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Review

An Overview of the Foundations of the Hypergroup Theory

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Abstract: This paper is written in the framework of the Special Issue of *Mathematics* entitled "Hypercompositional Algebra and Applications", and focuses on the presentation of the essential principles of the hypergroup, which is the prominent structure of hypercompositional algebra. In the beginning, it reveals the structural relation between two fundamental entities of abstract algebra, the group and the hypergroup. Next, it presents the several types of hypergroups, which derive from the enrichment of the hypergroup with additional axioms besides the ones it was initially equipped with, along with their fundamental properties. Furthermore, it analyzes and studies the various subhypergroups that can be defined in hypergroups in combination with their ability to decompose the hypergroups into cosets. The exploration of this far-reaching concept highlights the particularity of the hypergroup theory versus the abstract group theory, and demonstrates the different techniques and special tools that must be developed in order to achieve results on hypercompositional algebra.

Keywords: group; hypergroup; subhypergroup; cosets

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1. Introduction

The early years of the 20th century brought the end of determinism and certainty to science. The emergence of quantum mechanics rocked the well-being of classical mechanics, which was founded by Isaac Newton in *Philosophiæ Naturalis Principia Mathematica*. In 1927, Werner Heisenberg developed his uncertainty principle while working on the mathematical foundations of quantum mechanics. On the other hand, in 1931 Kurt Gödel published his two incompleteness theorems, thus giving an end to David Hilbert's mathematical dreams and to the attempts that are culminating with *Principia Mathematica* of Bertrand Russell. In 1933, Andrey Kolmogorov published his book *Foundations of the Theory of Probability*, establishing the modern axiomatic foundations of probability theory. In the same decade the uncertainty invaded algebra as well. A young French mathematician, Frédéric Marty (1911–1940), during the 8th Congress of Scandinavian Mathematicians, held in Stockholm in 1934, introduced an algebraic structure in which the rule of synthesizing elements results to a set of elements instead of a single element. He called this structure hypergroup. Marty was killed at the age of 29, when his airplane was hit over the Baltic Sea, while he was in the military during World War II. His mathematical heritage on hypergroups was only three papers [1–3]. However, other mathematicians such as M. Krasner [4–8], J. Kuntzmann [8–10], H. Wall [11], O. Ore [12–14], M. Dresher [13], E. J. Eaton [14,15], and L. W. Griffiths [16] gradually started working on hypergroups shortly thereafter (see the classical book [17] for further bibliography). Thus, hypercompositional algebra came into being.

Hypercompositional algebra is the branch of abstract algebra that deals with structures equipped with multivalued operations. Multivalued operations, also called hyperoperations or hypercompositions, are laws of synthesis of the elements of a nonempty set, which associates a set of elements, instead of a single element, to every pair of elements.

The fundamental structure of hypercompositional algebra is the hypergroup. This paper enlightens the structural relation between the groups and the hypergroups. The study of such relationships is at the heart of structuralism. Structuralism is based on the idea that the elements of a system under study are not important, and only the relationships and structures among them are significant. As it is proved in this paper, the axioms of groups and hypergroups are the same, while these algebraic entities' difference is based on the relationship between their elements, which is created by the law of synthesis. In groups, the law of synthesis of any two elements is a composition, i.e., a single element, while in hypergroups it is a hypercomposition, that is, a set of elements.

The next section of this paper generalizes the notion of magma, which was introduced in *Éléments de Mathématique, Algèbre* [18] by Nicolas Bourbaki, and so will include algebraic structures with hypercomposition. The third section presents a unified definition of the group and the hypergroup. This definition of the group is not included in any group theory book, and its equivalence to the already-known ones is proved in the fourth section. The fifth section presents another, equivalent definition of the hypergroup, while certain of its fundamental properties are proved. As these properties derive directly from the axioms of the hypergroup, they outline the strength of these axioms. So, for instance it is shown that the dominant in the bibliography definition of the hypercomposition includes redundant assumptions. The restriction that a hypercomposition is a mapping from $E \times E$ into the family of nonempty subsets of E is needless, since, in the hypergroups, the result of the hypercomposition is proved to be always a nonvoid set. The sixth section deals with different types of hypergroups. The law of synthesis imposes a generality on the hypergroup, which allows its enrichment with more axioms. This creates a multitude of special hypergroups with many and interesting properties and applications. The join space is one of them. It was introduced by W. Prenowitz in order to study geometry with the tools of hypercompositional algebra, and many other researchers adopted this approach [19–34]. Another one is the fortified join hypergroup, which was introduced by G. Massouros in his study of the theory of formal languages and automata [35–42], and he was followed by other authors who continued in this direction e.g., [42–52]. One more is the canonical hypergroup, which is the additive part of the hyperfield that was used by M. Krasner as the proper algebraic tool in order to define a certain approximation of complete valued fields by sequences of such fields [53]. This hypergroup was used in the study of geometry as well e.g., [32,33,54–60]. Moreover, the canonical hypergroup became part of other hypercompositional structures like the hypermodule [61] and the vector hyperspace [62]. In [61], it is shown that analytic projective geometries and Euclidean spherical geometries can be considered as special hypermodules. Furthermore, the hyperfields were connected to the conic sections via a number of papers [63–65], where the definition of an elliptic curve over a field F was naturally extended to the definition of an elliptic hypercurve over a quotient Krasner's hyperfield. The conclusions obtained in [63–65] can be applied to cryptography as well. Moreover, D. Freni in [66] extended the use of the hypergroup in more general geometric structures, called geometric spaces; [67] contains a detailed presentation of the above. Also, hypergroups are used in many other research areas, like the ones mentioned in [68], and recently, in social sciences [69–73] and in an algebraization of logical systems [74,75]. The seventh section refers to subhypergroups. A far-reaching concept of abstract group theory is the idea of the decomposition of a group into cosets by any of its subgroups. This concept becomes much more complicated in the case of hypergroups. The decomposition of the hypergroups cannot be dealt with in a similar uniform way as in the groups. So, in this section, and depending on its specific type, the decomposition of a hypergroup to cosets is treated with the use of invertible, closed, reflexive, or symmetric subhypergroups.

Special notation: In the following, in addition to the typical algebraic notations, we use Krasner's notation for the complement and difference. So, we denote with $A \cdot\cdot B$ the set of elements that are in the set A, but not in the set B.

2. Magma

In *Éléments de Mathématique, Algèbre* [18], Nicolas Bourbaki used the Greek word *magma*, which comes from the verb $\mu\acute{\alpha}\sigma\sigma\omega$ (= "knead"), to indicate a set with a law of composition. The following definition extends this notion in order to incorporate more general laws of synthesizing the elements in a set.

Definition 1. *Let E be a nonvoid set. A mapping from $E \times E$ into E is called a composition on E and a mapping from $E \times E$ into the power set $P(E)$ of E is called a hypercomposition on E. A set with a composition or a hypercomposition is called a magma.*

The notation $(E, \perp)$, where $\perp$ is the composition or the hypercomposition, is used when it is required to write the law of synthesis in a magma. The image $\perp(x, y)$ of (x, y) is written $x \perp y$. The symbols + and $\cdot$ are the most commonly used instead of $\perp$. A law of synthesis denoted by the symbol + is called *addition* and $x + y$ is called the *sum* of x and y if the synthesis is a composition, and the *hypersum* of x and y if the synthesis is a hypercomposition. A law of synthesis denoted by the symbol $\cdot$ is called *multiplication*, and $x \cdot y$ is called the *product* of x and y if the synthesis is a composition and the *hyperproduct* of x and y if the synthesis is a hypercomposition; when there is no likelihood of confusion, the symbol $\cdot$ can be omitted and we write xy instead of $x \cdot y$.

Example 1. *The power set $P(E)$ of a set $E \neq \varnothing$ is a magma if*

$$(X, Y) \to X \cup Y \text{ or } (X, Y) \to X \cap Y.$$

Example 2. *The set $\mathbb{N}$ of natural numbers is a magma under addition or multiplication.*

Example 3. *A nonvoid set E becomes a magma under the following law of synthesis:*

$$(x, y) \to \{x, y\}$$

The above law of synthesis is called b-hypercomposition. E also becomes a magma if

$$(x, y) \to E.$$

This law of synthesis is called total hypercomposition.

Two significant types of hypercompositions are the closed and the open ones. A hypercomposition is called *closed* [76] (or *containing* [77], or *extensive* [78]) if the two participating elements always belong to the result of the hypercomposition, while it is called *open* if the result of the hypercomposition of any two different elements does not contain these two elements.

Example 4. *Let S be the set of points of a Euclidian geometry. In S we define the following law of synthesis "$\cdot$": if a and b are distinct points of S, then $a \cdot b$ is the set of all elements of the segment ab; while $a \cdot a$ is taken to be the point a, for any point a of S (Figure 1). Then, the set S of the points of the Euclidian geometry becomes a magma. Usually, $a \cdot b$ is written simply as ab and it is called the join of a and b. It is worth noting that we can actually define two laws of synthesis: an open and a closed hypercomposition, depending on whether we consider the open or the closed segment ab.*

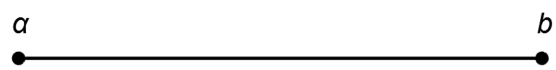

Figure 1. Example 4.

Any finite magma E can be explicitly defined by its synthesis table. If E consists of n elements, then the synthesis table is a $n \times n$ square array heading both to the left and above by a list of the n elements of E. In this table (Cayley table), the entry in the row headed by x and the column headed by y is the synthesis $x \perp y$.

Example 5. *Suppose that* $E = \{1, 2, 3, 4\}$. *Then the law of the synthesis is a composition in the first table and a hypercomposition in the second.*

$\perp$	1	2	3	4
1	1	3	2	3
2	3	2	1	4
3	1	2	1	1
4	2	3	4	1

$\perp$	1	2	3	4
1	{1,2,3}	{1,3}	{2,4}	{1,3,4}
2	{3}	{1,2,3,4}	{1,3}	{2,4}
3	{1,2,3}	{1,2}	{1,2,3,4}	{1}
4	{2,3}	{2,3}	{3,4}	{1,3}

Let $(E, \perp)$ be a magma. Given any two nonvoid subsets X, Y of E, then

$$X \perp Y = \{x \perp y \in E \mid x \in X,\ y \in Y\}, \quad \text{if } \perp \text{ is a composition}$$

and

$$X \perp Y = \bigcup_{x \in X,\ y \in Y} (x \perp y), \quad \text{if } \perp \text{ is a hypercomposition.}$$

If X or Y is empty, then $X \perp Y$ is empty. If $a \in E$ we usually write $a \perp Y$ instead of $\{a\} \perp Y$ and $X \perp a$ instead of $X \perp \{a\}$. In general, the singleton $\{a\}$ is identified with its member a. Sometimes it is convenient to use the relational notation $A \approx B$ to assert that subsets A and B have a nonvoid intersection. Then, as the singleton $\{a\}$ is identified with its member a, the notation $a \approx A$ or $A \approx a$ is used as a substitute for $a \in A$. The relation $\approx$ may be considered as a weak generalization of equality, since, if A and B are singletons and $A \approx B$, then $A = B$. Thus, $a \approx b \perp c$ means $a = b \perp c$, if the synthesis is a composition and $a \in b \perp c$, if the synthesis is a hypercomposition.

Definition 2. *Let* $(E, \perp)$ *be a magma. The law of synthesis*

$$(x, y) \to x \perp^{op} y = y \perp x$$

is called the opposite of $\perp$. *The magma* $(E, \perp^{op})$ *is called the opposite magma of* $(E, \perp)$. *When* $\perp^{op} = \perp$, *the law of synthesis is called commutative and the magma is called commutative magma.*

Definition 3. *Let E be a magma and X a subset of E. The set of elements of E that commute with each one of the elements of X is called the centralizer of X. The centralizer of E is called the center of E. An element of the center of E is called the central element of E.*

Every law of synthesis in a magma induces two new laws of synthesis. If the law of synthesis is written multiplicatively, then the two induced laws are:

$$a/b = \{x \in E \mid a \approx xb\} \text{ and } b\backslash a = \{x \in E \mid a \approx bx\}.$$

Thus, $x \approx a/b$ if and only if $a \approx xb$ and $x \approx b \backslash a$ if and only if $a \approx bx$. In the case of a multiplicative magma, the two induced laws are named *inverse laws* and they are called the *right* and *left division*, respectively. It is obvious that if the magma is commutative, then the right and left divisions coincide.

Proposition 1. *If the law of synthesis in a magma* $(E, \cdot)$ *is an open hypercomposition, then* $a/a = a\backslash a = a$ *for all* a *in* E*, while* $a/a = a\backslash a = E$ *for all* a *in* E*, when the law of synthesis is a closed (containing) hypercomposition.*

Example 6. *On the set* $\mathbb{Q}$ *of rational numbers, multiplication is a commutative law of synthesis. The inverse law is as follows:*

$$a/b = \begin{cases} \frac{a}{b} & if \;\; b \neq 0 \\ \varnothing & if \;\; b = 0,\; a \neq 0 \\ \mathbb{Q} & if \;\; b = 0,\; a = 0 \end{cases}$$

Here, the law of synthesis is a composition and the inverse law is a hypercomposition.

Example 7. *In Example 4, a law of synthesis was defined on the set* S *of the points of a Euclidian geometry. If we consider the open hypercomposition, then the inverse law is the following: If* a *and* b *are two distinct points in* S*, then* a/b *is the set of the points of the open halfline with endpoint* a*, that is opposite to point* b*, while* $a/a = a$*, for any point* a *of* S *(Figure 2). Usually, this law is called the extension of* a *over* b*, or "*a *from* b*".*

Figure 2. Example 7.

Example 8. *Let* $(E, \cdot)$ *be a magma and let* $/$ *and* $\backslash$ *be the right and left division. A new law of synthesis, called the extensive enlargement of "*$\cdot$*", can be defined on* E *as follows:*

$$a \odot b = a \cdot b \cup \{a, b\}, for\; all\; a, b \in E$$

Denoting the two induced laws of synthesis by ${}_{\odot}/$ *and* $\backslash_{\odot}$*, it is immediate that:*

$$a \,{}_{\odot}/\, b = \begin{cases} a/b \cup \{a\}, & if\; a \neq b \\ E, & if\; a = b \end{cases} \quad and \quad b \backslash_{\odot} a = \begin{cases} b\backslash a \cup \{a\}, & if\; a \neq b \\ E, & if\; a = b \end{cases}$$

Obviously, the extensive enlargement is a closed (containing) hypercomposition.

