

On the Algebraic Structure of Transposition Hypergroups with Idempotent Identity

Christos G. Massouros and Gerasimos G. Massouros
Technological Institute of Chalkis, GR34400, Evia, Greece

E-mail: Ch.Massouros@gmail.com

E-mail: masouros@gmail.com

ABSTRACT. This paper studies the algebraic structure of transposition hypergroups with idempotent identity. Their subhypergroups and their properties are examined. Right, left and double cosets are defined through symmetric subhypergroups and their properties are studied. Furthermore, this paper examines the homomorphisms, the behaviour of attractive and non-attractive elements through them, as well as the relation of their kernels and images to symmetric subhypergroups.

Keywords: hypergroups, transposition hypergroups, subhypergroups, symmetric subhypergroups, attractive elements.

2000 Mathematics subject classification: 20N20.

1. INTRODUCTION

An *operation* or *composition* in a non void set H is a function from $H \times H$ to H , while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H . An algebraic structure that satisfies the axioms

- i. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associative axiom) and
- ii. $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductive axiom).

is called *group* if “ \cdot ” is a composition (see remark in p. 72 of [25]) and *hypergroup* if “ \cdot ” is a hypercomposition [9]. When there is no likelihood of confusion “ \cdot ” can be omitted. If A and B are subsets of H , then AB signifies the union

$\bigcup_{(a,b) \in A \times B} ab$. In particular if either $A = \emptyset$ or $B = \emptyset$, then $AB = \emptyset$ and vice versa. Ab and aB have the same meaning as $A\{b\}$ and $\{a\}B$. In general, the singleton $\{a\}$ is identified with its member a . In [9] F. Marty also defined the two *induced hypercompositions* (right and left division) that follow from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{\mid b} = \{x \in H \mid a \in xb\} \text{ and } \frac{a}{\mid b} = \{x \in H \mid a \in bx\}.$$

It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, a/b or $a : b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b \backslash a$ or $a..b$ is used to denote the left division [6, 14, 17].

Since the hypergroup, is a very general structure it was progressively enriched with additional axioms, either more or less powerful, thus creating a significant number of specific hypergroups. e.g. [6, 7, 13, 18, 19, 23, 30, 37, 38, 39, 40]. Moreover some of these hypergroups constituted a constructive origin for the development of other new hypercompositional structures (e.g. see [1, 8, 10, 11, 21, 34, 43]). Thus, W. Prenowitz enriched hypergroups with an axiom, in order to use them in the study of geometry [5, 16, 17, 41, 42]. More precisely, he introduced into the commutative hypergroup, the *transposition axiom*

$$a/b \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \text{ in } H$$

and named this new hypergroup *join space* [41, 42]. It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata [20, 28, 31, 36]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

$$b \backslash a \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \text{ in } H.$$

He named this particular hypergroup *transposition hypergroup* [6]. Subsequently, this axiom was also introduced into H_V -groups [27] and in other hypercompositional structures [24]. So, the *transposition H_V -group*, the *transposition hypergroupoid*, the *transposition quasi-hypergroup*, and the *transposition semi-hypergroup* were defined. Clearly, if A, B, C and D are subsets of H , then $B \backslash A \cap C/D \neq \emptyset$ implies that $AD \cap BC \neq \emptyset$. In what follows, the relational notation $A \approx B$ (read A meets B) is used to assert that sets A and B have a non-void intersection.

The study of transposition hypergroups is not as extensive as that of join hypergroups (e.g. see [3, 4, 6, 7, 19, 24, 37]). In [26] the transposition hypergroups with idempotent identity were introduced and their fundamental properties were presented. It was proved that the elements of these hypergroups are separated into two classes: the set $A = \{x \in H \mid e \in ex = xe\}$, including e , of attractive elements and the set of non-attractive elements. A study of these elements is also conducted in [26]. This paper contributes in the direction of further and deeper study of transposition hypergroups with idempotent identity, by analyzing their algebraic structure.

2. PRELIMINARIES

Consequences of the hypergroup's definition axioms are [24, 25]:

- i. $ab \neq \emptyset$, for all a, b in H ,
- ii. $a/b \neq \emptyset$ and $a \setminus b \neq \emptyset$, for all a, b in H ,
- iii. $H = H/a = a/H$ and $H = a \setminus H = H \setminus a$, for all a in H ,
- iv. the non-empty result of the induced hypercompositions is equivalent to the reproductive axiom.

It has been proven in [6, 14] that in any hypergroup the following properties are valid:

Proposition 2.1. *In any hypergroup*

- i. $(a/b)/c = a/(cb)$ and $c \setminus (b \setminus a) = (bc) \setminus a$ (mixed associativity),
- ii. $(b \setminus a)/c = b \setminus (a/c)$,
- iii. $b \in (a/b) \setminus a$ and $b \in a/(b \setminus a)$.

Corollary 2.2. *In any hypergroup H , if A, B, C are non-empty subsets of H , then:*

- i. $(A/B)/C = A/(CB)$ and $C \setminus (B \setminus A) = (BC) \setminus A$,
- ii. $(B \setminus A)/C = B \setminus (A/C)$,
- iii. $B \subseteq (A/B) \setminus A$ and $B \subseteq A/(B \setminus A)$.

Proposition 2.3. [6, 14, 18] *The following are true in any transposition hypergroup:*

- i. $a(b/c) \subseteq ab/c$ and $(c \setminus b)a \subseteq c \setminus ba$,
- ii. $a/(c/b) \subseteq ab/c$ and $(b \setminus c) \setminus a \subseteq c \setminus ba$.
- iii. $(b \setminus a)(c/d) \subseteq (b \setminus ac)/d = b \setminus (ac/d)$,
- iv. $(b \setminus a)/(c/d) \subseteq (b \setminus ad)/c = b \setminus (ad/c)$,
- v. $(b \setminus a) \setminus (c/d) \subseteq (a \setminus bc)/d = a \setminus (bc/d)$.

