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ON CERTAIN FUNDAMENTAL PROPERTIES of HYPERGROUPS and FUZZY HYPERGROUPS – MIMIC FUZZY HYPERGROUPS

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ABSTRACT

In this paper, properties of hypergroups pertaining to the relationship between the hypercomposition and induced hypercompositions with the empty set are presented. Analogous fuzzy hypergroup properties are also proven. Finally, the study of these properties leads to the introduction of the mimic fuzzy hypergroup (fuzzy$_M$ - hypergroup).

KEYWORDS

hypergroup, fuzzy hypergroup.

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1. INTRODUCTION

Almost all fields of science desire to and attempt to utilize mathematical models in the process of not only describing the various phenomena under study, but in also predicting, wherever and whenever possible, the results generated through the influence of various causes. Several of the mathematical models thus utilized are non-deterministic, as several phenomena incorporate numerous uncertainties. Theory of probability, stochastic processes, fuzzy set theory, soft set theory, vague set theory, hypercompositional algebra, etc., are all different ways of expressing uncertainty. The self-contained development, as well as the cross-linking of the above theories hold great attraction for mathematicians. For example, A. Maturo, in a series of articles of his, (e.g. [39, 40, 41, 42] delved into linking the theory of probability with hypercompositional algebra. On the other hand, several mathematicians worked on linking fuzzy set theory with hypercompositional algebra.

One can distinguish three approaches, which were employed in order to connect these two topics. One approach is to consider a certain hyperoperation defined through a fuzzy set (P. Corsini [4], P. Corsini - V. Leoreanu, [6], I. Cristea e.g. [7, 8, 9], I. Cristea - S. Hoskova [10], M. Stefanescu - I. Cristea [49], K. Serafimidis et al. [48] etc.). Another is to consider fuzzy hyperstructures in a similar way as Rosenfeld did for fuzzy groups [46] (Zahedi, A.

Hasankhani [12, 54, 55], B. Davvaz [11] and others). The third approach is employed in the pioneering papers by P. Corsini - I. Tofan [5] and by I. Tofan - A. C. Volf [50, 51], which introduce fuzzy hyperoperations that induce *fuzzy hypergroups*. This approach was further adopted by other researchers (Ath. Kehagias e.g. [15, 16, 17, 18], V. Leoreanu-Fotea e.g. [22, 23] etc.). M. K. Sen, R. Ameri and G. Chowdhury utilized this concept in defining fuzzy hypersemigroups [47].

This paper deals with the algebra of hypergroups and fuzzy hypergroups. In researching all relevant bibliography, we came to realize that there exists a certain amount of confusion in some fundamental matters. For example, in attempting to define the hypergroup, some authors view the hyperoperation on a non-void set H as a function from $H \times H$ to the power set $P(H)$ of H, while others view it as a function from $H \times H$ to $P^*(H)$, i.e. to the set of all non-empty subsets of H. Certain fundamental properties of hypergroups and fuzzy hypergroups, which are proven herein, afford us a clearer view of such matters.

2. ON CRISP HYPERCOMPOSITIONS

Hypercompositional algebra was born in 1934, when F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [24]. The hypergroup is an algebraic structure in which the result of the composition of two elements is not an element but a set of elements. To make this paper self-contained, we begin by listing some definitions from the theory of hypercompositional structures (see also [35]). A (*crisp*) *hypercomposition* or *hyperoperation* in a non-empty set H is a function from $H \times H$ to the power set $P(H)$ of H. A non-void set H endowed with a hypercomposition "·" is called *hypergroupoid* if $ab \neq \varnothing$ for any a,b in H, otherwise it is called *partial hypergroupoid*. Note that, if A, B are subsets of H, then AB signifies the union $\bigcup\limits_{(a,b) \in A \times B} ab$. Since $A \times B = \varnothing \Leftrightarrow A = \varnothing$ or $B = \varnothing$, one can observe that if $A = \varnothing$ or $B = \varnothing$, then $AB = \varnothing$ and vice versa. aA and Aa have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a.