Definition 4. *An element* 0 *of a magma* $(E, \perp)$ *is called left absorbing (resp. right absorbing) if* $0 \perp x = 0$ *(resp.* $x \perp 0 = 0$*) for all* $x \in E$*. An element of a magma is called absorbing if it is a bilaterally absorbing element.*

A direct consequence of the above definition is Proposition 2:

Proposition 2. *If a magma has a left (resp. right) absorbing element, then the relevant induced law of synthesis is a hypercomposition.*

Definition 5. *A law of synthesis* $(x, y) \rightarrow x \perp y$ *on a set* E *is called associative if the property*

$$(x \perp y) \perp z = x \perp (y \perp z)$$

is valid, for all elements x, y, z *in* E*. A magma whose law of synthesis is associative is called an associative magma.*

Example 9. *If the law of synthesis is the b-hypercomposition denoted by +, then*

$$(x+y)+z=\{x,y\}+z=(x+z)\cup(y+z)=\{x,z\}\cup\{y,z\}=\{x,\ y,\ z\}$$

and $$x+(y+z)=x+\{y,z\}=(x+y)\cup(x+z)=\{x,y\}\cup\{x,z\}=\{x,\ y,\ z\}$$

Hence $(x+y)+z=x+(y+z),$ *for all* x,y,z *in* E*. Thus, the b-hypercomposition is associative.*

Example 10. *The law of synthesis defined in Example 4 on the set S of the points of a Euclidian geometry is associative. For the verification, it is required to consider many cases. The following Figure 3 presents the general case for the open hypercomposition, in which the points* a,b,c *are not collinear. The result of both* $a(bc)$ *and* $(ab)c$ *is the interior of the triangle* abc*.*

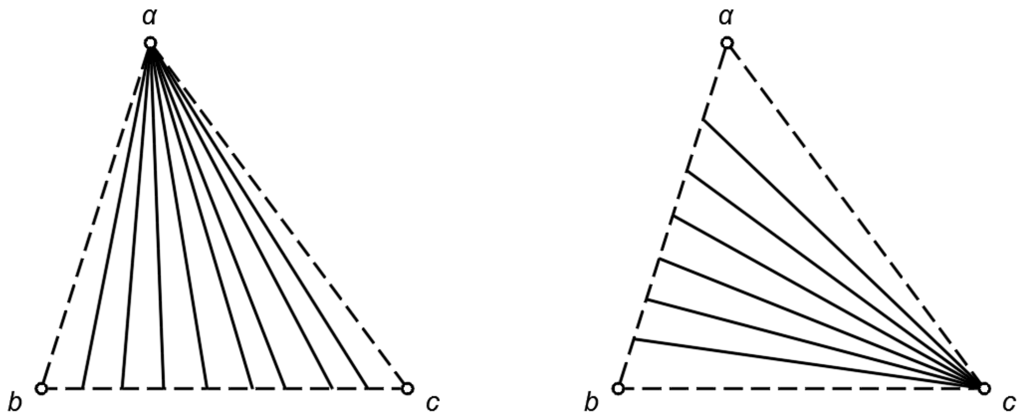

Figure 3. Example 10.

The interaction of the law of synthesis with the two induced laws in an associative magma gives the mixed associativity.

Proposition 3. *In an associative magma* $(E,\cdot)$ *the properties*

$$\begin{array}{ll}(a/b)/c=a/(cb) & \textit{right mixed associativity}\\ c\backslash(b\backslash a)=(bc)\backslash a & \textit{left mixed associativity}\\ (b\backslash a)/c=b\backslash(a/c) & \end{array}$$

are valid, for all $a,b,c\in E$*.*

Proof. Let $x\approx(a/b)/c$. Then we have the following sequence of equivalent statements:

$$x\approx(a/b)/c\Leftrightarrow xc\approx a/b\Leftrightarrow a\approx(xc)b\Leftrightarrow a\approx x(cb)\Leftrightarrow x\approx a/cb,$$

therefore $(a/b)/c=a/(cb)$. Similar is the proof of the left mixed associativity. Next, let $x\approx(b\backslash a)/c$. Then we have the sequence of implications:

$$x\approx(b\backslash a)/c\Leftrightarrow xc\approx b\backslash a\Leftrightarrow a\approx b(xc)\Leftrightarrow a\approx(bx)c\Leftrightarrow bx\approx a/c\Leftrightarrow x\approx b\backslash(a/c),$$

hence $(b\backslash a)/c=b\backslash(a/c)$. □

Corollary 1. *In an associative magma* $(E,\cdot)$ *the equalities*

$$\begin{array}{l}(A/B)/C=A/(CB)\\ C\backslash(B\backslash A)=(BC)\backslash A\\ (B\backslash A)/C=B\backslash(A/C)\end{array}$$

are valid for all nonvoid subsets A,B,C *of* E*.*

Proposition 4. *In an associative magma* $(E,\cdot)$ *it holds that*

$$b \approx (a/b)\backslash a \quad and \quad b \approx a/(b\backslash a)$$

for all $a, b \in E$.

Proof. Let $x \approx a/b$. Then, $a \approx xb$, therefore $a \approx (a/b)b$, hence $b \approx (a/b)\backslash a$. The proof of the second relation is similar. □

Corollary 2. *In an associative magma* $(E,\cdot)$ *the inclusions*

$$B \subseteq (A/B)\backslash A \quad and \quad B \subseteq A/(B\backslash A)$$

are valid for all nonvoid subsets A, B *of* E.

Definition 6. *A hypercomposition* $(x,y) \to x\perp y$ *on a set E is called weakly associative if for all elements* x, y, z *in E,*

$$(x\perp y)\perp z \approx x\perp(y\perp z).$$

A magma whose law of synthesis is weakly associative is called a weakly associative magma.

Example 11. *Suppose that the law of synthesis on a magma E, with more than three elements, is the following one:*

$$\begin{array}{ll} x\perp y = E\cdot\cdot\{x,y\} & \text{for every } x,y \in E \text{ with } x \neq y \\ x\perp x = x & \text{for all } x \in E \end{array}$$

Then E is not an associative magma, because

$$(x\perp x)\perp y = x\perp y = E\cdot\cdot\{x,y\}, \text{ while, } x\perp(x\perp y) = x\perp[E\cdot\cdot\{x,y\}] = E\cdot\cdot\{x\}.$$

However, E is a weakly associative magma, since

$$(x\perp y)\perp z \cap x\perp(y\perp z) \neq \varnothing, \text{for all } x,y,z \text{ in } E.$$

Proposition 5. *The result of the hypercomposition in a weakly associative magma E is always a nonempty set.*

Proof. Suppose that $x\perp y = \varnothing$ for some $x, y \in E$. Then, $(x\perp y)\perp z = \varnothing$ for any $z \in E$. Therefore, $(x\perp y)\perp z \cap x\perp(y\perp z) = \varnothing$, which is absurd. Hence, $x\perp y$ is nonvoid. □

Definition 7. *A hypercomposition* $(x,y) \to x\perp y$ *on a set E is called weakly commutative if, for all elements* x, y *in E,*

$$x\perp y \approx y\perp x.$$

A magma whose law of synthesis is weakly commutative is called a weakly commutative magma.

Example 12. *Let* $(E,\cdot)$ *be a magma and let* $\odot$ *be the extensive enlargement of the law of synthesis. Then* $(E,\odot)$ *is a weakly commutative magma, since*

$$\{x,y\} \subseteq x\odot y \cap y\odot x$$

for all elements x, y *in E.*

Two statements of magma theory are dual statements if each one results from the other by interchanging the order of the law of synthesis $\perp$, that is, interchanging $a\perp b$ with $b\perp a$. Observe that the axiom of associativity is self-dual. The two induced laws of synthesis $_\perp/$

and $\backslash_\perp$ have dual definitions, hence they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds:

> *Given a theorem, the dual statement, which results from the interchange of the order of the synthesis $\perp$ (and necessarily interchange ${}_\perp/$ and $\backslash_\perp$), is also a theorem.*

A direct consequence of the principle of duality is the following proposition:

Proposition 6. *The opposite law of an associative law of synthesis is associative.*

Proposition 7. *The extensive enlargement of an associative law of synthesis is associative.*

Proof. Let $\odot$ denote the extensive enlargement of an associative law of synthesis, which is written multiplicatively, then:

$$\begin{aligned}(a \odot b) \odot c &= [\{a,b\} \cup ab] \odot c = a \odot c \cup b \odot c \cup (ab) \odot c = \\ &= \{a,c\} \cup ac \cup \{b,c\} \cup bc \cup ab \cup \{c\} \cup (ab)c = \\ &= \{a,b,c\} \cup ac \cup bc \cup ab \cup a(bc) = a \odot (b \odot c) \ \square\end{aligned}$$

Proposition 8. *Let $(E, \perp)$ be an associative magma. If an element a of E commutes with the elements b and c of E, then it commutes with their synthesis as well.*

Proof. $a\perp(b\perp c) = (a\perp b)\perp c = (b\perp a)\perp c = b\perp(a\perp c) = b\perp(c\perp a) = (b\perp c)\perp a$. □

Definition 8. *A law of synthesis $(x,y) \to x\perp y$ on a set E is called reproductive if the equality*

$$x\perp E = E\perp x = E$$

is valid for all elements x in E. A magma whose law of synthesis is reproductive is called a reproductive magma.

Example 13. *The following laws of synthesis in $E = \{1,2,3\}$ are reproductive:*

$\perp$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$\perp$	1	2	3
1	{1,3}	{3}	{2}
2	{2}	{1,2,3}	{2}
3	{2}	{1}	{1,2,3}

3. Groups and Hypergroups

Definition 9. *An associative and reproductive magma is called a group if the law of synthesis on the magma is a composition, while it is called a hypergroup if the law of synthesis is a hypercomposition.*

We observe that the two algebraic structures we just defined are equipped with the same operating rules, that is, the axioms of associativity and reproductivity, but with different synthesizing laws for their elements. Of course, if the singletons are identified with their members, then the groups are special cases of the hypergroups. The term *group* was first used in a technical sense by the French mathematician Évariste Galois (25 October 1811–31 May 1832). His brilliant paper "Une Mémoire sur les Conditions de Résolubilité des Equations par Radicaux" was submitted in January 1831, for the third time, to the French Academy of Sciences. It was followed by his famous letter describing his discoveries, written the night before he was killed in a duel at the age of 21, in May 1832. Galois's manuscripts, with annotations by Joseph Liouville, were published in 1846 in the *Journal de Mathématiques Pures et Appliquées*. Évariste Galois discovered the notion of normal subgroups and realized their importance. Another young French mathematician, Frédéric Marty (23 June 1911–14 June 1940), who was born a hundred years later, while working on cosets determined by not normal subgroups, introduced the *hypergroup*. His mathematical heritage on hypergroups was only three papers [1–3], as he was killed at the age of 29 during World War II, in the Gulf of Finland, while serving in the French Air Force as a lieutenant. For the record, Frédéric Marty's father, Joseph Marty, was also a mathematician who was killed in 1914, when Frédéric was only 3 years old, during World War I.

Example 14. *Let $E = \{a\}$. There is only one associative and reproductive magma, whose law of synthesis is defined by the following table:*

$\perp$	a
a	a

Example 15. *Let $E = \{a, b\}$. There is only one group with two elements. The following table shows this group.*

$\perp$	a	b
a	a	b
b	b	a

On the other hand, there exist eight nonisomorphic hypergroups with two elements. The following tables display these hypergroups. Observe that the first one is the above group.

$\perp$	a	b
a	a	b
b	b	a

$\perp$	a	b
a	a	b
b	b	$\{a, b\}$

$\perp$	a	b
a	a	b
b	$\{a, b\}$	$\{a, b\}$

$\perp$	a	b
a	a	$\{a, b\}$
b	b	$\{a, b\}$

$\perp$	a	b
a	a	$\{a, b\}$
b	$\{a, b\}$	b

$\perp$	a	b
a	a	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$

$\perp$	a	b
a	b	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$

$\perp$	a	b
a	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$

Example 16. *Let $E = \{a, b, c\}$. There is only one group with three elements. The following table displays this group.*

$\perp$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

However, there are 3999 nonisomorphic hypergroups with three elements; [79] presents a software that can calculate and print all these 3999 hypergroups.

Example 17. *A magma endowed with the b-hypercomposition is a hypergroup, called b-hypergroup.*

Example 18. *A magma endowed with the total hypercomposition is a hypergroup, called total hypergroup.*

Theorem 1. *Let $(G, \cdot)$ be a group or a hypergroup. If "$\odot$" is the extensive enlargement of "$\cdot$", then $(G, \odot)$ is a hypergroup.*

Proof. By Proposition 7, the extensive enlargement of $\cdot$ is associative. Moreover:

$$x \odot H = H \odot x = \{x\} \cup H = H$$

and the theorem holds. □

4. The Reproductive Axiom in Groups

Recall the definition of the group that is mentioned in the previous section:

Definition 10. (FIRST DEFINITION OF A GROUP). *An associative and reproductive magma is called a group, if the law of synthesis on the magma is a composition.*

In other words, a group is a set of elements equipped with a law of composition that is associative and reproductive. The next theorems give some important properties of the group structure. In this section, unless otherwise indicated, the law of synthesis is a composition that will be written multiplicatively, and G will denote a multiplicative group.

Theorem 2. *Let G be a group. Then*:

i. *There exists an element $e \in G$ such that $ea = a = ae$, for all $a \in G$.*
ii. *For each element $a \in G$ there exists an element $a' \in G$ such that $a'a = e = aa'$.*

Proof. (i) Let $x \in G$. By reproductive axiom, $x \in xG$. Consequently, there exists $e \in G$, for which $xe = x$. Next, let y be an arbitrary element in G. Since the composition is reproductive $y \in Gx$, therefore, there exists $z \in G$ such that $y = zx$. Consequently, $ye = (zx)e = z(xe) = zx = y$. In an analogous way, there exists an element $\hat{e}$ such that $\hat{e}y = y$ for all $y \in G$. Then the equality $e = \hat{e}e = \hat{e}$ is valid. Therefore, there exists $e \in G$ such that $ea = a = ae$, for all $a \in G$.

(ii) Let $a \in G$. By reproductive axiom, $e \in a\,G$. Thus, there exists $a' \in G$, such that $e = aa'$. Also by reproductive axiom, $e \in\ Ga$. Therefore, there exists $a'' \in G$, such that $e = a''a$. However:

$$a' = ea' = \left(a''a\right)a' = a''\left(aa'\right) = a''e = a''.$$

Hence a' and a'' coincide. Therefore $aa' = e = a'a$. □

The element e is called the *neutral element* of G or the *identity* of G. Moreover, a' is called the *symmetric* of a in G. If the composition is written multiplicatively, then e is called the *unit* of G and it is denoted by 1. Furthermore, a' is called the *inverse* of a and it is denoted by a^{-1}. If the composition is written additively, then e is called the *zero* of G and it is denoted by 0. Also a' is called the *opposite* of a and it is denoted by $-a$.

Corollary 3. *For each $a, b \in G$,*

$$a/b = ab^{-1} \ \textit{and}\ \ b\backslash a = b^{-1}a.$$

Corollary 4. *Let e be the identity of a group G. Then:*

$$e/b = b^{-1} \ \textit{and}\ \ b\backslash e = b^{-1}$$

for all $b \in G$.

Corollary 5. *Let e be the identity of a group G. Then:*

$$b/b = e = b\backslash b \ \textit{and}\ \ (b/b)/b = b^{-1} = b\backslash(b\backslash b)$$

for all $b \in G$.