Corollary 2.4. *The following is true in any transposition hypergroup*

$$(b \setminus a)(c/d) \cup (b \setminus a)/(d/c) \cup (a \setminus b) \setminus (c/d) \subseteq (b \setminus ac)/d = b \setminus (ac/d).$$

Proposition 2.5. [12, 18] *The following are true in any join hypergroup*

- i. $a(b/c) \cup b(a/c) \cup a/(c/b) \cup b/(c/a) \subseteq ab/c$,
- ii. $(a/b)(c/d) \cup (a/d)(c/b) \cup (a/b)/(d/c) \cup (a/d)/(b/c) \cup (c/d)/(b/a) \cup (c/b)/(d/a) \subseteq ac/bd$.

Corollary 2.6. *The relations of Propositions 2.3, 2.5 and of Corollary 2.4 are also valid if the elements a, b, c, d are replaced by non-empty subsets A, B, C, D of the transposition hypergroup.*

In [6] and then in [7] a principle of duality is established in the theory of hypergroups and in the theory of transposition hypergroups as follows:

Given a theorem, the dual statement which results from the interchanging of the order of the hypercomposition “.” (and necessarily interchanging of the left and the right division), is also a theorem.

Since we are working in transposition hypergroups, this principle is used throughout this paper.

An element e is called *right identity*, if $x \in xe$ for all x in H . If $x \in ex$ for all x in H , then x is called *left identity*, while x is called *identity* if it is both right and left identity. If equality $e = ee$ is valid for an identity e , then e is called *idempotent identity*. If $x = xe = ex$ for all x in H , then e is a *scalar identity*. When a scalar identity exists in H , then it is unique. An identity e is a *strong identity*, if $x \in xe = ex \subseteq \{e, x\}$ for all x in H . The strong identity need not be unique [7]. Both scalar and strong identities are idempotent identities.

Proposition 2.7. *If e is a strong identity in H and $x \neq e$, then $x/e = e \setminus x = x$.*

Proposition 2.8. *If e is a scalar identity in H , then $x/e = e \setminus x = x$.*

A hypergroup H is called *semi-regular*, if every $x \in H$ has at least one right and one left identity. An element x' is called *right e -inverse* or *right e -symmetric* of x , if a right identity $e \neq x'$ exists such that $e \in x \cdot x'$. The definition of the *left e -inverse* or *left e -symmetric* is analogous to the above, while x' is called *e -inverse* or *e -symmetric* of x , if it is both right and left inverse with regard to the same identity e . If e is an identity in a hypergroup H , then the set of left inverses of $x \in H$, with regard to e , will be denoted by $S_{el}(x)$, while $S_{er}(x)$ will denote the set of right inverses of $x \in H$ with regard to e . The intersection $S_{el}(x) \cap S_{er}(x)$ will be denoted by $S_e(x)$. A semi-regular hypergroup H is called *regular*, if it has at least one identity e and if each element has at least one right and one left e -inverse. H is called *strictly e -regular*, if for the identity e the equality $S_{el}(x) = S_{er}(x)$ is valid for all $x \in H$. In a strictly e -regular hypergroup, the inverses of x are denoted by $S_e(x)$ and, when there is no likelihood of confusion, e can be omitted. H has *semistrict e -regular structure*, if $S_{el}(x) \cap S_{er}(x) \neq \emptyset$ for any $x \in H$ is true for the identity e . Obviously, in commutative hypergroups only strict e -regular structures exist.

A subset h of H is called a *subhypergroup* of H , if $xh = hx = h$ for all $x \in h$. A subhypergroup h of H is *central* if $xy = yx$ for all $x \in h$ and $y \in H$.

Proposition 2.9. *If H is a hypergroup with strong identities, then the set E of these identities is a central subhypergroup of H .*

Let e be an identity element in a hypergroup H and x an element in H . Then, x will be called *right e -attractive*, if $e \in ex$, while it will be called *left e -attractive* if $e \in xe$. If x is both left and right e -attractive, then it will be called *e -attractive*. When there is no likelihood of confusion, then e can be omitted. When the identity is strong, then $ex = xe = \{e, x\}$ is valid, if x is

attractive; if x is non-attractive, then $ex = xe = x$ is valid. In the case of strong identity, non-attractive elements are called canonical. See [33] for the origin of the terminology.

Proposition 2.10. *In a hypergroup H , $e \setminus e$ is the set of right e -attractive elements of H and e/e is the set of left e -attractive elements of H .*

Proof. Suppose that x is a right attractive element in H . Then $e \in ex$. Thus $x \in e \setminus e$. Also, if $x \in e \setminus e$, then $e \in ex$. Hence $e \setminus e$ consists of the right attractive elements of H . The rest follows per duality. \square

In the following some properties of attractive elements, essential for the next paragraphs, which are proven in [26] are provided.

Proposition 2.11. *i. If x is not a right (resp. left) e -attractive element in a hypergroup with idempotent identity e , then ex consists of non-right (resp. left) e -attractive elements.*

ii. If x is a right (resp. left) e -attractive element in a transposition hypergroup with idempotent identity e , then all the elements of $x e$ are right (resp. left) e -attractive.

Proposition 2.12. *i. If x is a right (resp. left) attractive element in a transposition hypergroup with idempotent identity e , then its right (resp. left) inverses are also right (resp. left) attractive elements.*

ii. If x is not a right (resp. left) attractive element in a transposition hypergroup with idempotent identity e , then its right (resp. left) inverses are not right (resp. left) attractive elements as well.

Proposition 2.13. *Let H be a strictly e -regular hypergroup, where e is a strong identity. Then:*

- i. $x \setminus e = eS(x) = \{e\} \cup S(x) = S(x)e = e/x$ for any attractive element $x \neq e$,
- ii. $x \setminus e = e/x = S(x)$ for any non attractive element x .

Proposition 2.14. *If x is not a right (resp. left) e -attractive element in a hypergroup H with strong identity e , then $xS_{er}(x)$ (resp. $S_{el}(x)x$) contains all the right (resp. left) attractive elements.*

In what follows, it is assumed that the identities are bilateral and idempotent. Examples of such transposition hypergroups, some of which are connected to the theory of languages and automata, can be found in [7, 19, 22, 28, 33, 36, 37]. Also, T will denote a strictly e -regular transposition hypergroup, where e is an idempotent identity. In T let A denote the set of attractive elements and C the set of non-attractive ones. Then $T = A \cup C$ and $A \cap C = \emptyset$.