A *hypergroup* is a non-void set H endowed with a hypercomposition which satisfies the following axioms:

    i. $(ab)c = a(bc)$ for every $a,b,c \in H$ (associativity) and

    ii. $aH = Ha = H$ for every $a \in H$ (reproduction).

If only (i) is valid, then the hypercompositional structure is called *semi-hypergroup*, while if only (ii) is valid, then it is called *quasi-hypergroup*. The quasi-hypergroups in which the weak associativity is valid, i.e. $(ab)c \cap a(bc) \neq \varnothing$ for every $a,b,c \in H$, were named H_V-*groups* [52].

Remark. If a non-void set H is endowed with a composition which satisfies the

associative and the reproduction axioms, then H is a group. Indeed let $x \in H$. Per reproduction $x \in xH$. Therefore there exists $e \in H$ such that $xe = x$. Next let y be an arbitrary element in H. Per reproduction there exists $z \in H$ such that $y = zx$. Consequently $ye = (zx)e = z(xe) = zx = y$. Hence e is a right neutral element. Now, per reproduction $e \in xH$. Thus there exists $x' \in H$, such that $e = xx'$. Hence any element in H has a right inverse.

Proposition 2.1. *The result of the hypercomposition of any two elements in a hypergroup is always non-void.*

Proof. Let H be a hypergroup and suppose that $ab = \varnothing$ for some $a, b \in H$. Per reproduction, $aH = H$ and $bH = H$. Hence, $H = aH = a(bH) = (ab)H = \varnothing H = \varnothing$, which is absurd.

Proposition 2.2. *If the weak associativity is valid in a hypercompositional structure, then the result of the hypercomposition of any two elements is always non-void.*

Proof. Let H be a non-void set endowed with a hypercomposition satisfying the weak associativity. Suppose that $ab = \varnothing$ for some $a, b \in H$. Then, $(ab)c = \varnothing$ for any $c \in H$. Therefore, $(ab)c \cap a(bc) = \varnothing$, which is absurd. Hence, ab is non-void.

Corollary 2.1. *The result of the hypercomposition of any two elements in a H_V-group is always a non-void set.*

F. Marty also defined in [24] the two induced hypercompositions (right and left division) resulting from the hypercomposition of the hypergroup, i.e.:

$$\frac{a}{\,|\,b} = \{x \in H \,/\, a \in xb\} \quad \text{and} \quad \frac{a}{b\,|} = \{x \in H \,/\, a \in bx\}.$$

It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, a/b or $a : b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b \backslash a$ or $a..b$ is used to denote the left division. F. Marty's life was cut short, as he was killed during a military mission in World War II. [24, 25, 26] are the only works on hypergroups he left behind. However, several relevant papers by other authors began appearing shortly thereafter (e.g. Krasner [19, 20], Kuntzmann [21] etc). Up to the present, a vast number of papers has been produced on this subject (e.g.: see [3, 6])

In [13] and then in [14], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition ab with the hypercomposition ba. One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds true:

Given a theorem, the dual statement resulting from interchanging the order of hypercomposition "·" (and, by necessity, interchanging of the left and the right divisions), is also a theorem.

This principle is used throughout this paper. The following properties are direct consequences of the hypergroup axioms and the principle of duality is used in their proofs.

Proposition 2.3. $a/b \neq \varnothing$ *and* $b \backslash a \neq \varnothing$ *for all the elements* a, b *of a quasi-hypergroup* H.

Proof. Per reproduction, $Hb = H$ for every $b \in H$. Hence, for every $a \in H$ there exists $x \in H$, such that $a \in xb$. Thus, $x \in a/b$ and, therefore, $a/b \neq \varnothing$. Dually, $b \backslash a \neq \varnothing$.

Proposition 2.4. *In a quasi-hypergroup* H, *the non-empty result of the induced hypercompositions is equivalent to the reproduction axiom.*

Proof. Suppose that $x/a \neq \varnothing$ for every $a, x \in H$. Thus, there exists $y \in H$, such that $x \in ya$. Therefore, $x \in Ha$ for every $x \in H$ and so $H \subseteq Ha$. Next, since $Ha \subseteq H$ for every $a \in H$, it follows that $H = Ha$. Per duality, $H = aH$. Conversely now, per Proposition 2.3, the reproduction axiom implies that $a/b \neq \varnothing$ and that $a \backslash b \neq \varnothing$ for every a, b in H.