Theorem 3. *Let G be an associative magma whose law of synthesis is a composition. Then G is a group if the following two postulates are fulfilled:*

i. *There exists an element $e \in G$ such that $ea = a = ae$, for all $a \in G$.*
ii. *For each element $a \in G$, there exists an element $a' \in G$ such that $a'a = e = aa'$.*

Proof. It must be proved that the reproductive axiom is valid for G. Since $aG \subseteq G$, for all $a \in G$, it has to be proved that $G \subseteq aG$. Suppose that $x \in G$. Then:

$$x = ex = (aa')x = a(a'x).$$

The product $a'x$ is an element of G, thus $x \in aG$. Hence $G \subseteq aG$, therefore $aG = G$. Similarly, $Ga = G$. □

Theorems 2 and 3 lead to another definition of the group.

Definition 11. (Second Definition of a Group)**.** *An associative magma G in which the law of synthesis is a composition is called a group if:*

i. *There exists an element $e \in G$ such that $ea = a = ae$, for all $a \in G$.*
ii. *For each element $a \in G$, there exists an element $a' \in G$ such that $a'a = e = aa'$.*

Yet, one-half of the above definition's postulates (i) and (ii) can be omitted by the following dual propositions:

Proposition 9. *The postulates (i) and (ii) of Definition 11 are equivalent to:*

i.* *There exists an element $e \in G$ with $ea = a$, for all $a \in G$.*
ii.* *For each element $a \in G$, there exists an element $a' \in G$ such that $a'a = e$.*

Proposition 10. *The postulates (i) and (ii) of Definition 11 are equivalent to:*

*i**.* *There exists an element $e \in G$ with $ae = a$, for all $a \in G$.*
*ii**.* *For each element $a \in G$, there exists an element $a' \in G$ such that $aa' = e$.*

We quote the following well known and important Theorems 4, 5, 6, 7, 8, and 9, which can be easily proved with the use of the second definition of the group, because we want to show in the next sections that similar theorems can be proved only in very specific types of hypergroups.

Theorem 4. *The neutral element of a group is unique.*

Theorem 5. *The symmetric of each element of a group is unique.*

Theorem 6. *The symmetric of the neutral element is the neutral element itself.*

Theorem 7. *For each $a \in G$,*

$$\left(a^{-1}\right)^{-1} = a.$$

Theorem 8. *For each $a, b \in G$,*

$$(ab)^{-1} = b^{-1}a^{-1}.$$

Theorem 9. *The cancellation law:*

$$ac = bc \ \ implies \ \ a = b$$
$$ca = cb \ \ implies \ \ a = b$$

Theorem 10. *A finite associative magma G is a group if the law of synthesis on the magma is a composition in which the cancellation law holds.*

Proof. Let $G = \{a_1, \cdots, a_n\}$ and let a be an arbitrary element in G. Then:

$$aG = \{aa_1, \cdots, aa_n\} \subseteq G.$$

From the cancellation law, it follows that the n elements of aG are all distinct. Therefore aG and G have the same cardinality. Consequently, since G is finite, $aG = G$ is valid. Duality gives $Ga = G$ and so the theorem. □

Theorem 11. *An associative magma G whose law of synthesis is a composition is a group if and only if the inverse laws are compositions.*

Proof. Let G be a group. We will prove that b/a and $a\backslash b$ are elements of G, for all pairs of elements $a,\ b$ of G. By reproduction, $Ga = G$ for all $a \in G$. Consequently, for every $b \in G$ there exists $x \in G$, such that $b = xa$. Thus, $x = b/a$. Dually, $a\backslash b$ is an element of G. Conversely now: suppose that the right quotient b/a exists for all pairs of elements $a,\ b$ of G. Thus, for each $a, b \in G$, there is an element x in G such that $b = xa$. Therefore $G \subseteq Ga$ for all $a \in G$. Next, since $Ga \subseteq G$, for all $a \in G$, it follows that $Ga = G$, for all $a \in G$. In a similar way, the existence of the left quotient $a\backslash b$ for all pairs of elements $a,\ b$ of G, yields $aG = G$, for all $a \in G$. Thus, the reproductive law is valid and so G is a group. □

Corollary 6. *An associative magma G is a group if and only if the equations*

$$xa = b \ \ and \ \ ay = b$$

are solvable for all pairs of elements $a,\ b$ of G.

Proof. By Theorem 11, G is a group if and only if b/a and $a\backslash b$ are elements of G, for all $a,\ b$ in G, which equivalently implies that the equations $xa = b$ and $ay = b$, respectively, are solvable for all pairs of elements $a,\ b$ of G. □

Having proved the above, another definition can be given for the group.

Definition 12. (THIRD DEFINITION OF A GROUP). *An associative magma G in which the law of synthesis is a composition is called a group if the right quotient b/a and the left quotient $a\backslash b$ result in a single element of G, for all $a,\ b \in G$.*

Or else:

An associative magma G in which the law of synthesis is a composition is called a group if the equations

$$xa = b \ \ and \ \ ay = b$$

are solvable for all pairs of elements $a,\ b$ of G.

Definition 13. *A group that has the additional property that for every pair of its elements*

$$a\,b = b\,a$$

is called an Abelian (after N.H. Abel, 1802-29) or commutative group.

In abstract algebra, we also consider structures that do not satisfy all the axioms of a group.

Definition 14. *An associative magma in which the law of synthesis is a composition is called a semigroup. A semigroup with an identity is called a monoid.*

Definition 15. *A reproductive magma in which the law of synthesis is a composition is called a quasigroup. A quasigroup with an identity is called loop.*

Definition 16. *A magma that is the union of a group with an absorbing element is called almost-group.*

5. Fundamental Properties of Hypergroups.

Recall the earlier mentioned definition for the hypergroup:

Definition 17. (First Definition of a Hypergroup)**.** *An associative and reproductive magma is called a hypergroup if the law of synthesis on the magma is a hypercomposition.*

In this section, unless otherwise indicated, the law of synthesis is a hypercomposition that will be written multiplicatively, and H will denote a multiplicative hypergroup.

Theorem 12. *$ab \neq \varnothing$ is valid for all the elements $a,\ b$ of a hypergroup H.*

Proof. Suppose that $ab = \varnothing$ for some $a, b \in H$. By the reproductive axiom, $aH = H$ and $bH = H$. Hence:

$$H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$$

which is absurd. □

Theorem 13. *$a/b \neq \varnothing$ and $b\backslash a \neq \varnothing$ is valid for all the elements a, b of a reproductive magma E.*

Proof. By the reproductive axiom, $Eb = E$ for all $b \in E$. Hence, for every $a \in E$ there exists $x \in E$, such that $a \in xb$. Thus, $x \in a/b$ and therefore $a/b \neq \varnothing$. Dually, $b\backslash a \neq \varnothing$. □

Theorem 14. *If $a/b \neq \varnothing$ and $b\backslash a \neq \varnothing$ for all pairs of elements a, b of a magma E, then E is a reproductive magma.*

Proof. Suppose that $x/a \neq \varnothing$ for all $a, x \in E$. Thus, there exists $y \in E$, such that $x \in ya$. Therefore $x \in Ea$ for all $x \in E$, and so $E \subseteq Ea$. Next, since $Ea \subseteq E$ for all $a \in E$, it follows that $E = Ea$. By duality, $E = aE$. □

Following Theorem 14, another definition of the hypergroup can be given:

Definition 18. (Second Definition of a Hypergroup)**.** *An associative magma is called a hypergroup if the law of synthesis is a hypercomposition and the result of each one of the two inverse hypercompositions is nonvoid for all pairs of elements of the magma.*

Theorem 15. *In a hypergroup H, the equalities*

i. $H = H/a = a/H$ *and*
ii. $H = a\backslash H = H\backslash a$

are valid for all a in H.

Proof. (i) By Theorem 12, the result of the hypercomposition in H is always a nonempty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y/a$. Hence $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore $H = H/a$. Next, let $x \in H$. Since

$H = xH$, there exists $y \in H$, such that $a \in xy$, which implies that $x \in a/y$. Hence $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore $H = a/H$. (ii) follows by duality. □

Likewise to the groups, certain axioms were removed from the hypergroup, thus revealing the following weaker structures.

Definition 19. *A magma in which the law of synthesis is a hypercomposition is called a hypergoupoid if for every two of its elements a, b it holds that $ab \neq \varnothing$, otherwise it is called a partial hypergroupoid.*

In the case of finite hypergroupoids, the ratio of the number of hypergroups over the number of hypergroupoids is exceptionally small. For instance, we come across one 3-element hypergroup in every 1740 hypergroupoids of three elements [79].

Definition 20. *An associative hypergoupoid is called a semihypergroup, while a reproductive hypergoupoid is called a quasihypergroup.*

Definition 21. *The magma which is the union of a hypergroup with an absorbing element is called almost-hypergroup.*

Definition 22. *A reproductive magma in which the law of synthesis is weakly associative is called H_V-group [80].*

Because of Proposition 5, the result of the hypercomposition in an H_V-group is always nonvoid.

Many papers have been written on the construction of examples of the above algebraic structures. Among them are the papers by P. Corsini [81,82], P. Corsini and V. Leoreanu [83], B. Davvaz and V. Leoreanu [84], I. Rosenberg [85], I. Cristea et al. [86–90], M. De Salvo and G. Lo Faro [91,92], C. Pelea and I. Purdea [93,94], C.G. Massouros and C.G. Tsitouras [95,96], and S. Hoskova-Mayarova and A. Maturo [70], in which hypercompositional structures, defined in terms of binary relations, are presented and studied.

It is worth mentioning that a generalization of the hypergroup is the fuzzy hypergroup, which was studied by a multitude of researchers [97–127]. An extensive bibliography on this subject can be found in [124]. It can be proved that similar fundamental properties as the aforementioned ones are valid in the fuzzy hypergroups as well [125,126]. For instance, $a \circ b \neq 0_H$ is valid for any pair of elements a, b in a fuzzy hypergroup $(H, \circ)$ [125]. Generalizations of the fuzzy hypergroups are the mimic fuzzy hypergroups [125,126] and the fuzzy multihypergroups [127].

6. Types of Hypergroups

The hypergroup being a very general structure, was equipped with further axioms, which are more or less powerful and lead to a significant number of special hypergroups. One such important axiom is the *transposition axiom*. Initially this axiom was introduced by W. Prenowitz in a commutative hypergroup, all the elements of which also satisfy the properties $aa = a$ and $a/a = a$. He named this hypergroup *join space* and used it in the study of geometry [19–24]. The transposition axiom in a commutative hypergroup is:

$$a/b \cap c/d \neq \varnothing \quad \text{implies} \quad ad \cap bc \neq \varnothing$$

A commutative hypergroup that satisfies the transposition axiom is called a *join hypergroup*. Later on, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup H as follows:

$$b\backslash a \cap c/d \neq \varnothing \quad \text{implies} \quad ad \cap bc \neq \varnothing \quad \text{for all} \quad a, b, c, d \in H$$

.

A hypergroup equipped with the transposition axiom is called *transposition hypergroup* [128].

These hypergroups attracted the interest of a large number of researchers, including I. Cristea et al. [102,107–112], P. Corsini [100–103,129,130], V. Leoreanu-Fortea [101,103–105,131,132], I. Rosenberg [85,132], S. Hoskova-Mayerova, [133–137], J. Chvalina [135–138], P. Rackova [135,136], Ch.G. Massouros [139–150], G.G. Massouros [144–152], J. Nieminen [153,154], A. Kehagias [115], R. Ameri [117,118,155], M. M Zahedi [117], and G. Chowdhury [123].

Proposition 11. [150] *The following are true in any transposition hypergroup:*

i. $a(b/c) \subseteq ab/c$ *and* $(c\backslash b)a \subseteq c\backslash ba$,
ii. $a/(c/b) \subseteq ab/c$ *and* $(b\backslash c)\backslash a \subseteq c\backslash ba$,
iii. $(b\backslash a)(c/d) \subseteq (b\backslash ac)/d = b\backslash(ac/d)$,
iv. $(b\backslash a)/(c/d) \subseteq (b\backslash ad)/c = b\backslash(ad/c)$,
v. $(b\backslash a)\backslash(c/d) \subseteq (a\backslash bc)/d = a\backslash(bc/d)$.

The hypergroups are much more general algebraic structures than the groups, to the extent that a theorem similar to Theorem 2 cannot be proved for the hypergroups. In fact, a hypergroup does not necessarily have an identity element. Moreover, in hypergroups there exist different types of identities [149,150,156]. In general, an element e of a hypergroup H is called *right identity*, if $x \in x \cdot e$ for all x in H. If $x \in e \cdot x$ for all x in H, then x is called *left identity*, while x is called *identity* if it is both a right and a left identity; i.e., if $x \in xe \cap ex$ for all $x \in H$. If the equality $e = ee$ is valid for an identity e, then e is called *idempotent identity*. A hypergroup H is called *semiregular* if every $x \in H$ has at least one right and one left identity. An identity is called *scalar* if $a = ae = ea$ for all $a \in H$, while it is called *strong* if $ae = ea \subseteq \{e, a\}$ for all $a \in H$. More generally, an element $s \in H$ is called *scalar* if the result of the hypercomposition of this element with any element in H is a singleton; that is, if $sa \in H$ and $as \in H$ for all $a \in H$. If only the first membership relation is valid, then s is called *left scalar*, while if only the second relation is valid, then s is called *right scalar*. When a scalar identity exists in H, then it is unique but the strong identity is not necessarily unique. Both scalar and strong identities are idempotent identities.

Remark 1. *If a hypercomposition has a scalar identity e, then it is neither open nor closed (containing) because* $e \notin ex$ *and* $x \approx ex$.

Proposition 12. *If e is a strong identity in H and* $x \neq e$, *then* $x/e = e\backslash x = x$.

Proof. Let $t \in x/e$. Then $x \in te \subseteq \{t, e\}$. Since $x \neq e$, it follows that $t = x$. Thus $x/e = x$. Similarly, it can be proven that $e\backslash x = x$. □

Corollary 7. *If e is a strong identity in H and X is a nonempty subset of H, not containing e, then* $X/e = e\backslash X = X$.

Proposition 13. *If e is a scalar identity in H, then* $x/e = e\backslash x = x$.

Corollary 8. *If X is a non-empty subset of H and if e is a scalar identity in H, then* $X/e = e\backslash X = X$.

Theorem 16. *i. If a hypergroup H contains a scalar element, then it contains a scalar identity e as well.*
ii. The set U of the scalar elements of a hypergroup H is a group.

Proof. (i) Let s be a scalar element. Then, per reproductivity, there exists an element $e \in H$ such that $se = s$. Also, because of the reproductive axiom, any element $y \in H$ can be written as $y = xs$, $x \in H$. Hence $ye = (xs)e = x(se) = xs = y$. Similarly, $ey = y$.