Proposition 2.15. [26] *In a strictly e -regular transposition hypergroup:*

- i. *The result of the hypercomposition of two attractive elements contains only attractive elements.*
- ii. *The result of the hypercomposition of an attractive element with a and non-attractive element consist of non-attractive elements.*
- iii. *If x, y are attractive elements in T , then $x/y \subseteq A$ and $y \setminus x \subseteq A$.*
- iv. *If x is a non-attractive element in T , then $A \subseteq xC \cap Cx$.*
- v. *The set C of non-attractive elements of T is not stable under the hypercomposition.*
- vi. *If either x or y are non-attractive elements, then $x/y \subseteq C$ and $y \setminus x \subseteq C$.*

Proposition 2.16. [26] *If the identity of T is strong, then:*

- i. *the result of the hypercomposition of two attractive elements contains these two elements (see also [10, 19, 31, 33, 35]),*
- ii. *the result of the hypercomposition of an attractive element with a canonical element is the canonical element (see also [10, 19, 31, 33, 35]).*

Corollary 2.17. *If the identity of T is strong, then:*

- i. $x \in x/y$ and $x \in y \setminus x$, for all $x, y \in A$,
- ii. $A = x/x = x \setminus x$, for all $x \in A$.

Proposition 2.18 (26). *If the identity of T is strong and*

- i. x, y are two attractive elements in T , such that $e \notin xS(y)$, then $xS(y) = x/y \cup S(y)$ and $S(y)x = y \setminus x \cup S(y)$,
- ii. x, y are two elements in T and any of these is non-attractive, then $xS(y) = x/y$ and $S(y)x = y \setminus x$.

Corollary 2.19. *If the identity of T is strong and:*

- i. X, Y are non-empty subsets of $A \subseteq T$ and $e \notin XS(Y)$, then $XS(Y) = X/Y \cup S(Y)$ and $S(Y)X = Y \setminus X \cup S(Y)$,
- ii. if X or Y are non-empty subsets of $C \subseteq T$, then $XS(Y) = X/Y$ and $S(Y)X = Y \setminus X$.

When identity is strong and $S(x)$ is singleton for all $x \in T$, properties of attractive elements are developed in [7, 30, 32, 33].

3. SUBHYPERGROUPS

A subhypergroup K of a hypergroup H is called *closed from the right* (resp. *from the left*) if $(Ka) \cap K = \emptyset$ (resp. $(aK) \cap K = \emptyset$) for every $a \in H - K$. K is called *closed* if it is both right and left closed (for more details see [29]). In [12, 14] it is proven that h is right closed (resp. left closed) if and only if $b \setminus a \subseteq K$ (resp. $a/b \subseteq K$) for all $a, b \in K$.

Proposition 3.1. *The set A of the attractive elements of T is a closed subhypergroup of T .*

Proof. According to Proposition 2.15.i, $xA \subseteq A$, if $x \in A$. Next, let y be an arbitrary element of A . We shall prove that $y \in xA$. Indeed, if x is an element of A , then its inverses are also in A (Prop. 2.12). Therefore, $x'y \subseteq A$, if $x' \in S(x)$ and $y \in ey \subseteq (xx')y = x(x'y)$. Thus, there exists $z \in x'y$, such that $y \in xz \subseteq xA$. Hence, $xA = A$. Dually, $Ax = A$ and, therefore, A is a subhypergroup of T . Now, if w belongs to $T - A$, i.e. if w is a non-attractive element, then, because of Proposition 2.15.ii, $wA \subseteq T - A$ is valid. Therefore, $(wA) \cap A = \emptyset$ and so A is closed from the right. Because of duality A is closed from the left and thus A is a closed subhypergroup of T . \square

From Propositions 3.1, 2.15.ii and v, it follows that:

Proposition 3.2. *The set of attractive elements is the minimum closed subhypergroup of T (in the sense of inclusion).*

A subhypergroup K of a transposition hypergroup with an identity e is called *symmetric* with respect to e , if for all $x \in K$ the right and the left inverses of x , with respect to e , are subsets of K (see also [7, 33]). From Proposition 2.12 it follows that:

Proposition 3.3. *The set of attractive elements is a symmetric subhypergroup of T .*

Proposition 3.4. *A non-empty subset K of T is a symmetric subhypergroup with respect to e of T , if and only if $xS_e(y) \subseteq K$ and $S_e(y)x \subseteq K$ for all $x, y \in K$.*

Proof. The above condition is obviously valid when K is a symmetric subhypergroup of T . Conversely now, suppose that x belongs to K . Then, $xS_e(x) \subseteq K$ and so $e \in K$, which implies $eS_e(x) \subseteq K$ and so $S_e(x) \subseteq K$. Next, for the proof of the reproductive axiom, suppose that y is an arbitrary element of K . Then, there exists $y \in S_e(y) \subseteq K$, such that $y \in S_e(y)$. Thus, $xy \subseteq xS_e(y) \subseteq K$. Therefore, $xK \subseteq K$. By duality, $Kx \subseteq K$. Also, $S_e(x)y \subseteq K \Rightarrow xS_e(x)y \subseteq xK \Rightarrow ey \subseteq xK \Rightarrow y \in xK$. Therefore, $K \subseteq xK$. Dually, $K \subseteq Kx$. Hence, $xK = Kx = K$ for all $x \in K$. \square

Corollary 3.5. *A non-empty subset K of T is a symmetric subhypergroup with respect to e of T , if and only if $KK = K$ and $S_e(K) = K$.*

Proposition 3.6. *Let e be an identity in T and let K_1, K_2 be any two symmetric subhypergroups of T with respect to e . Then, their intersection $K_1 \cap K_2$ is a symmetric subhypergroup of T .*

Proof. $e \in K_1 \cap K_2$ and $S_e(x) \subseteq K_1 \cap K_2$ for all $x \in K_1 \cap K_2$. Next, let x be an arbitrary element of $K_1 \cap K_2$. Then, $x(K_1 \cap K_2) \subseteq xK_1 = K_1$ and $x(K_1 \cap K_2) \subseteq xK_2 = K_2$. Hence, $x(K_1 \cap K_2) \subseteq K_1 \cap K_2$. Now, let y be an element in $K_1 \cap K_2$ and x' an inverse of x . Then, $y \in ey \subseteq (xx')y = x(x'y) \subseteq x(K_1 \cap K_2)$, thus $K_1 \cap K_2 \subseteq x(K_1 \cap K_2)$ and therefore, $K_1 \cap K_2 = x(K_1 \cap K_2)$. \square