Based on Proposition 2.4, we are now in a position to give an equivalent definition of the hypergroup.

Definition 2.1. *A hypergoup is a non-void (crisp) set* H *endowed with a (crisp) hypercomposition, i.e. a function from* $H \times H$ *to the powerset* $P(H)$ *of* H, *which satisfies the following axioms:*

    i. $(ab)c = a(bc)$ *for every* $a, b, c \in H$ (*associativity*) and

    ii. $a/b \neq \varnothing$ *and* $b \backslash a \neq \varnothing$ *for every* $a, b \in H$.

Proposition 2.5. *In a hypergroup* H, *equalities* (i) $H = H/a = a/H$ *and* (ii) $H = a \backslash H = H \backslash a$ *are valid for every* a *in* H.

Proof. (i) Per Proposition 2.1, the result of hypercomposition in H is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$, such that $y \in xa$, which implies that $x \in y/a$. Hence, $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore, $H = H/a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a/y$. Hence, $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore, $H = a/H$. (ii) follows by duality.

The hypergroup is a very general structure, which was progressively enriched with additional axioms, either more or less powerful. This created a significant number of specific hypergroups. Moreover, some of these hypergroups constituted a constructive origin for the development of other new hypercompositional structures (e.g.: see [3, 6, 27, 28, 29, 32, 34, 36, 43]). Thus, W. Prenowitz enriched hypergroups with an axiom, in order to utilize them in the study of geometry [e.g.: see 44, 45]. More precisely, he introduced into the commutative hypergroup the *transposition axiom*

$$a/b \cap c/d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for every } a, b, c, d \in H$$

and named this new hypergroup *join space* [44]. For the sake of terminology unification, join spaces are also called *join hypergroups* [30]. It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata [34, 37]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

$$b \backslash a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for every } a,b,c,d \in H .$$

He named this particular hypergroup *transposition hypergroup* [13]. Clearly, if A, B, C and D are subsets of H, then $B \backslash A \cap C / D \neq \varnothing$ implies that $AD \cap BC \neq \varnothing$. In [31, 38] specialized *transposition hypergroups* were studied and, in [33], the transposition axiom was introduced into H_V-groups and the *transposition H_V-group* was thus defined.

3. ON FUZZY HYPERCOMPOSITIONS

Zadeh, in 1965, in order to provide «*a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership, rather than the presence of random variables*» [53] introduced the notion of fuzzy sets. If H is a non-void crisp set, then a fuzzy subset of H is a mapping from H to the interval of real numbers [0,1]. If $A \subseteq H$, then the *characteristic function X_A* of A

$$X_A : H \to [0,1], \; x \to X_A(x) = \begin{matrix} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{matrix}$$

is a fuzzy subset of H. Thus, if $A = \varnothing$, then its characteristic function is the identically zero function 0_H, i.e. $X_\varnothing(x) = 0_H(x) = 0$ for every $x \in H$. Moreover, if $A = H$, $H \neq \varnothing$, then the characteristic function of the entire set H, is $X_H(x) = 1_H(x) = 1$ for every $x \in H$. Thus, we can consider crisp sets as special case of fuzzy sets and identify every set (crisp or fuzzy) with its membership function. The collection of all fuzzy subsets of H is denoted by $F(H)$. A *fuzzy hypercomposition* maps the pairs of elements of the Cartesian product $H \times H$ to fuzzy subsets of H, i.e. $\circ : H \times H \to F(H)$. Hence, if $\circ$ is a fuzzy hyperoperartion, then $a \circ b$ is a function and the notation $(a \circ b)(x)$ means the value of $a \circ b$ at the element *x*. The definition of the fuzzy hyperoperartion subsumes the relevant one of crisp hyperoperation as a special case, since the later results from the former using the characteristic function.