(ii) Let $s \in U$. Then, per reproductivity, there exists an element $s' \in H$ such that $ss' = e$. If $y \in H$ then, because of the reproductive axiom, $y = xs$, $x \in H$. Hence $ys' = (xs)s' = x(ss') = xe = x$, and therefore s' is a right scalar element. Similarly, there exists a left scalar element s'' such that $s''s = e$. But s' is equal to s'' since $s'' = s''e = s''(ss') = \left(s''s\right)s' = es' = s'$. Consequently, U is a subgroup of the hypergroup H. □

The group of the scalars was named the *nucleus* of H by Wall [11].

An element x' is called *right e-symmetric* of x, or *right e-inverse* in the multiplicative case, if there exists a right identity $e \neq x'$ such that $e \in x \cdot x'$. The definition of the *left e-symmetric* or *left e-inverse* is analogous to the above, while x' is called the *e-inverse* or *e-symmetric* of x, if it is both right and left inverse with regard to the same identity e. If e is an identity in a multiplicative hypergroup H, then the set of the left inverses of $x \in H$, with regard to e, will be denoted by $S_{el}(x)$, while $S_{er}(x)$ will denote the set of the right inverses of $x \in H$ with regard to e. The intersection $S_{el}(x) \cap S_{er}(x)$ will be denoted by $S_e(x)$. A semiregular hypergroup H is called *regular* if it has at least one identity e and if each element has at least one right and one left e-inverse. H is called *strictly e-regular* if for the identity e the equality $S_{el}(x) = S_{er}(x)$ is valid for all $x \in H$. In a strictly e-regular hypergroup, the inverses of x are denoted by $S_e(x)$ and, when there is no likelihood of confusion, e can be omitted, and the notation $S(x)$ is used for the inverses of x. We say that H has *semistrict e-regular structure* if $S_{el}(x) \approx S_{er}(x)$ is valid for any $x \in H$. Obviously, in the commutative hypergroups there exist only the strict e-regular structures.

Proposition 14. *If e is an identity in a hypergroup H, then* $S_{el}(x) = (e/x) \cdot \cdot \{e\}$ *and* $S_{er}(x) = (x \backslash e) \cdot \cdot \{e\}$.

Corollary 9. *If* $S_{el}(x) \cap S_{er}(x) \neq \varnothing$, *then* $x \backslash e \cap e/x \neq \varnothing$.

Definition 23. *A regular hypergroup is called reversible if it satisfies the following conditions:*

i. $z \in xy \Rightarrow x \in zy', \text{ for some } y' \in S(y)$,
ii. $z \in xy \Rightarrow y \in x'z, \text{ for some } x' \in S(x)$.

The enrichment of a hypergroup with an identity creates different types of hypergroups, depending on the type of the identity.

Proposition 15. *If H is a transposition hypergroup with a scalar identity e, then, for any x in H, the quotients* e/x *and* $x \backslash e$ *are singletons and equal to each other.*

Proof. Obviously $e/e = e\backslash e = e$. Let $x \neq e$. Because of reproduction, there exist x' and x'', such that $e \in x'x$ and $e \in xx''$. Thus $x \in x'\backslash e$ and $x \in e/x''$. Hence $x'\backslash e \approx e/x''$. Therefore, because of transposition, $ex'' \approx x'e$ is valid. Since e is a scalar identity, the following is true: $x'' = ex''$ and $x'e = x'$. Thus $x' = x''$. However, $x' \in e/x$ and $x'' \in x\backslash e$. Therefore e/x and $x\backslash e$ are equal, and since this argument applies to any $y', y'' \in H$, such that $e \in y'x$ and $e \in xy''$, it follows that e/x and $x\backslash e$ are singletons. □

Definition 24. *A transposition hypergroup that has a scalar identity e is called quasicanonical hypergroup [157,158] or polygroup [159–161].*

The connection of quasicanonical hypergroups with color schemes, relation algebras, and finite permutation groups, as well as with weak cogroups, produces a lot of examples of quasicanonical hypergroups (e.g., see [160]). In [162], quasicanonical hypergroups appear as Pasch geometries. In a Pasch geometry (A,Δ,e), A becomes a quasicanonical hypergroup with scalar identity e and $a^{-1} = a^{\#}$, when $ab = \{x | (a, b, x^{\#}) \in \Delta\}$ (see also [128]). The following example from [158] shows the structural relation of groups with the quasicanonical hypergroups.

Example 19. *Let* $(G, \cdot)$ *be a group and e its identity. The following hypercomposition is defined on G:*

$$\begin{array}{ll} x \circ y = \{x, y, x \cdot y\}, & if \;\; y \neq x^{-1},\; x, y \neq e \\ x \circ x^{-1} = x^{-1} \circ x = G, & if \;\; x \neq e \\ x \circ e = e \circ x = x, & for\; all \;\; x \in G \end{array}$$

Then $(G, \circ)$ *becomes a quasicanonical hypergroup. Note that this construction can be used to produce new quasicanonical hypergroups from other quasicanonical hypergroups.*

In the quasicanonical hypergroups, there exist properties analogous to (i) and (ii) of Theorem 2, which are valid in the groups:

Theorem 17. [128,141] *If* $(Q, \cdot, e)$ *is a quasicanonical hypergroup, then:*

i. *For each* $x \in Q$ *there exists one and only one* $x' \in Q$ *such that* $e \in xx' = x'x$.
ii. $z \in xy \Rightarrow x \in zy' \Rightarrow y \in x'z$.

Corollary 10. *A quasicanonical hypergroup is a reversible hypergroup.*

The inverse of Theorem 17 is also true:

Theorem 18. [128,141] *If a hypergroup Q has a scalar identity e and:*

i. *For each* $x \in Q$ *there exists one and only one* $x' \in Q$ *such that* $e \in xx' = x'x$
ii. $z \in xy \Rightarrow x \in zy' \Rightarrow y \in x'z$

then the transposition axiom is valid in Q.

When the hypercomposition is written multiplicatively, x' is denoted by x^{-1} and it is called the *inverse* of x, while, if the hypercomposition is written additively, the identity is denoted by 0 and the unique element x' is called *opposite* or *negative*, and it is denoted by $-x$.

Proposition 16. *In a quasicanonical hypergroup Q,*

i. $x^{-1} = e/x = x\backslash e$ *and* $x = e/x^{-1} = x^{-1}\backslash e$ *for all* $x \in Q$.
ii. $x/y = xy^{-1}$ *and* $y\backslash x = y^{-1}x$ *for all* $x, y \in Q$.

Proof. (i) is a direct consequence of Proposition 15. Next, for (ii), applying (i), Proposition 11, and Corollary 8, we have:

$$x/y = x/\left(e/y^{-1}\right) \subseteq xy^{-1}/e = xy^{-1}$$

and

$$xy^{-1} = x(e/y) \subseteq xe/y = x/y.$$

Therefore, the first equality is proved. The second one arises from duality. □

The aforementioned Theorems 4, 5, 6, 7, and 8, which are valid in the groups, are also valid in the quasicanonical hypergroups. The cancellation law (Theorem 9), though, is valid in the quasicanonical hypergroups as follows:

$$\begin{array}{l} ac = bc \;\text{ implies }\; b^{-1}a \cap cc^{-1} \neq \varnothing \\ ca = cb \;\text{ implies }\; ba^{-1} \cap cc^{-1} \neq \varnothing \end{array}$$

More generally the following theorem is valid:

Theorem 19. *If Q is a quasicanonical hypergroup, then* $ab \approx cd$ *implies that* $bd^{-1} \approx a^{-1}c$ *and* $c^{-1}a \approx db^{-1}$.

Proof. As it is mentioned above, Theorem 8 is valid in the quasicanonical hypergroups as well. Indeed, the equality $(ab)^{-1} = b^{-1}a^{-1}$ follows from the sequence of implications:

$$x \in ab\ ;\ a \in xb^{-1}\ ;\ aa^{-1} \subseteq xb^{-1}a^{-1}\ ;\ e \in xb^{-1}a^{-1}\ ;\ x^{-1} \in b^{-1}a^{-1}.$$

Next, let $x \in ab \cap cd$. Then $xx^{-1} \in ab(cd)^{-1} = abd^{-1}c^{-1}$. Hence $e \in abd^{-1}c^{-1}$, thus $c \in abd^{-1}$. Therefore, there exists $y \in bd^{-1}$ such that $c \in ay$. Reversibility implies that $y \in a^{-1}c$. Consequently, $bd^{-1} \approx a^{-1}c$. By duality, $c^{-1}a \approx db^{-1}$. □

A commutative quasicanonical hypergroup is called *canonical hypergroup*. The canonical hypergroup owes its name to J. Mittas [163,164] while it was first used by M. Krasner for the construction of the hyperfield, which is a hypercompositional structure that he introduced in order to define a certain approximation of complete valued fields by sequences of such fields [53]. The hyperfields, which were constructed afterward, contain interesting examples of canonical hypergroups [165–170]. An example of such a canonical hypergroup is presented in the following construction by J. Mittas [171].

Example 20. *Let E be a totally ordered set and 0 its minimum element. The following hypercomposition is defined on E:*

$$x + y = \begin{cases} \max\{x, y\} & if\ x \neq y \\ \{z \in E \mid z \leq x\} & if\ x = y \end{cases}$$

Then (E,+) is a canonical hypergroup.

We cite the above example because the hyperfield, which J. Mittas constructed based on this canonical hypergroup, is now called a tropical hyperfield (see, e.g., [54–58]) and it is used in the development of the tropical geometry. Example 19 also gives a canonical hypergroup, when $(G, \cdot)$ is an abelian group. The hyperfield that is constructed based on this canonical hypergroup leads to open problems in both hyperfield and field theories [168,169].

J. Mittas studied the canonical hypergroup in depth [163,164,172–180]. Also motivated by the valuated hyperfield theory, he introduced ultrametric distances to the canonical hypergroups, thus defining the valuated and the hypervaluated canonical hypergroups. Next, he proved that the necessary and sufficient condition for a canonical hypergroup to be valuated or hypervaluated is the validity of certain additional properties of a purely algebraic type; that is, properties that are expressed without the intervention of the valuation or the hypervaluation, respectively. Thus, three special canonical hypergroups came into being:

(a) The *strongly canonical hypergroup*, which also satisfies the axioms:
S_1: If $x \in x + a$, then $x + a = x$, for all $x, a \in H$.
S_2: If $(x + y) \cap (z + w) \neq \varnothing$, then either $x + y \subseteq z + w$ or $z + w \subseteq x + y$.

(b) The *almost strongly canonical hypergroup*, which also satisfies the above axiom S_2 and the axiom:
AS: If $x \neq y$, then either $(x - x) \cap (y - x) = \varnothing$ or $(y - y) \cap (y - x) = \varnothing$.

(c) The *superiorly canonical*, which is a strongly canonical hypergroup that also satisfies the axioms:
S_3: If $z, w \in x + y$ and $x \neq y$, then $z - z = w - w$.
S_4: If $x \in z - z$ and $y \notin z - z$, then $x - x \subseteq y - y$.

J. Mittas has presented a very deep and extensive study on these hypergroups, with a great number and variety of results, among which we mention the following theorem [178]:

Theorem 20. *The necessary and sufficient condition for a canonical hypergroup to be hypervaluated is to be superiorly canonical.*

In all the above cases, the neutral element is scalar. Let us now consider hypergroups that are equipped with a strong identity.

Definition 25. *A fortified transposition hypergroup (FTH) is a transposition hypergroup T with a unique strong identity e, which satisfies the axiom:*

For every $x \in T\cdot\cdot\{e\}$ there exists one and only one element $x' \in T\cdot\cdot\{e\}$, the symmetric of x, such that: $e \in xx'$ and furthermore, for x' it holds that $e \in x'x$.

If the hypercomposition is commutative, the hypergroup is called a fortified join hypergroup (FJH).

The fortified join hypergroup was introduced for the study of languages and automata with tools of hypercompositional algebra [35–43].

It has been proved that every fortified transposition hypergroup consists of two types of elements, the *canonical* (*c-elements*) and the *attractive* (*a-elements*) [141,144,149]. An element x is called canonical if $ex = xe$ is the singleton $\{x\}$, while it is called attractive if $ex = xe = \{e, x\}$. A denotes the set of the attractive elements (a-elements) and C denotes the set of the canonical elements (c-elements). By convention, $e \in A$.

Proposition 17. *If $(T, \cdot, e)$ is a fortified transposition hypergroup, then the following are valid:*

i. If $x \neq e$, then $x/e = e\backslash x = x$
ii. $e/e = e\backslash e = A$
iii. If $a \in A$, then $a/a = a\backslash a = A$
iv. If x, y are attractive elements, then $\{x, y\} \subseteq xy$
v. If x, y are attractive elements, then $x \in x/y$ and $x \in y\backslash x$
vi. If $a \in A$ and $c \in C$, then $ac = ca = c$
vii. If $a \in A$ and $a \neq e$, then $e/a = ea^{-1} = \{e, a\} = a^{-1}e = a\backslash e$
viii. If x, y are attractive elements, then $xy^{-1} = x/y \cup \{y^{-1}\}$ and $y^{-1}x = y\backslash x \cup \{y^{-1}\}$.

A detailed and thorough study of the properties of the a-elements and c-elements is presented in [141,144,149]. Theorems 4, 5, 6, and 7, which are valid for the groups, are also valid for the fortified transposition hypergroups (FTH). Theorem 8, though, is not valid in the FTH, as generally $\left(aa^{-1}\right)^{-1} \neq aa^{-1}$ (or $-(a-a) \neq a-a$ in the additive case). This led to the definition of two types of elements: those that satisfy the equality $\left(aa^{-1}\right)^{-1} = aa^{-1}$ (or $-(a-a) = a-a$ in the additive case), which are called *normal* and for which Theorem 8 is valid, and the rest, which are called *abnormal* [141,144,149].

Theorem 21. *If $(Q, \cdot)$is a quasicanonical hypergroup and "$\odot$" is the extensive enlargement of "$\cdot$", then$(Q, \odot)$ is a fortified transposition hypergroup consisting of attractive elements only.*

Proof. Per Proposition 7, the extensive enlargement of an associative law of synthesis is also associative. Next, for the proof of the transposition axiom, we observe that:

$$b\backslash_{\odot} a \cap c_{\odot}/d = [b\backslash a \cup \{a\}] \cap [c/d \cup \{c\}] \text{ and}$$
$$a \odot d \cap b \odot c = [ad \cup \{a, d\}] \cap [bc \cup \{b, c\}]$$

Therefore, if $b\backslash_{\odot} a \approx c_{\odot}/d$, we distinguish the cases:
if $b\backslash a \approx c/d$, then $ad \approx bc$, thus $a \odot d \approx b \odot c$;
if $c \in b\backslash a$, then $a \in bc$, thus $a \in a \odot d \cap b \odot c$, consequently $a \odot d \approx b \odot c$;
if $a \in c/d$, then $c \in ad$, thus $c \in a \odot d \cap b \odot c$, consequently $a \odot d \approx b \odot c$;
if $a = c$, then $a \odot d \approx b \odot c$.