From Proposition 3.6 above and from the fact that the intersection of two symmetric subhypergroups with respect to e is non-empty (as it always contains the identity e) it follows that:

Proposition 3.7. *In a strictly regular transposition hypergroup, the set of its symmetric subhypergroups with respect to e forms a complete lattice.*

Proposition 3.8. *Let K be a symmetric subhypergroup of T . If $x \notin K$, then $x/K \cap K = \emptyset$ and $K \setminus x \cap K = \emptyset$.*

Proof. Suppose that x does not belong in K and let y be an element in K , such that $x/y \cap K \neq \emptyset$. Then, $x \in Ky = K$, which contradicts the assumption above. Thus, $x/K \cap K = \emptyset$. \square

Proposition 3.9. *Suppose that T has a strong identity and that K is a symmetric subhypergroup of T . Then:*

- i. if $x \in A$, $K \subseteq A$ and $x \notin K$, then $xK = x/K \cup K$ and $Kx = K \setminus x \cup K$,*
- ii. if $x \in C$ or $K \subseteq C$, then $xK = x/K$ and $Kx = K \setminus x$.*

Proof. Since K is symmetric, $S(K) = K$. Thus: (i) $e \notin xK$, since $x \notin K$. So, according to Corollary 2.19.i, $xK = xS(K) = x/K \cup S(K) = x/K \cup K$. (ii) Using Corollary 2.19.ii, we get $xK = xS(K) = x/K$. The rest in (i), (ii) follows by duality. \square

Proposition 3.10. *Suppose that T has a strong identity, K is a symmetric subhypergroup of T and x is an element of T , but not an element K . Then $K/x = KS(x)$ and $x \setminus K = S(x)K$.*

Proof. According to Proposition 2.13, $S(x)$ is a subset of e/x . Moreover, e/x is a subset of K/x . Thus, $S(x) \subseteq K/x$. Since $x \notin K$, we have $e \notin KS(x)$. Thus, Corollary 2.19 implies either that $KS(x) = K/x \cup S(x) = K/x \cup e/x = K/x$, whenever $K \subseteq A$ and $x \in A$, or that $KS(x) = K/x$, whenever $K \subseteq C$ or $x \in C$. The rest follows by duality. \square

Proposition 3.11. *Suppose that T has a strong identity and K is a symmetric subhypergroup of T . If $x \notin K$, then $(x/K)K = xK$ and $K(K \setminus x) = Kx$.*

Proof. Since $x \in x/K$, it follows that $xK \subseteq (x/K)K$. Also, because of Proposition 3.9, $x/K \subseteq xK$ is valid. Thus, $xK \subseteq (x/K)K \subseteq (xK)K = xK$. Duality yields the rest. \square

Proposition 3.12. *Suppose that T has a strong identity and K is a symmetric subhypergroup of T . If $x, y \notin K$, then:*

- i. $x/K \approx y/K$ implies $x/K = y/K$,
- ii. $K \setminus x \approx K \setminus y$ implies $K \setminus x = K \setminus y$,
- iii. $K \setminus (x/K) \approx K \setminus (y/K)$ implies $K \setminus (x/K) = K \setminus (y/K)$.

Proof. (i) $x/K \cap y/K \neq \emptyset$ implies that $x \in (y/K)K$. Since $y \notin K$, from Propositions 3.11 and 3.9 follows that $(y/K)K = yK \subseteq y/K \cup K$. Thus, $x \in y/K \cup K$. Since $x \notin K$, it follows that $x \in y/K$. Thus, $x/K \subseteq (y/K)/K = y/(KK) = y/K$. By symmetry, $y/K \subseteq x/K$. Hence, $x/K = y/K$. Duality gives (ii).

(iii) Per Propositions 2.1, 2.10 and 3.11:

$$\begin{aligned} K \setminus (x/K) \approx K \setminus (y/K) &\Rightarrow (K \setminus x)/K \approx K \setminus (y/K) \Rightarrow K \setminus x \approx [K \setminus (y/K)]K \Rightarrow \\ &\Rightarrow K \setminus x \approx K \setminus [(y/K)K] \Rightarrow K \setminus x \approx K \setminus yK \Rightarrow x \in yK \Rightarrow y \in x/K \Rightarrow \\ &\Rightarrow y/K \subseteq (x/K)/K \Rightarrow y/K \subseteq x/(KK) \Rightarrow y/K \subseteq x/K \Rightarrow K \setminus (y/K) \subseteq K \setminus (x/K). \end{aligned}$$

By symmetry, $K \setminus (x/K) \subseteq K \setminus (y/K)$, thus equality is valid. \square

Proposition 3.13. *The symmetric subhypergroup K of a strictly regular transposition hypergroup T , generated by a subset X of T , is the union of all products $x_1 \dots x_n$ of any $n > 0$ elements, each of which is either an element of X or the inverse of an element of X .*

Proposition 3.14. *For any two symmetric subhypergroup K_1, K_2 of a strictly regular transposition hypergroup T , there exists a least symmetric subhypergroup, which contains both K_1 and K_2 ; i.e it is a symmetric subhypergroup K of T with $K_1 \subseteq K$, $K_2 \subseteq K$ and for which the inclusions $K_1 \subseteq N$, $K_2 \subseteq N$ imply $K \subseteq N$ for any symmetric subhypergroup N of T .*

Proof. Let U be the set of all symmetric subhypergroups of T which contain both K_1 and K_2 . Then, according to Proposition 3.6, the intersection of all symmetric subhypergroups in U is a symmetric subhypergroup with the desired property. \square

The symmetric subhypergroup of Proposition 3.14 is denoted by $K_1 \vee K_2$ and is usually larger than the union of the sets K_1 and K_2 , since $K_1 \vee K_2$ is the set of all those elements of T which belong for some j in a hyperproduct $a_1 b_1 \dots a_j b_j$, $a_i \in K_1$, $b_i \in K_2$. $K_1 \vee K_2$ is the lowest symmetric subhypergroup situated above both K_1 and K_2 in the lattice of symmetric subhypergroups.

