 13. The Cayley tables of the additive canonical hypergroups of the non-quotient hyperfields whose multiplicative group is the Vierergruppe.

$NQHF_5^1$	0	1	a	b	c
0	0	1	a	b	c
1	1	{0, 1}	{b, c}	{a, c}	{a, b}
a	a	{b, c}	{0, a}	{1, c}	{1, b}
b	b	{a, c}	{1, c}	{0, b}	{1, a}
c	c	{a, b}	{1, b}	{1, a}	{0, c}

$NQHF_5^2$	0	1	a	b	c
0	0	1	a	b	c
1	1	{a, b, c}	{1, a, b, c}	{1, a, b, c}	{0, a, b}
a	a	{1, a, b, c}	{1, b, c}	{0, 1, c}	{1, a, b, c}
b	b	{1, a, b, c}	{0, 1, c}	{1, a, c}	{1, a, b, c}
c	c	{0, a, b}	{1, a, b, c}	{1, a, b, c}	{1, a, b}

$NQHF_5^3$	0	1	a	b	c
0	0	1	a	b	c
1	1	{0, a, b, c}	{1, a, b, c}	{1, a, b, c}	{1, a, b, c}
a	a	{1, a, b, c}	{0, 1, b, c}	{1, a, b, c}	{1, a, b, c}
b	b	{1, a, b, c}	{1, a, b, c}	{0, 1, a, c}	{1, a, b, c}
c	c	{1, a, b, c}	{1, a, b, c}	{1, a, b, c}	{0, 1, a, b}

$NQHF_5^4$	0	1	a	b	c
0	0	1	a	b	c
1	1	{0, a, b, c}	{1, a}	{1, b}	{1, c}
a	a	{1, a}	{0, 1, b, c}	{a, b}	{a, c}
b	b	{1, b}	{a, b}	{0, 1, a, c}	{b, c}
c	c	{1, c}	{a, c}	{b, c}	{0, 1, a, b}

The following is an alternative proof that the hyperfields $NQHF_5^2$ and $NQHF_5^4$ are non-quotient hyperfields, which is not based on Corollary 1. Indeed:

(α) For $NQHF_5^2$ observe that the opposite of 1 is c, the opposite of a is b and moreover that:

$$(1+c)\cap(a+b)=\{0,\ a,\ b\}\cap\{0,\ 1,\ c\}=\{0\}$$

Therefore, according to Lemma 1, if $NQHF_5^2$ were isomorphic to a quotient hyperfield, then the sums of any two non-opposite elements should have the same cardinality. However, this is not the case because, for example:

$$card(1+a)=4 \quad \text{while} \quad card(1+1)=3.$$

(β) For $NQHF_5^4$ observe that it is the hyperfield constructed via Theorem 10, when the Vierergruppe is used. Since the Vierergruppe is periodic, Theorem 10 implies that the hyperfield $NQHF_5^4$ cannot be isomorphic to a quotient hyperfield.

The classification of the remaining 7 hyperfields that appear in [71] is a hitherto open problem, and it also raises the question of whether there exist quotient hyperfields that have the Vierergruppe as their multiplicative group. It is worth mentioning here that the hypercompositions in all 7 unclassified hyperfields are closed, and so $x-x$ contains all the elements of the hyperfield for each x in the Vierergruppe.

5.4.2. Hyperfields Having as Multiplicative Group the Cyclic Group C_4

In [71], it is shown that there exist 16 hyperfields whose multiplicative group is the cyclic group C_4. Some of them have been identified as quotient hyperfields. Their classification is completed in the following, starting with the quotient hyperfields.

5.4.2.i. Quotient Hyperfields with Multiplicative Group Being the Cyclic Group C_4

We begin with the field $\mathbb{Z}_5$ and then we continue with the quotient hyperfields of finite fields, along with their augmented hyperfields which, according to Theorems 4 and 5, are quotient hyperfields as well (Table 14).

Table 14. The Cayley tables of the additive group of $\mathbb{Z}_5$ and of the canonical hypergroup of its augmented hyperfield $[\mathbb{Z}_5]$, which is also denoted by QHF_5^1.