Finally, if e is the neutral element of the quasicanonical hypergroup, then:

$$a \odot e = ae \cup \{a,\ e\} = \{a,\ e\},\ \text{for all } a \in Q.\ \square$$

Example 21. *Let H be a totally ordered set, dense and symmetric around a center denoted by $0 \in H$. The partition $H = H^{-} \cup \{0\} \cup H^{+}$ is defined with regard to this center and according to it, for every $x \in H^{-}$ and $y \in H^{+}$ it is $x < 0 < y$; and for every $x,\ y \in H,\ x \le y \Rightarrow -y \le -x$, where –x is the symmetric of x with regard to 0. Then H, equipped with the hypercomposition:*

$$x + y = \{x, y\},\ if\ y \neq -x$$

and $$x + (-x) = [\,0,\ |x|\,] \cup \{-|x|\}$$

becomes an FJH in which $x - x \neq -(x - x)$, for every $x \neq 0$. So, all the elements of $(H, +)$ are abnormal.

The fortified transposition hypergroup is closely related to the quasicanonical hypergroup as per the following structure theorem:

Theorem 22. [141] *A transposition hypergroup H containing a strong identity e is isomorphic to the expansion of the quasicanonical hypergroup $C \cup \{e\}$ by the transposition hypergroup A of all attractive elements with regard to the identity e.*

Definition 26. *A transposition polysymmetrical hypergroup (TPH) is a transposition hypergroup $(P, \cdot)$ with an idempotent identity e, which satisfies the axioms:*

i. $x \in xe = ex$

ii. *For every $x \in P \cdot\cdot \{e\}$ there exists $x' \in P \cdot\cdot \{e\}$, the symmetric of x, such that $e \in xx'$, and furthermore, x' satisfies $e \in x'x$.*

The set of the symmetric elements of x is denoted by $S(x)$. A commutative transposition polysymmetrical hypergroup is called a join polysymmetrical hypergroup (JPH).

A direct consequence of this definition is that for a nonidentity element x, $S(x) \cup \{e\} = x \backslash e = e/x$, when x is attractive, and $S(x) = x \backslash e = e/x$, when x is non attractive.

Example 22. *Let K be a field and G a subgroup of its multiplicative group. In K we define a hypercomposition "$\dot{+}$" as follows:*

$$x \dot{+} y = \{\, z \in K \mid z = xp + yq,\quad p, q \in G \,\}$$

Then $(K, \dot{+})$ is a join polysymmetrical hypergroup having the 0 of K as its neutral element. Since $x \dot{+} 0 = \{\, xq,\quad q \in G \,\}$, the neutral element 0 is neither scalar nor strong. The symmetric set of an element x of K is $S(x) = \{-xp \mid p \in G\}$.

Example 23. *Let $(A_i, \cdot)$, $i \in I$, be a family of fortified transposition hypergroups that consist only of attractive elements, and suppose that the hypergroups A_i, $i \in I$ have a common identity e. Then $T = \bigcup\limits_{i \in I} A_i$ becomes a transposition polysymmetrical hypergroup under the hypercomposition:*

$a \odot b = ab$ *if a, b are elements of the same hypergroup A_i*

$a \odot b = \{a,\ e,\ b\}$ *if $a \in A_i \cdot\cdot \{e\},\ b \in A_j \cdot\cdot \{e\}$ and $i \neq j$*

Observe that e is a strong identity in T. Moreover, if $a \in A_i$, then $S(a) = (T \cdot\cdot A_i) \cup \{a'\}$ where a' is the inverse of a in A_i.

Example 24. *Let Δ_i, $i \in I$ be a family of totally ordered sets that have a common minimum element e. The set $\Delta = \bigcup_{i \in I} \Delta_i$ with hypercomposition:*

$$xy = \begin{cases} [min\{x,y\}\,,\; max\{x,y\}] & if\;\; x,\, y \in \Delta_i\,,\; i \in I \\ [e,\, x] \cup [\,e,\, y\,] & if\;\; x \in \Delta_i,\;\; y \in \Delta_j \quad and \quad i \neq j, \quad i,\, j \in I \end{cases}$$

becomes a JPH with neutral element e.

Proposition 18. *If x is an attractive element of a transposition polysymmetrical hypergroup, then $S(x)$ consists of attractive elements.*

Proof. Let $e \in ex$. Then $x \in e \backslash e$. Moreover, $x \in e/x'$ for any $x' \in S(x)$. Thus, $e/x' \cap e \backslash e \neq \varnothing$, which, by the transposition axiom, gives $ee \cap ex' \neq \varnothing$, or $e \in ex'$. Hence x' is attractive. □

Proposition 19. *The result of the hypercomposition of two attractive elements in a transposition polysymmetrical hypergroup consists of attractive elements only, while the result of the hypercomposition of an attractive element with a non attractive element consists of non attractive elements.*

Proposition 20. *If the identity in a transposition polysymmetrical hypergroup is strong, then $\{x, y\} \subseteq xy$ and $x \in x/y$, $x \in y \backslash x$, for any two attractive elements x, y.*

The algebraic properties of transposition polysymmetrical hypergroups are studied in [145,146].

Definition 27. *A quasicanonical polysymmetrical hypergroup is a hypergroup H with a unique scalar identity e, which satisfies the axioms:*

i. *For every $x \in H$ there exists at least one element $x' \in H$, called symmetric of x, such that $e \in xx'$ and $e \in x'x$.*

ii. *If $z \in xy$, there exist x', $y' \in S(x)$ such that $x \in zy'$ and $y \in x'z$.*

A commutative quasicanonical polysymmetrical hypergroup is called a canonical polysymmetrical hypergroup.

Example 25. *Let H be a set that is totally ordered and symmetric around a center, denoted by $0 \in H$. Then H, equipped with the hypercomposition:*

$$x + y = \begin{cases} y, & if \;\; |x| < |y| \;\; for\, every \;\; x, y \in H^- \cup \{0\} \;\; or \;\; x, y \in \{0\} \cup H^+ \\ [x,\, y], & if\; x \in H^- \;\; and \;\; y \in H^+ \end{cases}$$

becomes a canonical polysymmetrical hypergroup. Suppose now that $x, y, a, b \in H^+$ and $x < y < a < b$. Then $x/y = a/b = H^-$. Thus, $x/y \cap a/b \neq \varnothing$. However, $x + b = \{b\}$ and $y + a = \{a\}$. Therefore $(x + b) \cap (y + a) = \varnothing$, and so the transposition axiom is not valid.

The canonical polysymmetrical hypergroup was introduced by J. Mittas [181]. In addition, J. Mittas and Ch. Massouros, while studying the applications of hypergroups in the linear spaces, defined the generalized canonical polysymmetrical hypergroup [31]. Moreover, J. Mittas, in his paper [174], motivated by an observation about algebraically closed fields, discovered a special type of completely regular polysymmetrical hypergroup, which, later on, C. Yatras called *M-polysymmetrical hypergroup*. C. Yatras, in a series of papers, studied this hypergroup and its properties in detail [182–184]. J. Mittas also defined the *generalized M-polysymmetrical hypergroups* that were studied by himself and by Ch. Massouros [185,186].

Definition 28. *A M-polysymmetrical hypergroup H is a commutative hypergroup with an idempotent identity e that also satisfies the axioms:*

i. $x \in xe = ex$,

ii. *For every* $x \in H \cdot \cdot \{e\}$ *there exists at least one element* $x' \in H \cdot \cdot \{e\}$, *such that* $e \in xx'$, *and* $e \in x'x$, *(symmetric of x),*

iii. *If* $z \in xy$ *and* $x' \in S(x)$, $y' \in S(y)$, $z' \in S(z)$, *then* $z' \in x'\, y'$.

Proposition 21. [182,183] *Every M-polysymmetrical hypergroup is a join hypergroup.*

In addition to the aforementioned hypergroups, other hypergroups have been defined and studied. Among them, are the complete hypergroups and the complete semihypergroups [17,102,107,112,187–189], the 1-hypergroups [102,107,190], the hypergroups of type U [191–198], the hypergroups of type C [199,200], and the cambiste hypergroup [28].

7. Subhypergroups and Cosets

Decompositions and partitions play an important role in the study of algebraic structures. Undoubtedly, this study is of particular interest in the theory of hypercompositional algebra. It has recently been addressed in various papers from different perspectives (e.g., [188,201–204]). Moreover, in [33] it is proved that general decomposition theorems that are valid in hypergroups give as corollaries well-known decomposition theorems in convex sets. A far-reaching concept of abstract group theory is the decomposition of a group into cosets by its subgroups. The hypergroup, though, being a more general structure than that of the group, has various types of subhypergroups. In contrast to groups, where any subgroup decomposes the group into cosets, in the hypergroups, not all the subhypergroups can define such a partition. This section presents the subhypergroups that can create a partition in the hypergroup and the relevant partitions.

Definition 29. *A nonempty subset K of H is a semi-subhypergroup when it is stable under the hypercomposition, i.e., it has the property* $xy \subseteq K$ *for all* $x, y \in K$. *K is a subhypergroup of H if it satisfies the reproductive axiom, i.e., if the equality* $xK = Kx = K$ *is valid for all* $x \in K$.

7.1. Closed and Ultra-Closed Subhypergroups

From the above Definition 29, it derives that when K is a subhypergroup and $a, b \in K$, the relations $a \in bx$ and $a \in yb$ always have solutions in K. If, for any two elements a and b in K, all the solutions of the relation $a \in yb$ lie inside K, then K is called *right closed* in H . Similarly, K is *left closed* when all the solutions of the relation $a \in bx$ lie in K. K is *closed* when it is both right and left closed. Note that the concepts subhypergroup and closed subhypergroup are self-dual. A direct consequence of the definition of the closed subhypergroup is the proposition:

Proposition 22. *The nonvoid intersection of two closed subhypergroups is a closed subhypergroup.*

The relevant property is not valid for every subhypergroup, since, although the nonvoid intersection of two subhypergroups is stable under the hypercomposition, the validity of the reproductive axiom fails. This was one of the reasons that led, from the very beginning of the hypergroup theory, to the consideration of more special types of subhypergroups, one of which is the above defined closed subhypergroup (e.g., see [5,13]). An equivalent definition of the closed hypergroup is the following one:

Definition 30. [205,206] *A subhypergroup K of H is called right closed if K is stable under the right division, i.e., if* $a/b \subseteq K$ *for all* $a, b \in K$. *K is called left closed if K is stable under the left division, i.e., if* $b \backslash a \subseteq K$, *for all* $a, b \in K$. *K is called closed when it is both right and left closed.*

Proposition 23. *If the hypercomposition in a hypergroup H is closed (containing), then H has no proper closed subhypergroups.*

Proof. According to Proposition 1, if a hypercomposition is closed, then $a/a = H$, for all $a \in H$. Consequently, the only closed subhypergroup of H is H itself. □

Proposition 24. *If K is a subset of a hypergroup H such that $a/b \subseteq K$ and $b\backslash a \subseteq K$, for all $a, b \in K$, then K is a closed subhypergroup of H.*

Proof. Initially, it will be proved that K is a hypergroup, i.e., that $aK = Ka = K$, for any a in K. Let $x \in K$. Then $a\backslash x \subseteq K$. Therefore $x \in aK$. Hence $K \subseteq aK$. For the reverse inclusion, now suppose that $y \in aK$. Then $K/y \subseteq K/aK$. So $K \cap (K/aK)y \neq \varnothing$. Thus, $y \in (K/aK)\backslash K$. Per mixed associativity, the equality $K/aK = (K/K)/a$ is valid. Thus:

$$(K/aK)\backslash K = ((K/K)/a)\backslash K \subseteq (K/a)\backslash K \subseteq (K/K)\backslash K \subseteq K\backslash K \subseteq K$$

Hence $y \in K$ and so $aK \subseteq K$. Therefore $aK = K$. The equality $Ka = K$ follows by duality. The rest comes from Definition 30. □

Proposition 25. *If K is a closed hypergroup of a hypergroup H*, then:

$$K = K/a = a/K = a\backslash K = K\backslash a$$

for all a in K.

Proposition 26. *If K is a subhypergroup of H, then:*

$$H \cdot \cdot K \subseteq (H \cdot \cdot K)x \text{ and } H \cdot \cdot K \subseteq x(H \cdot \cdot K)$$

for all $x \in K$.

Proof. Let $y \in H \cdot \cdot K$ and $y \notin (H \cdot \cdot K)x$. Per reproductive axiom, $y \in Hx$ and since $y \notin (H \cdot \cdot K)x$, y must be a member of Kx. Thus, $y \in Kx \subseteq KK = K$. This contradicts the assumption, and so $H \cdot \cdot K \subseteq (H \cdot \cdot K)x$. The second inclusion follows by duality. □

Proposition 27. *i. A subhypergroup K of H is right closed in H if and only if $(H \cdot \cdot K)x = H \cdot \cdot K$, for all $x \in K$.*
ii. A subhypergroup K of H is left closed in H if and only if $x(H \cdot \cdot K) = H \cdot \cdot K$, for all $x \in K$.
iii. A subhypergroup K of H is closed in H if and only if $x(H \cdot \cdot K) = (H \cdot \cdot K)x = H \cdot \cdot K$, for all $x \in K$.

Proof. (i) Let K be right closed in H. Suppose that $z \in H \cdot \cdot K$ and $zx \cap K \neq \varnothing$. Then there exists an element y in K such that $y \in zx$, or equivalently $z \in y/x$. Therefore $z \in K$, which is absurd. Hence $(H \cdot \cdot K)x \subseteq H \cdot \cdot K$. Next, because of Proposition 26, $H \cdot \cdot K \subseteq (H \cdot \cdot K)x$ and therefore $H \cdot \cdot K = (H \cdot \cdot K)x$. Conversely now: suppose that $(H \cdot \cdot K)x = H \cdot \cdot K$ for all $x \in K$. Then $(H \cdot \cdot K)x \cap K = \varnothing$ for all $x \in K$. Hence $t \notin rx$ and so $r \notin t/x$ for all $x, t \in K$ and $r \in H \cdot \cdot K$. Therefore $t/x \cap (H \cdot \cdot K) = \varnothing$ which implies that $t/x \subseteq K$. Thus K is right closed in H. (ii) follows by duality and (iii) is an obvious consequence of (i) and (ii). □

Corollary 11. *i. If K is a right closed subhypergroup in H, then $xK \cap K = \varnothing$, for all $x \in H \cdot \cdot K$.*
ii. If K is a left closed subhypergroup in H, then $Kx \cap K = \varnothing$, for all $x \in H \cdot \cdot K$.
iii. If K is a closed subhypergroup in H, then $xK \cap K = \varnothing$ and $Kx \cap K = \varnothing$, for all $x \in H \cdot \cdot K$.

Proposition 28. *Let A be a nonempty subset of a hypergroup H and suppose that*

$$A_0 = A \cup AA \cup A/A \cup A\backslash A$$
$$A_1 = A_0 \cup A_0A_0 \cup A_0/A_0 \cup A_0\backslash A_0$$
$$\vdots$$
$$A_n = A_{n-1} \cup A_{n-1}A_{n-1} \cup A_{n-1}/A_{n-1} \cup A_{n-1}\backslash A_{n-1}$$

Then $\langle A \rangle = \underset{n\geq 0}{\cup} A_n$ is the least closed subhypergroup of H, which contains A.