Definition 3.1. [17, 18] *If $\circ : H \times H \to F(H)$ is a fuzzy hypercomposition, then for every $a \in H$, $B \in F(H)$, the fuzzy sets $a \circ B$ and $B \circ a$ are defined respectively by*

$$(a \circ B)(z) = \vee_{y \in H}\left(\left[(a \circ y)(z)\right] \wedge B(y)\right)$$

$$(B \circ a)(z) = \vee_{y \in H}\left(\left[(y \circ a)(z)\right] \wedge B(y)\right).$$

Per definition 3.1, if $a,b,c \in H$, then:

$$\left(a\circ(b\circ c)\right)(z)=\vee_{y\in H}\left[(a\circ y)(z)\wedge(b\circ c)(y)\right]$$

and

$$\left((a\circ b)\circ c\right)(z)=\vee_{y\in H}\left[(y\circ c)(z)\wedge(a\circ b)(y)\right]$$

Definition 3.2. [17, 18] *If $\circ: H{\times}H \rightarrow F(H)$ is a fuzzy hypercomposition, then, for every $A,B \in F(H)$, the fuzzy set $A\circ B$ is defined by*

$$(A\circ B)(z)=\vee_{x,y\in H}\left(\left[(x\circ y)(z)\right]\wedge A(x)\wedge B(y)\right).$$

As mentioned above, these definitions subsume the relevant ones of crisp hyperoperations as special cases. For example, if X_A is the characteristic function of the crisp set A, then, per Definition 2.1:

$$(x\circ X_A)(z)=\vee_{y\in H}\left(\left[(x\circ y)(z)\right]\wedge X_A(y)\right)=\vee_{\{y:z\in x\circ y\}}X_A(y).$$

Hence $x\circ X_A$ is the characteristic function of the crisp set $xA=\cup_{y\in A}xy$ (see also [17]).

Definition 3.3. *If $a \in H$ and $A,\ B \in P(H)$, then $a\circ B=a\circ X_B$, $B\circ a=X_B\circ a$ and $A\circ B=X_A\circ X_B$.*

Definition 3.4. [5, 51] *If $\circ: H{\times}H \rightarrow F(H)$ is a fuzzy hypercomposition, then H is called fuzzy hypergroup, if the following two axioms are valid:*

i. $(a\circ b)\circ c=a\circ(b\circ c)$ *for every $a,b,c \in H$ (associativity),*
ii. $a\circ H=H\circ a=X_H$ *for every $a \in H$ (reproduction).*
If (i) is only valid, then H is called a fuzzy semi-hypergroup [47] *while if (ii) is only valid, then H is called a fuzzy quasi-hypergroup.*

Lemma 3.1. *For every $a,\ b \in H$ and $C \in F(H)$, the following is true:*

$$(a\circ b)\circ C=a\circ(b\circ C).$$

Proof. $\left[(a\circ b)\circ C\right](z)=\vee_{x,y\in H}\left[(y\circ x)(z)\wedge(a\circ b)(y)\wedge C(x)\right]=$

$=\vee_{x\in H}\left[\vee_{y\in H}\left[(y\circ x)(z)\wedge(a\circ b)(y)\right]\wedge C(x)\right]=$

$=\vee_{x\in H}\left[\left[(a\circ b)\circ x\right](z)\wedge C(x)\right]=\vee_{x\in H}\left[\left[a\circ(b\circ x)\right](z)\wedge C(x)\right]=$

$=\vee_{x\in H}\left[\vee_{y\in H}\left[(a\circ y)(z)\wedge(b\circ x)(y)\right]\wedge C(x)\right]=$

$=\vee_{x,y\in H}\left[(a\circ y)(z)\wedge(b\circ x)(y)\wedge C(x)\right]=$

$$= \vee_{y \in H} \left[(a \circ y)(z) \wedge \left[\vee_{x \in H} (b \circ x)(y) \wedge C(x) \right] \right] =$$

$$= \vee_{y \in H} \left[(a \circ y)(z) \wedge (b \circ C)(y) \right] = \left[a \circ (b \circ C) \right](z).$$