4. COSETS

In [6] it is proven that, if K is a closed subhypergroup of a join hypergroup H , then the sets $\{x_K = xK \mid x \in H\}$ and $\{K/x \mid x \in H\}$ of the classes modulo K are equal. The set of these classes is denoted by $H : K$. The family of the

cosets $H : K$ becomes a canonical hypergroup [18, 38], if it is endowed with the hypercomposition $xK \cdot yK = \{zK \mid z \in xy\}$ [6]. K is the scalar identity in $(H : K, \cdot)$ and the inverse of x_K is K/x . In [6] it is proven that, if K is a closed subhypergroup of a transposition hypergroup H for which the equality $x \setminus K = K/x$ holds for all $x \in H$, then $H : K$ is quasicanonical hypergroup [13]. This paragraph studies the cosets which are defined from a nonempty symmetric subhypergroup in a strictly e-regular transposition hypergroup T , where e is a strong identity. If $x \in T$ and K is a nonempty symmetric subhypergroup of T , then $x_{\overleftarrow{K}}$ (i.e. the *left coset* of K determined by x) and dually, $x_{\overrightarrow{K}}$ (i.e. the *right coset* of K determined by x) are given by:

$$x_{\overleftarrow{K}} = \begin{cases} K & \text{if } x \in K \\ x/K & \text{if } x \notin K \end{cases} \quad \text{and} \quad x_{\overrightarrow{K}} = \begin{cases} K & \text{if } x \in K \\ K \setminus x & \text{if } x \notin K \end{cases}$$

For $Q \subseteq T$, $Q_{\overleftarrow{K}}$ and $Q_{\overrightarrow{K}}$ denote the unions $\cup \{x_{\overleftarrow{K}} \mid x \in Q\}$ and $\cup \{x_{\overrightarrow{K}} \mid x \in Q\}$ respectively. Propositions 3.8 and 3.12 assure that distinct left cosets and right cosets, are disjoint.

Remembering that, per Corollary 2.2, equality $(B \setminus A)/C = B \setminus (A/C)$ is valid in any hypergroup, the *double coset* of K determined by x can be defined by:

$$x_K = \begin{cases} K & \text{if } x \in K \\ K \setminus (x/K) = (K \setminus x)/K & \text{if } x \notin K \end{cases}$$

Following the above notation, if Q is a non-void subset of T , then Q_K denotes the union $\cup \{x_K \mid x \in Q\}$.

Proposition 4.1. *Let K be a symmetric subhypergroup of T . Then:*

- i. $x \in x_{\overleftarrow{K}}$, $x \in x_{\overrightarrow{K}}$ and $x \in x_K$,
- ii. $x_{\overleftarrow{K}} \subseteq x_K$ and $x_{\overrightarrow{K}} \subseteq x_K$,
- iii. $x_K = (x_{\overleftarrow{K}})_{\overrightarrow{K}} = (x_{\overrightarrow{K}})_{\overleftarrow{K}}$.

Proposition 3.12 assures that distinct left cosets and right cosets, as well as double cosets, are disjoint. Thus:

Proposition 4.2. *Each of the families $T : \overleftarrow{K} = \{x_{\overleftarrow{K}} \mid x \in T\}$, $T : \overrightarrow{K} = \{x_{\overrightarrow{K}} \mid x \in T\}$ and $T : K = \{x_K \mid x \in T\}$ of left, right and double cosets are partitions of T .*

Since the identity of T is strong, if K contains a non-attractive element, then, because of Proposition 2.14, K contains all the attractive elements. In this case, cosets are determined only by non-attractive elements. Herein, Proposition 2.18 implies that $xK = xS(K) = x/S(K) = x/K$ and that $Kx = S(K)x = S(K) \setminus x = K \setminus x$. Next if K consists of attractive elements and x is a non-attractive element, then Proposition 2.18 again gives $xK = x/K$ and $Kx =$

$K \setminus x$. On the other hand, if K consists of attractive elements and x is also an attractive element, not belonging in K , then Proposition 2.18.i implies that $xK = x/K \cup K$ and $Kx = K \setminus x \cup K$. The latter case, which is the most interesting, will be studied here. Hereunder, T_A will denote a strictly regular transposition hypergroup with strong identity, consisting only of attractive elements.

Proposition 4.3. *Let K be a symmetric subhypergroup of T_A . Then:*

- i. $x \xleftarrow{K} K = xK = x \xleftarrow{K} \cup K$,
- ii. $Kx \xrightarrow{K} = Kx = x \xrightarrow{K} \cup K$.

Proof. (i) If $x \in K$, then equalities (i) and (ii) above are valid, since every part of each equality equals K . If $x \notin K$, per Proposition 3.11, $x \xleftarrow{K} K = (x/K)K = xK$; per Proposition 2.18.i, $xK = x/K \cup K = x \xleftarrow{K} \cup K$. Duality gives (ii). \square

Corollary 4.4. *If Q is a non-empty subset of T_A and K is a symmetric subhypergroup of T_A , then:*

$$Q \xleftarrow{K} K = QK = Q \xleftarrow{K} \cup K \quad \text{and} \quad KQ \xrightarrow{K} = KQ = Q \xrightarrow{K} \cup K.$$

Proposition 4.5. *Let K be a symmetric subhypergroup of T_A . Then:*

$$Kx_K = Kx \xleftarrow{K} = x_K \cup K = KxK = x \xrightarrow{K} K = x_K K.$$

Proof. Per Proposition 4.3.i: $KxK = K(x \xleftarrow{K} \cup K) = Kx \xleftarrow{K} \cup K = Kx \xleftarrow{K}$ and per duality: $KxK = x \xrightarrow{K} K$. Next, per Proposition 4.1.iii and Corollary 4.4: $KxK = K(x \xleftarrow{K}) \xrightarrow{K} = Kx \xleftarrow{K} = (x \xleftarrow{K}) \xrightarrow{K} \cup K = x_K \cup K$. Duality gives the rest. \square

Corollary 4.6. *If Q is a nonempty subset of T_A and K is a symmetric subhypergroup of T_A , then:*

$$KQ_K = KQ \xleftarrow{K} = Q_K \cup K = KQK = Q \xrightarrow{K} K = Q_K K.$$

Proposition 4.7. *Let K be a symmetric subhypergroup of T_A . Then:*

- i. $(xy) \xleftarrow{K} \subseteq x \xleftarrow{K} y \xleftarrow{K} \cup K$,
- ii. $(xy) \xrightarrow{K} \subseteq x \xrightarrow{K} y \xrightarrow{K} \cup K$.