If A is a singleton, then the above procedure constructs the monogene closed subhypergroup of the hypergroup H. It is easy to see that if the hypercomposition is open, then $\langle a \rangle = \{a\}$, while if it is closed, $\langle a \rangle = H$ for all $a \in H$. The notion of the monogene subhypergroups was introduced by J. Mittas in [177] for the case of the canonical hypergroups. In [67], there is a detailed study of the monogene symmetric subhypergroups of the fortified join hypergroups. These studies highlight the existence of two types of the order of an element: the *principal order*, which is an integer, and the *associated order*, which is a function. A subcategory of the monogene subhypergroups is the cyclic subhypergroups; that is, subhypergroups of the form:

$$\underset{k\in N}{\cup} a^k,\ a \in H$$

It is easy to observe that in the case of the open hypercompositions, the cyclic subhypergroup, which is generated by an element a, is the element a itself. The study of cyclic subhypergroups has attracted the interest of many researchers [207–218]. A detailed study and thorough review of cyclic hypergroups is given in [219].

The closed subhypergroups are directly connected with the ultra-closed subhypergroups [17,220]. The properties of these subhypergroups, together with their ability to create cosets in special hypergroups, were studied in a long series of joint papers by M. De Salvo, D. Freni, D. Fasino, and G. Lo Faro [191–195]; D. Freni [196–198]; M. Gutan et al. [198–200]; and L. Haddad and Y. Sureau [221,222]. A new definition for the ultra-closed subhypergroups, with the use of the induced hypercompositions, is given below. After that, Theorem 24 proves that this definition is equivalent to Sureau's definition.

Definition 31. *A subhypergroup K of a hypergroup H is called right ultra-closed if it is closed and $a/a \subseteq K$ for each $a \in H$. K is called left ultra-closed if it is closed and $a\backslash a \subseteq K$ for each $a \in H$. If K is both right and left ultra-closed, then it is called ultra-closed.*

Theorem 23. *i. If K is right ultra-closed in H, then, either $a/b \subseteq K$, or $a/b \cap K = \varnothing$, for all $a, b \in H$. Moreover, if $a/b \subseteq K$, then $b/a \subseteq K$.*
ii. If K is left ultra-closed in H, then, either $b\backslash a \subseteq K$, or $b\backslash a \cap K = \varnothing$, for all $a, b \in H$. Moreover, if $b\backslash a \subseteq K$, then $a\backslash b \subseteq K$.

Proof. Suppose that $a/b \cap K \neq \varnothing$, $a, b \in H$. Then $a \in sb$, for some $s \in K$. Next, assume that $b/a \cap (H \cdot \cdot K) \neq \varnothing$. Then $b \in ta$, $t \in H \cdot \cdot K$. Thus $a \in s(ta) = (st)a$. Since K is closed, per Proposition 27, $st \subseteq H \cdot \cdot K$. So $a \in ra$, for some $r \in H \cdot \cdot K$. Therefore $a/a \cap (H \cdot \cdot K) \neq \varnothing$, which is absurd. Hence $b/a \subseteq K$. Now let x be an element in K such that $b \in xa$. If $a/b \cap (H \cdot \cdot K) \neq \varnothing$, there exists $y \in H \cdot \cdot K$ such that $a \in yb$. Therefore $b \in x(yb) = (xy)b$. Since K is closed, per Proposition 27, $xy \subseteq H \cdot \cdot K$. So, $b \in zb$ for some $z \in H \cdot \cdot K$. Therefore $b/b \cap (H \cdot \cdot K) \neq \varnothing$, which is absurd. Hence $a/b \subseteq K$. Duality gives (ii). □

Theorem 24. *Let H be a hypergroup and K a subhypergroup of H. Then:*
i. K is a right ultra-closed subhypergroup of H if and only if $Ka \cap (H \cdot \cdot K)a = \varnothing$, for all $a \in H$.
ii. K is a left ultra-closed subhypergroup of H if and only if $aK \cap a(H \cdot \cdot K) = \varnothing$, for all $a \in H$.

Proof. Suppose that K is a right ultra-closed subhypergroup of H. Then $a/a \subseteq K$ for all $a \in H$. Since K is right closed, then $(a/a)/s \subseteq K$ is valid, or equivalently, $a/(as) \subseteq K$ for all $s \in K$. Theorem 23 yields $(as)/a \subseteq K$ for all $s \in K$. If $Ka \cap (H \cdot \cdot K)a \neq \varnothing$, then there exist $s \in K$ and $r \in H \cdot \cdot K$, such that $sa \cap ra \neq \varnothing$, which implies that $r \in as/a$. But $(as)/a \subseteq K$, hence $r \in K$, which is absurd. Conversely now: let $Ka \cap (H \cdot \cdot K)a = \varnothing$ for all $a \in H$. If $a \in K$, then $K \cap (H \cdot \cdot K)a = \varnothing$. Therefore $s \notin ra$, for every $s \in K$ and $r \in H \cdot \cdot K$. Equivalently, $s/a \cap (H \cdot \cdot K) = \varnothing$, for all $s \in K$. Hence $s/a \subseteq K$ for all $s \in K$ and $a \in K$. So K is right closed. Next, suppose that $a/a \cap (H \cdot \cdot K) \neq \varnothing$ for some $a \in H$. Then $a \in (H \cdot \cdot K)a$, or $Ka \subseteq K(H \cdot \cdot K)a$. Since K is closed, per Proposition 27, $K(H \cdot \cdot K) \subseteq H \cdot \cdot K$ is valid. Thus $Ka \subseteq (H \cdot \cdot K)a$, which contradicts the assumption. Duality gives (ii). □

7.2. Invertible Subhypergroups and their Cosets

Definition 32. *A subhypergroup K of a hypergroup H is called right invertible if $a/b \cap K \neq \varnothing$ implies $b/a \cap K \neq \varnothing$, with $a, b \in H$, while it is called left invertible if $b\backslash a \cap K \neq \varnothing$ implies $a\backslash b \cap K \neq \varnothing$, with $a, b \in H$. K is invertible when it is both right and left invertible.*

Note that the concept of the invertible subhypergroup is self-dual. Theorem 4 in [134] gives an interesting example of an invertible subhypergroup in a join hypergroup of partial differential operators.

Theorem 25. *If K is invertible in H, then K is closed in H.*

Proof. Let $x \in K/K$. Then $K \approx xK$, thus $K \approx x\backslash K$. Since K is invertible, $K \approx x\backslash K$ implies $K \approx K\backslash x$ from which it follows that $x \in KK$. Since K is a subhypergroup, $KK = K$. Therefore $x \in K$. Consequently $K/K \subseteq K$. Since invertibility is self-dual, $K\backslash K \subseteq K$ is valid as well. □

The converse of Theorem 25 is not true. In [17] there are examples of closed hypergroups that are not invertible. On the other hand, the following proposition is a direct consequence of Theorem 23.

Proposition 29. *i. The right (left) ultra-closed subhypergroups of a hypergroup are right (left) invertible.*
ii. The ultra-closed subhypergroups of a hypergroup are invertible.

Theorem 25 and Proposition 23 give the following result:

Proposition 30. *A hypergroup H has no proper invertible subhypergroups if its hypercomposition is closed (containing).*

Proposition 31. *A hypergroup H has no proper invertible subhypergroups if the hypercomposition is open.*

Proof. Suppose that K is an invertible subhypergroup of H and a is an element of H that does not belong to K. Because of the reproductive axiom, there exists $b \in H$ such that $a \in Kb$. Therefore $a/b \cap K \neq \varnothing$, which, per invertibility of K, implies that $b/a \cap K \neq \varnothing$. Hence $b \in Ka$, and so

$$a \in Kb \subseteq K(Ka) = (KK)a = Ka.$$

Thus, $a/a \cap K \neq \varnothing$. But, according to Proposition 1, $a/a = a$. Therefore $a \in K$, which contradicts the assumption. □

Proposition 32. *i. K is right invertible in H, if and only if the implication $a \in Kb \Rightarrow b \in Ka$ is valid for all $a, b \in H$.*
ii. K is left invertible in H if and only if the implication $b \in aK \Rightarrow a \in bK$ is valid for all $a, b \in H$.

Lemma 1. *i. If the implication* $Ka \neq Kb \Rightarrow Ka \cap Kb = \varnothing$ *is valid for all* $a, b \in H$*, then* $c \in Kc$*, for each* $c \in H$*.*
ii. If the implication $aK \neq bK \Rightarrow aK \cap bK = \varnothing$ *is valid for all* $a, b \in H$*, then* $c \in cK$*, for each* $c \in H$*.*

Proof. (i) There exists $x \in H$ such that $c \in Kx$. Hence $Kc \subseteq Kx$. Therefore $Kc \cap Kx \neq \varnothing$. So, $c \in Kx = Kc$. (ii) is the dual of (i). □

Theorem 26. *i. K is right invertible in H, if and only if the implication* $Ka \neq Kb \Rightarrow Ka \cap Kb = \varnothing$ *is valid for all* $a, b \in H$*.*
ii. K is left invertible in H, if and only if the implication $aK \neq bK \Rightarrow aK \cap bK = \varnothing$ *is valid for all* $a, b \in H$*.*

Proof. (i) Let K be right invertible in H. Assume that $Ka \neq Kb$ and that $c \in Ka \cap Kb$. Then $Kc \subseteq Ka$ and $Kc \subseteq Kb$. Moreover, $c \in Ka \cap Kb$ implies that $c/a \cap K \neq \varnothing$ and $c/b \cap K \neq \varnothing$. But K is invertible, consequently we have the sequence of the following equivalent statements:

$$a/c \cap K \neq \varnothing \text{ and } b/c \cap K \neq \varnothing; \;\; a \in Kc \text{ and } b \in Kc; \;\; Ka \subseteq Kc \text{ and } Kb \subseteq Kc$$

Therefore, $Ka = Kc$ and $Kb = Kc$. Thus, $Ka = Kb$, which contradicts the assumption. So, $Ka \cap Kb = \varnothing$. Conversely now: suppose that $Ka \neq Kb \Rightarrow Ka \cap Kb = \varnothing$ is valid for all $a, b \in H$ and moreover, assume that $a/b \cap K \neq \varnothing$. Then $a \in Kb$, consequently $Ka \subseteq Kb$. Since $Ka \cap Kb \neq \varnothing$, it derives that $Ka = Kb$. Per Lemma 1, $b \in Kb$. Therefore $b \in Ka$, which implies that $b/a \cap K \neq \varnothing$. (ii) is the dual of (i). □

Theorem 27. *i. If K is right invertible in H, then* $H \nearrow K = \{Kx \mid x \in H\}$ *is a partition in H.*
ii. If K is left invertible in H, then $H \swarrow K = \{xK \mid x \in H\}$ *is a partition in H.*
iii. If K is invertible in H, then $H \swarrow\!\!\!\nearrow K = \{KxK \mid x \in H\}$ *is a partition in H.*

Proof. The proofs of (i) and (ii) are similar to the proof of (iii). So, we prove (iii).
(iii) According to Lemma 1, if c is an arbitrary element in H, then $c \in Kc$ and $c \in cK$. $c \in Kc$ implies $cK \subseteq KcK$, and since $c \in cK$, it derives that $c \in KcK$. Now suppose that there exist $a,\ b \in H$ such that $KaK \cap KbK \neq \varnothing$. Let $c \in KaK \cap KbK$. $c \in KaK$ implies that there exists $y \in Ka$ such that $c \in yK$. However, due to the invertibility of K, it follows that $a \in Ky$ and $y \in cK$. Hence $a \in KcK$ and therefore $KaK \subseteq KcK$. Moreover, $c \in KaK$ implies that $KcK \subseteq KaK$. Consequently $KaK = KcK$. In the same way $KbK = KcK$, and therefore $KaK = KbK$. □

Theorem 28. *Let K be a right invertible subhypergroup of a hypergroup H and let* $X = Kx$*,* $x \in H$*. Then* $X * Y = XY \nearrow K$ *is a hypercomposition in* $H \nearrow K = \{Kx \mid x \in H\}$ *and* $(H \nearrow K, *)$ *is a hypergroup.*

Corollary 12. *If K is a subgroup of a group G, then* $(G \nearrow K, *)$ is a hypergroup.

Theorem 29. *Suppose that K is a subgroup of a group G, and* $K \subseteq M \subseteq G$*. Then* $(M \nearrow K, *)$ *is a subhypergroup of* $(G \nearrow K, *)$ *if and only if M is a subgroup of a group G.*

Proof. Suppose that $M \nearrow K$ is a subhypergroup of $G \nearrow K$. Then:

$$Kx * (M \nearrow K) = M \nearrow K = (M \nearrow K) * Kx, \text{ for all } Kx \in M \nearrow K$$

Therefore $KxM = M = MKx$, for all $x \in M$. Since $e \in K$, it follows that $xM = M$ and $Mx = M$ for all $x \in M$. Thus, M is a subgroup of G. Conversely now, let M be a subgroup of G and $x \in M$. Then:

$$Kx * (M \nearrow K) = KxM \nearrow K = KM \nearrow K = M \nearrow K$$

and $$(M \nearrow K) * Kx = MKx \nearrow K = Mx \nearrow K = M \nearrow K.$$

Consequently, $(M \nearrow K, *)$ is a subhypergroup of $(G \nearrow K, *)$. □

7.3. Reflexive, Closed Subhypergroups and their Cosets

As shown above, when the hypercomposition is open or closed, there do not exist proper invertible subhypergroups. So, in such cases, the hypergroup cannot be decomposed into cosets with the use of the previous techniques, and different methods need to be developed in order to solve the decomposition problem. Such a method that uses a special type of closed subhypergroups is presented below, for the case of the transposition hypergroups.

Definition 33. *A subhypergroup N of a hypergroup H is called normal or invariant if $aN = Na$, for all $a \in H$.*

Proposition 33. *N is an invariant subhypergroup of a hypergroup H if and only if $N \backslash a = a/N$, for all $a \in H$.*

Proof. Suppose that N is invariant in H. Then we have the sequence of equivalent statements:

$$x \in N\backslash a;\ a \in Nx;\ a \in xN;\ x \in a/N$$

Therefore $N\backslash a = a/N$. Conversely now, suppose that $N\backslash a = a/N$. Then we have the following equivalent statements:

$$x \in aN;\ a \in x/N;\ a \in N\backslash x;\ x \in Na$$

Consequently, N is invariant. □

Definition 34. *A subhypergroup R of a hypergroup H is called reflexive if $a\backslash R = R/a$, for all $a \in H$.*

Obviously, all the subhypergroups of the commutative hypergroups are invariant and reflexive. The closed and reflexive subhypergroups have a special interest in transposition hypergroups because they can create cosets.