$\mathbb{Z}_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 14. *Cont.*

QHF_5^1	0	1	2	3	4
0	0	1	2	3	4
1	1	{2, 1}	{1, 2, 3}	{1, 3, 4}	{0, 1, 2, 3, 4}
2	2	{1, 2, 3}	{2, 4}	{0, 1, 2, 3, 4}	{2, 4, 1}
3	3	{1, 3, 4}	{0, 1, 2, 3, 4}	{1, 3}	{3, 4, 2}
4	4	{0, 1, 2, 3, 4}	{2, 4, 1}	{3, 4, 2}	{4, 3}

Next, G = {1,3,9} is the multiplicative subgroup of order 4 of the field $\mathbb{Z}_{13}$ and $2G$, 2^2G, 2^3G, are its cosets. Denoting by 1, a, a^2, a^3 the group G and its cosets, respectively, Table 15 gives the Cayley tables for the additive canonical hypergroups of $\mathbb{Z}_{13}/\mathrm{G}$ and of its augmented hyperfield:

Table 15. The Cayley tables of the additive canonical hypergroup of the hyperfield $\mathbb{Z}_{13}/G$, which is also denoted by QHF_5^2 and of its augmented hyperfield $[\mathbb{Z}_{13}/G]$, which is also denoted by QHF_5^3.

QHF_5^2	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{a,a^2\}$	$\{1,a,a^3\}$	$\{0,a,a^3\}$	$\{1,a^2,a^3\}$
a	a	$\{1,a,a^3\}$	$\{a^2,a^3\}$	$\{1,a,a^2\}$	$\{0,1,a^2\}$
a^2	a^2	$\{0,a,a^3\}$	$\{1,a,a^2\}$	$\{1,a^3\}$	$\{a,a^2,a^3\}$
a^3	a^3	$\{1,a^2,a^3\}$	$\{0,1,a^2\}$	$\{a,a^2,a^3\}$	$\{1,a\}$

QHF_5^3	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{1,a,a^2\}$	$\{1,a,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a^2,a^3\}$
a	a	$\{1,a,a^3\}$	$\{a,a^2,a^3\}$	$\{1,a,a^2\}$	$\{0,1,a,a^2,a^3\}$
a^2	a^2	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2\}$	$\{1,a^2,a^3\}$	$\{a,a^2,a^3\}$
a^3	a^3	$\{1,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{a,a^2,a^3\}$	$\{1,a,a^3\}$

Theorem 14 will continue to be used for the classification of the next classes of quotient hyperfields of order 5. Thus, in addition to the above, the fields with cardinality less than or equal to 4·11+1=45 are the following ones:

$$GF[3^2],\ GF[5^2],\ \mathbb{Z}_{17},\ \mathbb{Z}_{29},\ \mathbb{Z}_{37},\ \text{and}\ \mathbb{Z}_{41}.$$

$GF[3^2]$ consists of the 9 polynomials in x of degree 0 or 1 with coefficients in the field $\mathbb{Z}_3$ and writing x^2 as 2 whenever it occurs. G = {1,2} is the multiplicative subgroup of index 4 in the field $GF[3^2]$. The hyperfield $GF[3^2]/\mathrm{G}$ is the following one:

$$GF\left[3^2\right]\Big/G = \{G,\ xG,\ (x+1)G,\ (x+2)G\} = \left\{ (x+1)^k G \,\middle|\, k = 0,1,2,3 \right\}$$

Denoting the coset (x+1)G by a and G by 1, the additive canonical hypergroup of the $GF[3^2]/\mathrm{G}$ is shown in Table 16:

Table 16. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $GF[3^2]/\mathrm{G}$, which is also denoted by QHF_5^4.

QHF_5^4	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0,1\}$	$\{a^2,a^3\}$	$\{a,a^3\}$	$\{a,a^2\}$
a	a	$\{a^2,a^3\}$	$\{0,a\}$	$\{1,a^3\}$	$\{1,a\}$
a^2	a^2	$\{a,a^3\}$	$\{1,a^3\}$	$\{0,a^2\}$	$\{1,a\}$
a^3	a^3	$\{a,a^2\}$	$\{1,a^2\}$	$\{1,a\}$	$\{0,a^3\}$

$GF[5^2]$ consists of the 25 polynomials in x of degree 0 or 1 with coefficients in the field $\mathbb{Z}_5$. Since x^2+3x+4 is the irreducible polynomial of degree 2 we are writing x^2 as $-$ 3$x-$ 4 = 2x + 1 whenever it occurs. G = {1, 4, 2x, 3x+4, 3x, 2x+1} is the multiplicative subgroup of index 4 in the field $GF[5^2]$. The hyperfield $GF[5^2]/\mathrm{G}$ is the following:

$$GF\left[5^2\right]\Big/G\ =\ \{G,\ 2G,\ (x+1)G,\ (2x+2)G\}\ =\ \left\{ (x+1)^k G \,\middle|\, k = 0,1,2,3 \right\}$$

Denoting the coset (x+1)G by a and G by 1, the Cayley table for the additive canonical hypergroup of the $GF[5^2]/\mathrm{G}$ is presented in Table 17:

Table 17. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $GF[5^2]/\mathrm{G}$, which is also denoted by QHF_5^5.