Proof. (i) Per Corollary 4.4. $(xy) \xleftarrow{K} \subseteq (xy) \xleftarrow{K} K = xyK$. Next, per Corollary 2.17: $xyK \subseteq (x/K)yK = x \xleftarrow{K} yK$. Now, per Proposition 4.2:

$$x \xleftarrow{K} yK = x \xleftarrow{K} (y \xrightarrow{K} \cup K) = x \xleftarrow{K} y \xrightarrow{K} \cup x \xleftarrow{K} K = x \xleftarrow{K} y \xrightarrow{K} \cup x \xleftarrow{K} \cup K.$$

Finally, per Proposition 2.16: $x \xleftarrow{K} y \xrightarrow{K} \cup x \xleftarrow{K} \cup K = x \xleftarrow{K} y \xrightarrow{K} \cup K$. Duality gives part (ii). \square

Corollary 4.8. *Let X, Y be non-empty subsets of T_A and K is a symmetric subhypergroup of T_A . Then:*

$$(XY)_{\overleftarrow{K}} \subseteq X_{\overleftarrow{K}} Y_{\overleftarrow{K}} \cup K \quad \text{and} \quad (XY)_{\overrightarrow{K}} \subseteq X_{\overrightarrow{K}} Y_{\overrightarrow{K}} \cup K.$$

Proposition 4.9. *Let K be a symmetric subhypergroup of T_A . Then $(xy)_K \subseteq x_K y_K \cup K$.*

Proof. Per Proposition 4.1.iii and Corollary 4.8:

$$\begin{aligned} (xy)_K &= \left((xy)_{\overleftarrow{K}} \right)_{\overrightarrow{K}} \subseteq \left[x_{\overleftarrow{K}} y_{\overleftarrow{K}} \cup K \right]_{\overrightarrow{K}} = \left(x_{\overleftarrow{K}} y_{\overleftarrow{K}} \right)_{\overrightarrow{K}} \cup K_{\overrightarrow{K}} \subseteq \left(x_{\overleftarrow{K}} \right)_{\overrightarrow{K}} \left(y_{\overleftarrow{K}} \right)_{\overrightarrow{K}} \cup K = \\ &= x_K y_K \cup K \end{aligned}$$

□

Corollary 4.10. *Let X, Y be non-empty subsets of T_A and K a symmetric subhypergroup of T_A . Then:*

$$(XY)_K \subseteq X_K Y_K \cup K.$$

Corollary 4.11. *Let X, Y be non empty subsets of T_A and K a symmetric subhypergroup of T_A . Then:*

- i. $K \cap X_K Y_K \neq \emptyset$ implies $(X_K Y_K)_K \subseteq X_K Y_K \cup K$,
- ii. $K \cap X_K Y_K = \emptyset$ implies $(X_K Y_K)_K = X_K Y_K$.

In each of the families $T_A : \overleftarrow{K}$, $T_A : \overrightarrow{K}$ and $T_A : K$ of cosets, a hypercomposition induced by the hypercomposition in T_A , can be defined. Thus in $T_A : K$ we have $x_K \cdot y_K = \{z_K \mid z \in x_K y_K\}$. As mentioned in [7], families $T : \overleftarrow{K}$ and $T : \overrightarrow{K}$ do not necessarily form a hypergroup, as associativity may fail. However, it was also proven in [7] that, when T is a fortified transposition hypergroup, the family of the double cosets form a fortified transposition hypergroup as well.

Proposition 4.12. *If K is a symmetric subhypergroup of T_A , then $(T_A : K)$ is a hypergroup.*

Proof. It is known that the associativity holds in $T_A : K$ if and only if $((x_K y_K)_K z_K)_K = (x_K (y_K z_K)_K)_K$ [6]. Equality $((x_K y_K)_K z_K)_K = (x_K y_K z_K)_K$ is shown to hold hereunder. If $K \cap x_K y_K = \emptyset$, then Corollary 4.11.ii yields $(x_K y_K)_K = x_K y_K$ and the above equality is obvious. If $K \cap x_K y_K \neq \emptyset$, then Corollary 4.11.i yields $(x_K y_K)_K = x_K y_K \cup K$. Hence:

$$\begin{aligned} (x_K y_K) z_K &= (x_K y_K \cup K) z_K = x_K y_K z_K \cup K z_K = x_K y_K z_K \cup z_K \cup K = \\ &= x_K y_K z_K \cup K \end{aligned}$$

Since $K \cap x_K y_K \neq \emptyset$ and $x_K y_K \subseteq x_K y_K z_K$, it follows that $K \subseteq (x_K y_K z_K)_K$ is valid. Therefore,

$$((x_K y_K)_K z_K)_K = (x_K y_K z_K \cup K)_K = (x_K y_K z_K)_K \cup K = (x_K y_K z_K)_K.$$

Duality yields $(x_K y_K z_K)_K = (x_K (y_K z_K)_K)_K$ and so the associativity is valid. Reproduction in $T_A : K$ derives directly from the reproduction in T_A , QED. \square

A consequence of Proposition 4.5 is that $K \cdot x_K = x_K \cdot K = \{x_K, K\}$ for every x_K in $T_A : K$. Hence:

Proposition 4.13. *K is a strong identity in hypergroup $T_A : K$, which consists only of attractive elements.*

Proposition 4.14. *The following are true in $T_A : K$*

- i. $\{x_K, y_K\} \subseteq x_K \cdot y_K$ for all $x_K, y_K \in T_A : K$,
- ii. $K \in x_K \cdot y_K, y \in S(x)$ for all $x_K \in T_A : K$.

5. HOMOMORPHISMS

According to the terminology introduced by M. Krasner, if H and H' are two hypergroups, then a *homomorphism* from H to H' is a mapping $\varphi : H \rightarrow P(H')$, such that $\varphi(xy) \subseteq \varphi(x)\varphi(y)$ for all $x, y \in H$. A homomorphism is called *strong* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in H$. A mapping $\varphi : H \rightarrow H'$ is called *strict homomorphism* if $\varphi(xy) \subseteq \varphi(x)\varphi(y)$ for all $x, y \in H$, while it is called *normal* $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in H$ [13, 15, 35].