Theorem 30. *If L is a closed and reflexive subhypergroup of a transposition hypergroup T, then the sets L/x form a partition in T.*

Proof. Let $L/x \cap L/y \neq \varnothing$. Then $x\backslash L \cap L/y \neq \varnothing$, and therefore $xL \cap Ly \neq \varnothing$. Consequently, $x \in Ly/L$. Next, using the properties mentioned in Propositions 11 and 3 successively, we have:

$$L/x \subseteq L/(Ly/L) \subseteq LL/Ly = L/Ly = (L/y)/L = L\backslash(L/y) = (L\backslash L)/y = L/y$$

Hence $L/x \subseteq L/y$. By symmetry, $L/y \subseteq L/x$, and so $L/x = L/y$. □

According to the above theorem, for each $x \in T$ there exists a unique class that contains x. This unique class is denoted by L_x. The set of the classes modulo L is denoted by T_L. Note that $L \in T_L$. Indeed, since L is closed, $L/x = L$ for any $x \in L$.

Proposition 34. *If* $xy \cap L \neq \varnothing$*, then* $L/x = L_y$ *and* $L/y = L_x$*.*

Proof. $xy \cap L \neq \varnothing$ implies both $x \in L/y$ and $y \in x\backslash L = L/x$. Therefore $L_x = L/y$ and $L_y = L/x$. □

From the above proposition, it becomes evident that L_x and L/x are different classes in T_L. The following theorems reveal the algebraic form of the class L_x, $x \in T$.

Theorem 31. *If* L_x *is a class, then* L/L_x *is also a class modulo* L *and* $L/L_x = L/y$ *for some* $y \in L_x$ *.*

Proof. Let $y \in L_x$. It suffices to prove that $L/L_x = L/y$, because this implies that L/L_x is a class modulo L. Since $y \in L_x$, it follows that $L/y \subseteq L/L_x$. For the proof of the reverse inclusion, let $z \in L/L_x$. Then $L \cap zL_x \neq \varnothing$, hence $L_x \cap z\backslash L = L_x \cap L/z \neq \varnothing$. By Theorem 30, $L_x = L/z$, and so $y \in L/z$. Then, $yz \cap L \neq \varnothing$ and $z \in y\backslash L = L/y$. Consequently, $L/L_x \subseteq L/y$, hence the theorem. □

Theorem 32. *If* L_x *is the class modulo* L *of an element* x *in* T*, then:*

$$L_x = Lx/L = L\backslash Lx = L/(L/x) = (x\backslash L)\backslash L.$$

Proof. Let $x \in T$. Since L is reflexive, $x\backslash L = L/x$. Applying the transposition axiom, we get $xL \approx Lx$, which implies that $x \in Lx/L$ and $x \in L\backslash xL$. Next, from Proposition 11 (ii), it follows that $L/(L/x) \subseteq Lx/L$. For the proof of the reverse inclusion, Corollary 2 and Proposition 11 (ii) sequentially give:

$$Lx/L \subseteq [L/(Lx/L)]\backslash L \subseteq (LL/Lx)\backslash L = (L/Lx)\backslash L$$

Since L is reflexive, the equality $(L/Lx)\backslash L = L/(L/Lx)$ holds. Therefore, Proposition 3 and Proposition 11 (ii) give:

$$L/(L/Lx) = L/[(L/x)/L] \subseteq LL/(L/x) = L/(L/x)$$

Consequently, $L/(L/x) = Lx/L$. Duality yields the rest. □

Corollary 13. *If* T *is a quasicanonical hypergroup, then* $L_x = xL$*.*

Proof. Per Proposition 16, $x/L = xL^{-1}$, thus:

$$L_x = (x/L)/L = \left(xL^{-1}\right)L^{-1} = (xL)L = x(LL) = xL. \ \square$$

Proposition 35. *If* L *is a reflexive closed subhypergroup of a transposition hypergroup* T*, then:*

$$(L/x)(L/y) \subseteq L/xy, \ \ x, y \in T.$$

Proof. The successive application of Proposition 11 (iii) and mixed associativity gives:

$$(L/x)(L/y) = (x\backslash L)(L/y) \subseteq (x\backslash LL)/y = (x\backslash L)/y = (L/x)/y = L/xy. \ \square$$

A direct consequence of the above proposition is that the partition which is defined by L is regular. Therefore, a hypercomposition "$\cdot$" is defined in T_L by

$$(L/x) \cdot (L/y) = \{L/z \mid z \in xy\}.$$

Next, it is apparent that L is the neutral element of the above hypercomposition and that the transposition is valid. Hence the theorem:

Theorem 33. *If L is a reflexive closed subhypergroup of a transposition hypergroup T, then $(T_L, \cdot, L)$ is a quasicanonical hypergroup.*

Example 26. *A three-dimensional Euclidian space becomes a hypergroup under the hypercomposition defined in Example 4. It is easy to verify that this is a commutative transposition hypergroup of idempotent elements; that is, a join space. The notation E_J^3 is used for this hypergroup. A line L of the Euclidian space is a reflexive closed subhypergroup of E_J^3. The cosets that L defines in E_J^3 are the halfplanes L/x, which are drawn in the following Figure 4. $\left(E_J^3\right)_L$ is a quasicanonical hypergroup, L is its neutral element, and the symmetric of any element L/x is the element $L/(L/x)$.*

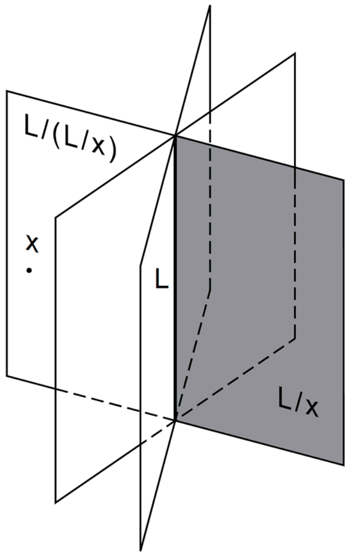

Figure 4. Example 26.

7.4. Symmetric Subhypergroups and their Cosets

As mentioned above, when the hypercomposition in a hypergroup is closed (containing), then there do not exist closed subhypergroups. In this case, other types of subhypergroups must be used for the decomposition of the hypergroup. An example of such a hypergroup is the transposition hypergroup with idempotent identity, which consists of attractive elements only. As shown in the sixth section of this paper, the transposition hypergroups can have one (fortified transposition hypergroups) or more (transposition polysymmetrical hypergroups) symmetric elements for each one of their elements. Obviously, a transposition polysymmetrical hypergroup becomes a fortified transposition hypergroup when the set $S(x)$ of the symmetric elements of any one of its elements x is a singleton, and moreover, when its identity is strong. The quite general case of the decomposition of the transposition polysymmetrical hypergroups with strong identity into cosets is pesented below. Since the result of the hypercomposition between two attractive elements contains these two elements, there do not exist proper closed subhypergroups. There exist, though, subhypergroups that contain the symmetric element of each one of their elements. These subhypergroups are the ones that decompose the transposition polysymmetrical hypergroups into cosets. In the following, P signifies a transposition polysymmetrical hypergroup with strong identity.

Definition 35. *A subhypergroup M of a transposition polysymmetrical hypergroup is called symmetric if $S(x) \subseteq M$, for all x in M .*

Proposition 36. *Let x, y be two elements in P such that $e \notin xS(y)$, then $xS(y) = x/y \cup S(y)$ and $S(y)x = y \backslash x \cup S(y)$.*

Proof. $xS(y) \subseteq x(e/y) \subseteq xe/y = \{x, e\}/y = x/y \cup e/y = x/y \cup \{e\} \cup S(y)$. Since e is not in $xS(y)$, it follows that $xS(y) \subseteq x/y \cup S(y)$. Moreover, since $e \in yS(y)$, it follows that $y \in e/S(y)$. Therefore $x/y \subseteq x/(e/S(y))$. Hence:

$$x/y \cup S(y) \subseteq x/(e/S(y)) \cup S(y) \subseteq xS(y)/e \cup S(y) = xS(y) \cup S(y) = xS(y)$$

Thus, $xS(y) = x/y \cup S(y)$. Dually, $S(y)x = y\backslash x \cup S(y)$. □

Proposition 37. *Let M be a symmetric subhypergroup of P. If $x \notin M$*, then:

i. $x/M \cap M = \varnothing$ *and* $M\backslash x \cap M = \varnothing$,
ii. $xM = x/M \cup M$ *and* $Mx = M\backslash x \cup M$,
iii. $M/x = M\,S(x)$ *and* $x\backslash M = S(x)M$,
iv. $(x/M)M = xM$ *and* $M(M\backslash x) = Mx$.

Proof. The two statements in each one of (i)–(iv) are dual, and therefore it is sufficient to prove only one of them.
(i) Let $x \notin M$ and $y \in M$ such that $x/y \cap M \neq \varnothing$. Then, $x \in My = M$, which contradicts the assumption for x. Thus, $x/M \cap M = \varnothing$.
(ii) Since M is symmetric, $S(M) = M$. Thus, $e \notin xM$, since $x \notin M$. So, via Proposition 36:

$$xM = xS(M) = x/M \cup S(M) = x/M \cup M.$$

(iii) According to Proposition 36, the equality $M\,S(x) = M/x \cup S(x)$ holds, therefore

$$M/x = M/x \cup e/x = M/x \cup S(x) = M\,S(x).$$

(iv) Since $x \in x/M$, it follows that $xM \subseteq (x/M)M$. In addition, because of (ii), $x/M \subseteq xM$, thus,

$$(x/M)M \subseteq (xM)M = x(MM) = xM. \ \square$$

Theorem 34. *Let M be a symmetric subhypergroup of P. If $x, y \notin M$, then:*

i. $x/M \approx y/M$ *implies* $x/M = y/M$,
ii. $M\backslash x \approx M\backslash y$ *implies* $M\backslash x = M\backslash y$,
iii. $M\backslash(x/M) \approx M\backslash(y/M)$ *implies* $M\backslash(x/M) = M\backslash(y/M)$,
iv. $(M\backslash x)/M \approx (M\backslash y)/M$ *implies* $(M\backslash x)/M = (M\backslash y)/M$.

Proof. (i) $x/M \cap y/M \neq \varnothing$ implies that $x \in (y/M)M$. Since $y \notin M$, from (iv) and (ii) of Proposition 37, it follows that $(y/M)M = yM \subseteq y/M \cup M$. Thus, $x \in y/M \cup M$. Since $x \notin M$, it follows that $x \in y/M$. Thus, $x/M \subseteq (y/M)/M = y/(MM) = y/M$. By symmetry, $y/M \subseteq x/M$. Hence $x/M = y/M$.
(ii) is the dual of (i).
(iii) Per Corollary 1, mixed associativity, and Propositions 11 and 37 (iv), we have:

$$M\backslash(x/M) \approx M\backslash(y/M) \Rightarrow (M\backslash x)/M \approx M\backslash(y/M) \Rightarrow M\backslash x \approx [M\backslash(y/M)]M \Rightarrow$$

$$\Rightarrow M\backslash x \approx M\backslash[(y/M)M] \Rightarrow M\backslash x \approx M\backslash yM \Rightarrow x \in yM \Rightarrow y \in x/M \Rightarrow$$

$$\Rightarrow y/M \subseteq (x/M)/M \Rightarrow y/M \subseteq x/(MM) \Rightarrow y/M \subseteq x/M \Rightarrow M\backslash(y/M) \subseteq M\backslash(x/M).$$

By symmetry, $M\backslash(x/M) \subseteq M\backslash(y/M)$, thus the equality is valid.
Finally, (iv) is true because $(M\backslash x)/M = M\backslash(x/M)$ and $(M\backslash y)/M = M\backslash(y/M)$. □

If $x \in P$ and M is a nonempty symmetric subhypergroup of P, then $x_{\underset{M}{\leftarrow}}$ (i.e., the *left coset* of M determined by x) and dually, $x_{\underset{M}{\rightarrow}}$ (i.e., the *right coset* of M determined by x) are given by:

$$x_{\underset{M}{\leftarrow}} = \begin{cases} M, & if\ x \in M \\ x/M, & if\ x \notin M \end{cases} \text{ and } x_{\underset{M}{\rightarrow}} = \begin{cases} M, & if\ x \in M \\ M\backslash x, & if\ x \notin M \end{cases}$$

Since, per Corollary 1, the equality $(B\backslash A)/C = B\backslash(A/C)$ is valid in any hypergroup, the double coset of M determined by x can be defined by:

$$x_M = \begin{cases} M, & if\ x \in M \\ M\backslash(x/M) = (M\backslash x)/M, & if\ x \notin M \end{cases}$$

Per Theorem 34, the distinct left cosets and right cosets, as well as the double cosets, are disjoint. Thus, each one of the families:

$$P \swarrow M = \left\{ x_{\underset{M}{\leftarrow}} \mid x \in P \right\},\ P \nearrow M = \left\{ x_{\underset{M}{\rightarrow}} \mid x \in P \right\} \text{ and } P \swarrow\!\!\!\!\nearrow M = \{ x_M \mid x \in P \}$$

forms a partition of P. If M is normal, then it follows that $x_{\underset{M}{\leftarrow}} = x_{\underset{M}{\rightarrow}} = x_M$. Therefore:

$$P \swarrow M = P \nearrow M = P \swarrow\!\!\!\!\nearrow M$$

Proposition 38. *Let M be a symmetric subhypergroup of P. Then:*

i. $x \in x_{\underset{M}{\leftarrow}},\ x \in x_{\underset{M}{\rightarrow}}$ *and* $x \in x_M,$

ii. $x_{\underset{M}{\leftarrow}} \subseteq x_M$ *and* $x_{\underset{M}{\rightarrow}} \subseteq x_M,$

iii. $x_M = \left(x_{\underset{M}{\leftarrow}}\right)_{\underset{M}{\rightarrow}} = \left(x_{\underset{M}{\rightarrow}}\right)_{\underset{M}{\leftarrow}}$

Proposition 39. *Let M be a symmetric subhypergroup of P. Then:*

$$xM = x/M \cup M \text{ and } Mx = M\backslash x \cup M.$$

Proof. The two equalities are dual. Per Proposition 36, the equality $xS(M) = x/M \cup S(M)$ is valid. But M is symmetric, so $S(M) = M$. Therefore, $xM = x/M \cup M$. □

Proposition 40. *Let M be a symmetric subhypergroup of P. Then:*

i. $x_{\underset{M}{\leftarrow}} M = xM = x_{\underset{M}{\leftarrow}} \cup M,$

ii. $Mx_{\underset{M}{\rightarrow}} = Mx = x_{\underset{M}{\rightarrow}} \cup M.$

Proof. (i) If $x \in M$, the equality is true, because each one of its parts equals to M. If $x \notin M$, then the sequential application of Propositions 37 (iv) and 39 gives:

$$x_{\underset{M}{\leftarrow}} M = (x/M)M = xM = x/M \cup M = x_{\underset{M}{\leftarrow}} \cup M.$$

Duality gives (ii). □

Corollary 14. *Let Q be a nonvoid subset of P and M a symmetric subhypergroup of P. Then:*

$$Q_{\underset{M}{\leftarrow}} M = QM = Q_{\underset{M}{\leftarrow}} M \cup M \text{ and } MQ_{\underset{M}{\rightarrow}} = MQ = Q_{\underset{M}{\rightarrow}} \cup M$$