QHF_5^5	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0,1,a^2,a^3\}$	$\{a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^2\}$
a	a	$\{a,a^2,a^3\}$	$\{0,1,a,a^3\}$	$\{1,a^2,a^3\}$	$\{1,a,a^2,a^3\}$
a^2	a^2	$\{1,a,a^2,a^3\}$	$\{1,a^2,a^3\}$	$\{0,1,a,a^2\}$	$\{1,a,a^3\}$
a^3	a^3	$\{1,a,a^2\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^3\}$	$\{0,a,a^2,a^3\}$

The multiplicative subgroup of index 4 in the field $\mathbb{Z}_{17}$ is G = {1,4,13,16} and $5G$, 5^2G, 5^3G, are its cosets. Denoting by 1, a, a^2, a^3 the group G and its three cosets, respectively, we have the following Cayley Table 18 for the additive canonical hypergroups of the quotient hyperfield $\mathbb{Z}_{17}/\mathrm{G}$:

Table 18. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_{17}/\{1,4,13,16\}$, which is also denoted by QHF_5^6.

QHF_5^6	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0,a,a^2\}$	$\{1,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{a,a^2,a^3\}$
a	a	$\{1,a^2,a^3\}$	$\{0,a^2,a^3\}$	$\{1,a,a^3\}$	$\{1,a,a^2,a^3\}$
a^2	a^2	$\{1,a,a^2,a^3\}$	$\{1,a,a^3\}$	$\{0,1,a^3\}$	$\{1,a,a^2\}$
a^3	a^3	$\{a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^2\}$	$\{0,1,a\}$

Notice that the hyperfields QHF_5^4, QHF_5^5 and QHF_5^6 have the same augmented hyperfield QHF_5^7. Because of Theorem 4, this hyperfield is a quotient hyperfield. Furthermore, it can be verified that this hyperfield is isomorphic to the quotient hyperfield $\mathbb{Z}_{53}/\{1,10,13,15,16,24,28,36,42,44,46,47,49\}$. The Cayley table of the additive canonical hypergroup of this hyperfield appears in Table 19:

Table 19. The Cayley table of the additive canonical hypergroup of the augmented hyperfield of QHF_5^4, QHF_5^5 and QHF_5^6 which is simultaneously the additive hypergroup of the quotient hyperfield $\mathbb{Z}_{53}$/{1,10,13,15,16,24,28,36,42,44,46,47,49}.

QHF_5^7	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$
a	a	$\{1,a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$
a^2	a^2	$\{1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$
a^3	a^3	$\{1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$

The multiplicative subgroup of index 4 in the field $\mathbb{Z}_{29}$ is G = {1,7,16,20,23,24,25} and $2G$, 2^2G, 2^3G, are its cosets. Denoting by 1, a, a^2, a^3 the group G and its three cosets, respectively, we have the following Cayley Table 20 for the additive canonical hypergroups of the quotient hyperfield $\mathbb{Z}_{29}$/G:

Table 20. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_{29}$/{1,7,16,20,23,24,25}.

QHF_5^8	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{1,a,a^3\}$	$\{1,a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$
a	a	$\{1,a,a^2,a^3\}$	$\{1,a,a^2\}$	$\{1,a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$
a^2	a^2	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$
a^3	a^3	$\{1,a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a,a^2,a^3\}$	$\{1,a^2,a^3\}$

G={1,7,9,10,12,16,26,33,34} is the multiplicative subgroup of index 4 in the field $\mathbb{Z}_{37}$ and $2G$, 2^2G, 2^3G, are its cosets. Denoting by 1, a, a^2, a^3 the group G and its three cosets, respectively, we have the following Cayley Table 21 for the additive canonical hypergroups of the quotient hyperfield $\mathbb{Z}_{37}$/G:

Table 21. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_{37}/\{1,7,9,10,12,16,26,33,34\}$.