Proposition 5.1. *If φ is a normal homomorphism from H to H' , then*

$$\varphi(b \setminus a) \subseteq \varphi(b) \setminus \varphi(a) \quad \text{and} \quad \varphi(a/b) \subseteq \varphi(a) \setminus \varphi(b).$$

Proof. If $y \in \varphi(b \setminus a)$, then $\varphi(x) = y$ for some $x \in b \setminus a$, which yields $a \in bx$. Thus, $\varphi(a) \in \varphi(bx) = \varphi(b)\varphi(x)$ and, consequently, $\varphi(x) \in \varphi(b) \setminus \varphi(a)$. Therefore, the first relation is established. The second relation follows by duality. \square

Now, let T and T' be two strictly regular transposition hypergroups with idempotent identities e and e' respectively. As usual, the kernel of φ , denoted by $\ker \varphi$, is the subset $\varphi^{-1}(\varphi(e))$ of T . Also, the homomorphic image $\varphi(T)$ of T is denoted by $Im\varphi$. A_T and $A_{T'}$ will denote the attractive elements of T and T' respectively, while C_T and $C_{T'}$ will signify the non-attractive elements of T and T' respectively.

Proposition 5.2. *If φ is a normal homomorphism from T to T' , then:*

- i. $\ker \varphi$ is a semisubhypergroup of T ,
- ii. $Im\varphi$ is a subhypergroup of T' , which generally does not contain the identity of T' , nevertheless $\varphi(e)$ is a neutral element in $Im\varphi$.

Proof. (i) If $x \in \ker \varphi$, then $\varphi(x \ker \varphi) = \varphi(e)$. Thus, $x \ker \varphi \subseteq \ker \varphi$.
(ii) Let $x \in T$. Then, $\varphi(x) \varphi(T) = \cup_{y \in T} \varphi(xy) = \varphi(xT) = \varphi(T)$. Similarly, $\varphi(T) \varphi(x) = \varphi(T)$. Thus, $Im \varphi$ is a subhypergroup of T' . Additionally, since $x \in ex = xe$, it holds that $\varphi(x) \in \varphi(e) \varphi(x) = \varphi(x) \varphi(e)$. \square

Proposition 5.3. *If φ is a normal homomorphism from T to T' and the identities e, e' of T and T' respectively are strong, then:*

- i. $S(x) \subseteq \ker \varphi$ for all $x \in C_T \cap \ker \varphi$,
- ii. if $C_T \cap \ker \varphi \neq \emptyset$, then $A_T \subseteq \ker \varphi$,
- iii. $\ker \varphi$ is a subhypergroup of T ,
- iv. if $\varphi(e) = e'$, then $\varphi(A_T) \subseteq A_{T'}$ and $\varphi(C_T) \subseteq C_{T'}$,
- v. if φ is an epimorphism, then $\varphi(e) = e'$.

Proof. (i) Per Proposition 2.12.ii, $S(x) \subseteq C_T$, if $x \in C_T$. Let $x' \in S(x)$. Then, for $\varphi(x')$ the following is valid: $\varphi(xx') = \varphi(x) \varphi(x') = \varphi(e) \varphi(x') = \varphi(ex') = \varphi(x')$. But $e \in xx'$, therefore $\varphi(e) \in \varphi(xx') = \varphi(x')$. Thus $\varphi(e) = \varphi(x')$. Hence, $x' \in \ker \varphi$.

(ii) Per Proposition 2.14, $A_T \subseteq xS(x)$, if $x \in C_T$. Therefore, $\varphi(A_T) \subseteq \varphi(xS(x)) = \varphi(x) \varphi(S(x)) = \varphi(e) \varphi(e) = \varphi(ee) = \varphi(e)$. Hence, $A_T \subseteq \ker \varphi$.

(iii) Per Proposition 5.2.i $\ker \varphi$ is a semisubhypergroup of T . Thus, if $x \in \ker \varphi$, then $x \ker \varphi \subseteq \ker \varphi$. Let y be an arbitrary element in $\ker \varphi$. It will be shown that $y \in x \ker \varphi$. Let x' be an element of $S(x)$ and suppose that $x' \in \ker \varphi$. Then, $y \in (xx')y = x(x'y) \subseteq x \ker \varphi$. Next, suppose that $x' \notin \ker \varphi$. Then, the previous part (i) and Proposition 2.12 imply that x and x' are attractive. Thus, if y is attractive, then, Proposition 2.16.i implies that $y \in xy \subseteq x \ker \varphi$, while, if y is canonical, Proposition 2.16.ii implies that $y = xy \subseteq x \ker \varphi$. Hence, $\ker \varphi \subseteq x \ker \varphi$ and so $\ker \varphi = x \ker \varphi$. Similarly, $\ker \varphi = (\ker \varphi)x$.

(iv) If $x \in A_T$, then

$$\varphi(x) e' = \varphi(x) \varphi(e) = \varphi(xe) = \varphi\{x, e\} = \{\varphi(x), \varphi(e)\} = \{\varphi(x), e'\}.$$

Hence, $\varphi(x) \in A_{T'}$. If $x \in C_T$, then $\varphi(x) e' = \varphi(x) \varphi(e) = \varphi(xe) = \varphi(x)$. Hence, $\varphi(x) \in C_{T'}$.

(v) Since φ is an epimorphism, for each $y \in S(\varphi(e))$ there exists $x \in T$, such that $\varphi(x) = y$. Thus, $e \in y\varphi(e) = \varphi(x) \varphi(e) = \varphi(xe) = \varphi\{x, e\} = \{\varphi(x), \varphi(e)\}$. Consequently, either $\varphi(e) = e$ or $\varphi(x) = e$. If $\varphi(x) = e$, then $y = e'$ for each $y \in S(\varphi(e))$. Therefore, $e = S(\varphi(e))$. Thus, $e = \varphi(e)$. \square

A homomorphism does not necessarily map attractive elements to attractive elements. A relevant example for fortified join hypergroups can be found in [15].