Proposition 41. *Let M be a symmetric subhypergroup of P. Then:*

$$Mx_M = Mx_{\underset{M}{\leftarrow}} = x_M \cup M = MxM = x_{\underset{M}{\rightarrow}} M = x_M M$$

Proof. Per Proposition 40 (i): $MxM = M\left(x_{\overleftarrow{M}} \cup M\right) = Mx_{\overleftarrow{M}} \cup MM = Mx_{\overleftarrow{M}} \cup M$, and since the hypercomposition is closed, $Mx_{\overleftarrow{M}} \cup M = Mx_{\overleftarrow{M}}$. Per duality: $MxM = x_{\overrightarrow{M}}M$. Next, per Propositions 38 (iii) and 40, we have:

$$Mx_M = M\left(x_{\overleftarrow{M}}\right)_{\overrightarrow{M}} = Mx_{\overleftarrow{M}} = \left(x_{\overleftarrow{M}}\right)_{\overrightarrow{M}} \cup M = x_M \cup M$$

Duality yields the rest. □

Proposition 42. *Let M be a symmetric subhypergroup of P. Then:*

$$(xy)_{\overleftarrow{M}}M \subseteq x_{\overleftarrow{M}}y_{\overleftarrow{M}} \cup M \;\; \textit{and} \;\; (xy)_{\overrightarrow{M}}M \subseteq x_{\overrightarrow{M}}y_{\overrightarrow{M}} \cup M$$

Proof. Per Corollary 14, $(xy)_{\overleftarrow{M}} \subseteq (xy)_{\overleftarrow{M}}M = (xy)M$. But $x \in x/M$, thus we have: $(xy)M \subseteq (x/M)yM = x_{\overleftarrow{M}}(yM)$. Now, per Proposition 40:

$$x_{\overleftarrow{M}}(yM) = x_{\overleftarrow{M}}\left(y_{\overrightarrow{M}} \cup M\right) = x_{\overleftarrow{M}}y_{\overrightarrow{M}} \cup x_{\overleftarrow{M}}M = x_{\overleftarrow{M}}y_{\overrightarrow{M}} \cup \left(x_{\overleftarrow{M}} \cup M\right) = \left(x_{\overleftarrow{M}}y_{\overrightarrow{M}} \cup x_{\overleftarrow{M}}\right) \cup M$$

Since the hypercomposition is closed, $x_{\overleftarrow{M}}y_{\overrightarrow{M}} \cup x_{\overleftarrow{M}} = x_{\overleftarrow{M}}y_{\overrightarrow{M}}$ is valid, and therefore:

$$\left(x_{\overleftarrow{M}}y_{\overrightarrow{M}} \cup x_{\overleftarrow{M}}\right) \cup M = x_{\overleftarrow{M}}y_{\overrightarrow{M}} \cup M.$$

Duality gives the second inclusion. □

Corollary 15. *If X, Y are nonvoid subsets of P and M is a symmetric subhypergroup of P, then:*

$$(XY)_{\overleftarrow{M}}M \subseteq X_{\overleftarrow{M}}Y_{\overleftarrow{M}} \cup M \;\; \textit{and} \;\; (XY)_{\overrightarrow{M}}M \subseteq X_{\overrightarrow{M}}Y_{\overrightarrow{M}} \cup M$$

Proposition 43. *Let M be a symmetric subhypergroup of P. Then:*

$$(xy)_M = x_M y_M \cup M$$

Proof. Per Proposition 38 (iii) and Corollary 15:

$$\begin{aligned}(xy)_M = \left((xy)_{\overleftarrow{M}}\right)_{\overrightarrow{M}} &\subseteq \left(x_{\overleftarrow{M}}y_{\overleftarrow{M}} \cup M\right)_{\overrightarrow{M}} = \left(x_{\overleftarrow{M}}y_{\overleftarrow{M}}\right)_{\overrightarrow{M}} \cup M_{\overrightarrow{M}} \subseteq \\ &\subseteq \left(x_{\overleftarrow{M}}\right)_{\overrightarrow{M}}\left(x_{\overleftarrow{M}}\right)_{\overrightarrow{M}} \cup M = x_M y_M \cup M \quad \square\end{aligned}$$

Corollary 16. *If X, Y are nonvoid subsets of P and M is a symmetric subhypergroup of P, then:*

$$(XY)_M = X_M Y_M \cup M$$

Corollary 17. *If X, Y are nonvoid subsets of P and M is a symmetric subhypergroup of P, then:*

i. $X_M Y_M \cap M \neq \varnothing$ *implies* $(X_M Y_M)_M = X_M Y_M \cup M$,
ii. $X_M Y_M \cap M = \varnothing$ *implies* $(X_M Y_M)_M = X_M Y_M$.

A hypercomposition that derives from P's hypercomposition can be defined in each one of the families $P \swarrow M$, $P \nearrow M$ and $P \swarrow\!\!\!\!\!\nearrow M$. Thus in $P \swarrow\!\!\!\!\!\nearrow M$, this hypercomposition is:

$$x_M \cdot y_M = \{z_M \mid z \in x_M y_M\}.$$

If the induced hypercompositions of $\cdot$ are denoted by $\nearrow$ and $\searrow$, then:

$$x_M \nearrow y_M = \{z_M \mid x_M \in z_M \cdot y_M\} = \{z_M \mid x \in z_M y_M\}$$

and
$$y_M \searrow x_M = \{z_M \mid x_M \in y_M \cdot z_M\} = \{z_M \mid x \in y_M z_M\}.$$

It is obvious that $x_M \nearrow y_M \neq \varnothing$ and $y_M \searrow x_M \neq \varnothing$. Therefore, according to Theorem 14, the following proposition is valid:

Proposition 44. *The reproductive axiom is valid in* $\left(P \swarrow\!\!\!\nearrow M, \cdot\right)$.

In the families $P \swarrow M$ and $P \nearrow M$, the associativity may fail. However, it is valid in $P \swarrow\!\!\!\nearrow M$, as per the next Proposition:

Proposition 45. *The associative axiom is valid in* $\left(P \swarrow\!\!\!\nearrow M, \cdot\right)$.

Proof. It must be proved that

$$(x_M \cdot y_M) \cdot z_M = x_M \cdot (y_M \cdot z_M).$$

This is true if and only if $((x_M y_M)_M z_M)_M = (x_M (y_M z_M)_M)_M$. So, if $x_M y_M \cap M = \varnothing$, then Corollary 17 (ii) yields $(x_M y_M)_M = x_M y_M$ and the above equality is obvious. If $x_M y_M \cap M \neq \varnothing$, then Corollary 17 (i) yields $(x_M y_M)_M = x_M y_M \cup M$. Hence:

$$\begin{aligned}(x_M y_M) z_M &= (x_M y_M \cup M) z_M = x_M y_M z_M \cup M z_M \\ &= x_M y_M z_M \cup z_M \cup M = x_M y_M z_M \cup M\end{aligned}$$

Since $x_M y_M \cap M \neq \varnothing$ and $x_M y_M \subseteq x_M y_M z_M$, it follows that $M \subseteq (x_M y_M z_M)_M$ is valid. Therefore:

$$((x_M y_M)_M z_M)_M = (x_M y_M z_M \cup M)_M = (x_M y_M z_M)_M \cup M = (x_M y_M z_M)_M.$$

Duality yields $(x_M y_M z_M)_M = (x_M (y_M z_M)_M)_M$, and so the associativity is valid. □

Proposition 46. *The transposition axiom is valid in* $\left(P \swarrow\!\!\!\nearrow M, \cdot\right)$.

Proof. Suppose that $y_M \searrow x_M \cap z_M \nearrow w_M \neq \varnothing$. Then:

$$\begin{aligned}&\{p_M \mid x_M \in y_M \cdot p_M\} \cap \{q_M \mid z_M \in q_M \cdot w_M\} \neq \varnothing \Leftrightarrow \\ &\Leftrightarrow \{p_M \mid x \in y_M p_M\} \cap \{q_M \mid z \in q_M w_M\} \neq \varnothing.\end{aligned}$$

Therefore $y' \backslash x \cap z / w' \neq \varnothing$ for some $y' \in y_M$ and $w' \in w_M$. Thus, $x w' \cap y' z \neq \varnothing$, and so

$$x_M \cdot w_M \cap z_M \cdot y_M \neq \varnothing. \square$$

Propositions 44, 45, and 46 give the theorem:

Theorem 35. *If M is a symmetric subhypergroup of P, then* $\left(P \swarrow\!\!\!\nearrow M, \cdot\right)$ *is a transposition hypergroup.*

A consequence of Proposition 41 is that $M \cdot x_M = x_M \cdot M = \{x_M, M\}$ for every x_M in $P \swarrow\!\!\!\nearrow M$. Hence:

Proposition 47. *M is a strong identity in the hypergroup* $P \swarrow\!\!\!\nearrow M$.

Proposition 48. *$P \swarrow\!\!\nearrow M$ consists only of attractive elements.*

Since $\{x_M, y_M\} \subseteq x_M \cdot y_M$ for all $x_M, y_M \in P \swarrow\!\!\nearrow M$, *the following is true:*

Proposition 49. *The hypercomposition in $\left(P \swarrow\!\!\nearrow M, \ \cdot\right)$ is closed.*

Proposition 50. *If $y \in S(x)$, then $M \in x_M \cdot y_M$, for all $x_M \in P \swarrow\!\!\nearrow M$.*

Propositions 47 and 50 give as a consequence the following theorem:

Theorem 36. *If M is a symmetric subhypergroup of P, then $\left(P \swarrow\!\!\nearrow M, \ \cdot\right)$ is a transposition polysymmetrical hypergroup.*

Corollary 18. *If M is a symmetric subhypergroup of a fortified transposition hypergroup of attractive elements T, then $\left(T \swarrow\!\!\nearrow M, \ \cdot\right)$ is a fortified transposition hypergroup, M is its strong identity, and each one of its elements is attractive.*

7.5. The Cosets in Quasicanonical Hypergroups

It is apparent that the subgroups of a group are symmetric, closed, and invertible. The same is true in the case of the quasicanonical hypergroups. Indeed, per Proposition 16, $x/y = xy^{-1}$ and $y \backslash x = y^{-1}x$, therefore a subhypergroup U of a quasicanonical hypergroup Q is symmetric if and only if it is closed. Moreover, since the reversibility is valid in the quasicanonical hypergroups, if U is a symmetric subhypergroup of a quasicanonical hypergroup Q, then the implications $a \in rb \Rightarrow b \in r^{-1}a$ and $a \in br \Rightarrow b \in ar^{-1}$ hold for every $a, b \in Q$, $r \in U$. Therefore, because of Proposition 32, U is invertible. Hence the left and right cosets of a subhypergroup U of a quasicanonical hypergroup Q are of the form aU and Ua, $a \in Q$, respectively, and create partitions in Q. The quotient hypergroup $Q \swarrow U$ defined by the left cosets and the quotient hypergroup $Q \nearrow U$ defined by the right cosets are transposition hypergroups, but not necessarily quasicanonical ones. For example, although U is a right scalar identity for the left cosets, since $aU \cdot U = aU$, it need not be a left scalar identity as well, since $aU \approx U \cdot aU$.

Double cosets have the form UaU and also create a partition in Q. It is easy to observe that $(UaU)^{-1} = Ua^{-1}U$ and that U is a bilateral scalar identity. Hence:

Theorem 37. *If U is a symmetric subhypergroup of a quasicanonical hypergroup Q, then the quotient hypergroup of the double cosets $Q \swarrow\!\!\nearrow U$ is a quasicanonical hypergroup with scalar identity U.*

Corollary 19. *If U is a normal symmetric subhypergroup of a quasicanonical hypergroup Q, then the left, right, and double cosets coincide and the quotient hypergroup $Q \swarrow\!\!\nearrow U$ is a quasicanonical hypergroup with U as its scalar identity.*

8. Conclusions and Open Problems

This paper presents the structural relation between the hypergroup and the group. As it has been proved, these two algebraic structures satisfy the exact same axioms and their only difference appears in the law of synthesis of their elements. The law of synthesis in the hypergroups is so general that it allowed this structure's enrichment with further axioms which created a significant number of special hypergroups. Many of these special hypergroups are presented in this paper, along with examples, their fundamental properties, and the different types of their subhypergroups and applications. Among them, the transposition hypergroup has important applications in geometry and computer science [67]. This hypergroup satisfies the following axiom:

$$b\backslash a \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing, \text{ for all } a, b, c, d \in H,$$

which was named *transposition axiom* [24]. If this axiom's implication is reversed, the following new axiom is created:

$$ad \cap bc \neq \varnothing \text{ implies } b\backslash a \cap c/d \neq \varnothing, \text{ for all } a, b, c, d \in H.$$

We will call this axiom *rev-transposition axiom* (i.e., *reverse transposition axiom*) and the relevant hypergroup *rev-transposition hypergroup* (i.e., *reverse transposition hypergroup*) or *rev-join hypergroup* (i.e., *reverse join hypergroup*) in the commutative case. The relation between these two hypergroups is very interesting. For example, the following proposition is valid in the rev-transposition hypergroups:

Proposition 51. *The following are true in any rev-transposition hypergroup:*

i. $ab/c \subseteq a(b/c)$ *and* $c\backslash ba \subseteq (c\backslash b)a$,
ii. $ab/c \subseteq a/(c/b)$ *and* $c\backslash ba \subseteq (b\backslash c)\backslash a$,
iii. $(b\backslash ac)/d = b\backslash(ac/d) \subseteq (b\backslash a)(c/d)$,
iv. $(b\backslash ad)/c = b\backslash(ad/c) \subseteq (b\backslash a)/(c/d)$,
v. $(a\backslash bc)/d = a\backslash(bc/d) \subseteq (b\backslash a)\backslash(c/d)$.

The comparison of Proposition 51 with Proposition 11 reveals that the reversal of the implication in the transposition axiom reverses the inclusion relations in properties (i)–(v). The study of this hypergroup and its potential applications present an interesting open problem of the theory of the hypercompositional algebra.

However, the following property also applies:

$$b\backslash a \cap c/d \neq \varnothing \quad \Leftrightarrow \quad ad \cap bc \neq \varnothing, \text{ for all } a, b, c, d \in H$$

We will name this axiom *bilateral transposition axiom*. Under this axiom, the inclusion relations of Propositions 11 and 51 become equalities. Examples of hypergroups verifying the bilateral transposition axiom are the quasicanonical hypergroups, the canonical hypergroups, and of course, the groups and the abelian groups. So, the following question arises: *Do there exist other hypergroups satisfying the bilateral transposition axiom apart from the quasicanonical and the canonical ones?*

In addition, as it is shown in Section 7, many questions remain open in the decomposition of hypergroups. Addressing these quesions necessitates the introduction of new tools and techniques, as hypercompositional algebra is an off-the-map region of abstract algebra, which requires great caution in the application of processes for achieving correct results. It is very easy to be led to wrong conclusions when relying on methods and results of classical algebra. Such a case is indicated in remark 2 in [67].

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