QHF_5^9	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{0, 1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$
a	a	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{0, 1, a, a^2, a^3\}$
a^2	a^2	$\{0, 1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$
a^3	a^3	$\{1, a, a^2, a^3\}$	$\{0, 1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$

G={1,4,10,16,18,23,25,31,37,40} is the multiplicative subgroup of index 4 in the field $\mathbb{Z}_{41}$ and $3G$, 3^2G, 3^3G, are its cosets. Denoting by $1, a, a^2, a^3$ the group G and its three cosets, respectively, we have the following Cayley Table 22 for the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_{41}/\mathrm{G}$:

Table 22. The Cayley table of the additive canonical hypergroup of the quotient hyperfield $\mathbb{Z}_{41}/\{1,4,10,16,18,23,25,31,37,40\}$.

QHF_5^{10}	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$
a	a	$\{1, a, a^2, a^3\}$	$\{0, 1, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$
a^2	a^2	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{0, 1, a, a^3\}$	$\{1, a, a^2, a^3\}$
a^3	a^3	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{0, 1, a, a^2\}$

5.4.2.ii. Non-Quotient Hyperfields with Multiplicative Group Being the Cyclic Group C_4

The first non-quotient hyperfield can be constructed via Theorem 10. The Cayley Table 23 presents its additive canonical hypergroup:

Table 23. The Cayley table of the additive canonical hypergroup of the non-quotient hyperfield constructed via Theorem 10.

$NQHF_5^1$	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0, a, a^2, a^3\}$	$\{1, a\}$	$\{1, a^2\}$	$\{1, a^3\}$
a	a	$\{1, a\}$	$\{0, 1, a^2, a^3\}$	$\{a, a^2\}$	$\{a, a^3\}$
a^2	a^2	$\{1, a^2\}$	$\{a, a^2\}$	$\{0, 1, a, a^3\}$	$\{a^2, a^3\}$
a^3	a^3	$\{1, a^3\}$	$\{a, a^3\}$	$\{a^2, a^3\}$	$\{0, 1, a, a^2\}$

Cayley Table 24 presents the additive canonical hypergroup of the second non-quotient hyperfield:

Table 24. The Cayley table of the additive canonical hypergroup of the second non-quotient hyperfield.

$NQHF_5^2$	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{a, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{0, a, a^3\}$	$\{1, a, a^2, a^3\}$
a	a	$\{1, a, a^2, a^3\}$	$\{1, a^2, a^3\}$	$\{1, a, a^2, a^3\}$	$\{0, 1, a^2\}$
a^2	a^2	$\{0, a, a^3\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^3\}$	$\{1, a, a^2, a^3\}$
a^3	a^3	$\{1, a, a^2, a^3\}$	$\{0, 1, a^2\}$	$\{1, a, a^2, a^3\}$	$\{1, a, a^2\}$

We will prove that $NQHF_5^2$ is a non-quotient hyperfield. Note that the opposite of 1 is a^2, the opposite of a is a^3 and that:

$$\left(1 + a^2\right) \cap \left(a + a^3\right) = \left\{0,\ a,\ a^3\right\} \cap \left\{0,\ 1,\ a^2\right\} = \{0\}$$

Therefore, according to Lemma 1, if $NQHF_5^2$ were isomorphic to a quotient hyperfield, then the sums of any two non-opposite elements should have had the same cardinality. However, this is not the case because, for example,

$$card(1{+}a){=}4 \quad \text{while} \quad card(1{+}1){=}3.$$

5.4.2.iii. Non-Classified Hyperfields Having as Multiplicative Group the Cyclic Group C_4

There remain three hyperfields whose multiplicative group is the cyclic group C_4. For these hyperfields the hypercompositions are defined as shown in Table 25:

Table 25. The Cayley tables of the additive canonical hypergroup of the three non-classified hyperfields with multiplicative group C_4.

HF_5^1	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{0,1,a,a^2,a^3\}$	$\{1,a\}$	$\{1,a^2\}$	$\{1,a^3\}$
a	a	$\{1,a\}$	$\{0,1,a,a^2,a^3\}$	$\{a,a^2\}$	$\{a,a^3\}$
a^2	a^2	$\{1,a^2\}$	$\{a,a^2\}$	$\{0,1,a,a^2,a^3\}$	$\{a,a^2,a^3\}$
a^3	a^3	$\{1,a^3\}$	$\{a,a^3\}$	$\{a,a^2,a^3\}$	$\{0,1,a,a^2,a^3\}$