Proposition 5.4. *Let φ be a normal homomorphism from T to T' and suppose that the identities e, e' of T and T' respectively are strong. Then:*

- i. if the image of an attractive element is a non-attractive element, then $Im\varphi \subseteq C_{T'}$,
- ii. if the image of a non-attractive element is an attractive element, then it belongs to $\ker \varphi$ and all the attractive elements are in $\ker \varphi$.

Proof. (i) Per Proposition 5.3.iv, $\varphi(e) \neq e'$. Let a be an attractive element and $\varphi(a)$ a non-attractive element. We distinguish the following cases:

(a) if $x \in C_T$, then, per Proposition 2.16.ii, $ax = xa = x$ is valid, thus: $e\varphi(x) = e\varphi(ax) = e[\varphi(a)\varphi(x)] = [e\varphi(a)]\varphi(x) = \varphi(a)\varphi(x) = \varphi(ax) = \varphi(x)$. Hence, $\varphi(x)$ is a non-attractive element.

(b) for $\varphi(e)$ it holds that:

$$\begin{aligned} \varphi(a)e' = \varphi(a) &\Rightarrow \varphi(e)\varphi(a)e' = \varphi(e)\varphi(a) \Rightarrow \varphi(ea)e' = \varphi(ea) \Rightarrow \\ &\Rightarrow \varphi(\{e, a\})e' = \varphi(\{e, a\}) \Rightarrow \{\varphi(e), \varphi(a)\}e' = \{\varphi(e), \varphi(a)\} \Rightarrow \\ &\Rightarrow [\varphi(e)e'] \cup [\varphi(a)e'] = \{\varphi(e), \varphi(a)\} \Rightarrow [\varphi(e)e'] \cup \{\varphi(a)\} = \{\varphi(e), \varphi(a)\} \end{aligned}$$

If $\varphi(a)e' = \{\varphi(e), e'\}$, then $e' \in \{\varphi(e), \varphi(a)\}$, which is absurd. Therefore, $\varphi(e)$ is a non-attractive element.

(c) Let $y \in A_T$, $y \neq e$. Since $\varphi(e) \in C_{T'}$, assuming that $\varphi(y)$ is an attractive element, then, per Proposition 2.15, $\varphi(y)\varphi(e)$ consists only of non-attractive elements. However $\varphi(y)\varphi(e) = \varphi(ye) = \varphi\{y, e\} = \{\varphi(y), \varphi(e)\}$. Hence, $\varphi(y)$ is a non-attractive element, which contradicts the assumption above. Thus, $\varphi(y)$ is a non-attractive element.

(ii) Suppose that a is an attractive element and x a non-attractive element, the image of which is an attractive element different from the identity. Then $\varphi(a) \in A_{T'}$; otherwise, according to (i) above, $Im\varphi \subseteq C_{T'}$, which is a contradiction. Next, per Proposition 2.16.ii, $\varphi(a)\varphi(x) = \varphi(ax) = \varphi(x)$. Hence, $\varphi(a) \neq e'$, because $\varphi(x)$ is an attractive element different from the identity and the result of the hypercomposition of an attractive element with the identity contains the identity. Therefore, $\varphi(a) = \varphi(x)$, which yields:

$$\begin{aligned} \varphi(a)\varphi(e) = \varphi(x)\varphi(e) &\Rightarrow \varphi(ae) = \varphi(xe) \Rightarrow \varphi\{a, e\} = \varphi(x) \Rightarrow \\ &\Rightarrow \{\varphi(a), \varphi(e)\} = \varphi(x) \end{aligned}$$

Therefore, $\varphi(a) = \varphi(e) = \varphi(x)$. Thus, $x \in \ker \varphi$, therefore, per of Proposition 5.3.ii, all the attractive elements belong to $\ker \varphi$. \square

As was shown in [15], the fact that an attractive element belongs in $\ker \varphi$ does not imply that its inverses also belong in $\ker \varphi$. This means that, even though $ker\varphi$ is a subhypergroup of T when φ is normal, generally $ker\varphi$ is not a symmetric subhypergroup of T . Therefore, the notion of the complete homomorphism, is introduced in [15]. This notion is generalized here as follows:

Definition 5.5. A homomorphism will be called *complete*, if $S(x) \subseteq ker\varphi$ for each $x \in \ker \varphi$.

Proposition 5.6. *If φ is a complete homomorphism, then $\ker \varphi$ is a symmetric subhypergroup of T .*

Proof. $x \in \ker \varphi$ implies $x \ker \varphi \subseteq \ker \varphi$, since $\ker \varphi$ is a semisubhypergroup of T . Next, let $y \in \ker \varphi$ and $x \in S(x)$. Then, $y \in (xx)y = x(xy) \in x \ker \varphi$. Thus, $\ker \varphi \subseteq x \ker \varphi$ and therefore $\ker \varphi = x \ker \varphi$. Similarly, $(\ker \varphi)x = \ker \varphi$, and therefore, $\ker \varphi$ is a subhypergroup of T . In addition, $\ker \varphi$ is a symmetric subhypergroup of T , since $x \in \ker \varphi$ implies that $S(x) \subseteq \ker \varphi$. \square

Proposition 5.7. *Let φ be a complete and normal homomorphism, for which $\varphi(e) = e$ is valid. Then $\varphi(S(x)) \subseteq S(\varphi(x))$.*

Proof. $e \in \text{Im} \varphi$, since $\varphi(e) = e$. Next, let $y \in \text{Im} \varphi$. Then, there exists $x \in T$, such that $y = \varphi(x)$. Let $x \in S(x)$. Then, $e = \varphi(e) \in \varphi(xx) = \varphi(x)\varphi(x)$. If $\varphi(x) \neq e$, then $\varphi(x) \neq e$, since φ is complete. Thus, $e \in \varphi(x)\varphi(x)$ implies that $\varphi(x) \in S(\varphi(x))$. Consequently, $\varphi(S(x)) \subseteq S(\varphi(x))$. \square

Corollary 5.8. *Let φ be a complete and normal homomorphism for which $\varphi(e) = e$ is valid and $\varphi(S(x)) = S(\varphi(x))$ for each $x \in T$. Then:*

- i. *$\text{Im} \varphi$ is a symmetric subhypergroup of T ,*
- ii. *the homomorphic image of every symmetric subhypergroup of T is a symmetric subhypergroup of T .*

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