HF_5^2	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	1	$\{1,a\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a^3\}$
a	a	$\{1,a\}$	a	$\{a,a^2\}$	$\{0,1,a,a^2,a^3\}$
a^2	a^2	$\{0,1,a,a^2,a^3\}$	$\{a,a^2\}$	a^2	$\{a^2,a^3\}$
a^3	a^3	$\{1,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{a^2,a^3\}$	a^3

HF_5^3	0	1	a	a^2	a^3
0	0	1	a	a^2	a^3
1	1	$\{1,a^2\}$	$\{1,a\}$	$\{0,1,a,a^2,a^3\}$	$\{1,a^3\}$
a	a	$\{1,a\}$	$\{a,a^3\}$	$\{a,a^2\}$	$\{0,1,a,a^2,a^3\}$
a^2	a^2	$\{0,1,a,a^2,a^3\}$	$\{a,a^2\}$	$\{1,a^2\}$	$\{a^2,a^3\}$
a^3	a^3	$\{1,a^3\}$	$\{0,1,a,a^2,a^3\}$	$\{a^2,a^3\}$	$\{a,a^3\}$

From the analysis and conclusions of the previous section, it follows that the above three hyperfields cannot be derived as a quotient of finite fields by subgroups of their multiplicative group. Thus, the question of whether they are isomorphic or not to quotient hyperfields of non-finite fields by multiplicative subgroups of index 4, still remains open.

6. Discussion

Marc Krasner introduced the hyperfield in 1956, and until 1983, no hyperfields other than the residuals ones were known in the wider mathematical society, regardless of the fact that Krasner had made his associates aware of the construction of the quotient hyperfields and hyperrings, which generalize the residual hyperfields. The criticism that he received was that if all hyperrings and hyperfields could be isomorphically embedded into the quotient hyperrings, then several conclusions of their theory would have been reached in a very straightforward manner, with the use of the theories of rings, fields, and modules, and it wouldn't have been necessary to develop new techniques, methods and methodologies for their proofs. Thus, in 1983, M. Krasner published the construction of the quotient hyperfields and hyperrings and raised the questions [24]:

Are all hyperrings which are not rings isomorphic to the subhyperrings of quotient hyperrings R/G of some ring R by some of its normal multiplicative subgroups G when they are not rings? Are all hyperfields isomorphic to a quotient K/G of a field K by some of its multiplicative subgroups G?

Negative answers to these questions first came from the works in [29,35] and then from [36,47], while Theorem 12 also constructs a new class of non-quotient hyperrings and hyperfields. The constructions thought of certain hyperfields which were introduced for answering Krasner's questions gave rise to the following problem in field theory:

When does a subgroup G of the multiplicative group of a field F possess the ability to generate F via the subtraction of G from itself?

So far, we do not have a clear and complete general solution to this problem. In the finite fields, we have sharp conclusions for the subgroups of indexes 2,3,4,5,6, as described in Theorems 14 and 15. Moreover, the construction of new hyperfields (Theorem 8) and the research on whether they belong to the quotient hyperfields introduced a new problem in the theory of fields:

Under what conditions can a field F's multiplicative subgroup G generate F··G via the subtraction of G from itself?

The question of the classification of hyperfields arose naturally as a follow-up to Krasner's question, and the Table 26 below summarizes the results of the classification of finite hyperfields with 2, 3, 4, 5 elements.

Table 26. Classification of the hyperfields of order 2,3,4,5.

Order of Hyperfields	Number of Hyperfields with Cyclic Multiplicative Subgroup	Number of Hyperfields with Non-Cyclic Multiplicative Subgroup	Fields	Quotient Hyperfields	Non-Quotient Hyperfields	Unclassified Hyperfields
2	2	–	1	1	–	–
3	5	–	1	4	–	–
4	7	–	1	4	2	–
5	–	11	–	–	4	7
	16	–	1	10	2	3

Evidently, the classification of the 10 unclassified finite hyperfields remains an open problem. For the infinite non-quotient hyperfields, note that besides the construction of finite non-quotient hyperfields, Theorems 9 and 10 give the construction of infinite non-quotient hyperfields as well. Additionally, Theorem 12 presents the construction of a class of such hyperfields. Evident examples of infinite quotient hyperfields are $\mathbb{R}/\mathbb{Q}^*$, $\mathbb{R}/\mathbb{Q}^+$, $\mathbb{C}/\mathbb{Q}^*$, $\mathbb{C}/\mathbb{Q}^+$, $\mathbb{C}/\mathbb{R}^*$, $\mathbb{C}/\mathbb{R}^+$ etc.

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