Proposition 3.1. $a \circ b \neq 0_H$ *is valid for any pair of elements a, b in a fuzzy hypergroup H.*

Proof. Suppose that $a \circ b = 0_H$ for some $a, b \in H$. Per reproduction, $a \circ H = X_H$ and $b \circ H = X_H$. Hence, $X_H = a \circ H = a \circ (b \circ H)$. Per Lemma 3.1, the equality $a \circ (b \circ H) = (a \circ b) \circ H$ is valid. Since $a \circ b = 0_H$, we have: $(a \circ b) \circ H = 0_H \circ X_H$. But $(0_H \circ X_H)(z) = \vee_{x,y \in H} \left(\left[(x \circ y)(z) \right] \wedge 0_H(x) \wedge X_H(y) \right) = 0_H(z)$. Therefore, $X_H = 0_H$, which is absurd because $H \neq \varnothing$.

The *fuzzy H_V-groups* were defined in [15]. If $A, B \in F(H)$ and $p \in (0,1]$, then we write $A \boxdot_p B$, if there exists $x \in H$ such that $A(x) \wedge B(x) \geq p$. The fuzzy quasi-hypergroups in which the weak associativity is valid, i.e. $(a \circ b) \circ c \boxdot_p a \circ (b \circ c)$ for every $a, b, c \in H$, are named *fuzzy H_V-groups.*

Proposition 3.2. *If a fuzzy hypercompositional structure* $(H, \circ)$ *is endowed with the weak associativity, then* $a \circ b \neq 0_H$ *is valid for every* $a, b \in H$.

Proof. Suppose that $a \circ b = 0_H$ for some $a, b \in H$. Then $(a \circ b) \circ c = 0_H \circ c$, for any $c \in H$. Since $(0_H \circ c)(z) = \vee_{y \in H} \left(\left[(y \circ c)(z) \right] \wedge 0_H(y) \right) = 0$, it follows that $0_H \circ c = 0_H$ and, therefore, $(a \circ b) \circ c = 0_H$. Hence, the weak associativity is not valid in $(H, \circ)$, which contradicts our supposition.

Corollary 3.1. *The result of the hypercomposition of any two elements in a fuzzy H_V-group is always a non-zero function.*

4. THE MIMIC FUZZY HYPERGROUP

If H is a non-void set endowed with a fuzzy hypercomposition $\circ$, then two new induced fuzzy hypercompositions "/" and "\" can be defined as follows:

$$(a / b)(x) = (x \circ b)(a) \text{ for every } a, b, x \in H \text{ and}$$

$$(b \backslash a)(x) = (b \circ x)(a) \text{ for every } a, b, x \in H.$$

As in the case of crisp hypercompositions, the two induced fuzzy hypercompositions will be called *fuzzy right division* and *fuzzy left division* respectively (see also [2]).

Proposition 4.1. *For any pair of elements a,b in a fuzzy hypergroup H, $a/b \neq 0_H$ and $a \backslash b \neq 0_H$ is valid.*

Proof. Per reproduction, $H \circ b = X_H$ is valid for every $b \in H$. Thus, equality $(H \circ b)(a) = X_H(a)$ is true for any $a \in H$. Since

$$(H \circ b)(a) = \vee_{y \in H} \big[(y \circ b)(a) \wedge X_H(y) \big],$$

it follows that there exists $y \in H$ such that $(y \circ b)(a) = 1$ or, equivalently, $(a/b)(y) = 1$. Therefore, $a/b \neq 0_H$. Dually, $a \backslash b \neq 0_H$.

It becomes obvious that a statement analogous to Proposition 2.4 is not valid in the case of fuzzy hypergroups. Indeed:

Example 4.1. Let $H = \{a,b\}$ and suppose that

$$(a \circ a)(a) \leq (a \circ a)(b) \leq (a \circ b)(a) \leq (b \circ b)(a)$$

and $\qquad (a \circ a)(b) = (a \circ b)(b) = (b \circ a)(a) = (b \circ a)(b) \leq (b \circ b)(a) \leq (b \circ b)(b)$

Then, $(H, \circ)$ is a fuzzy semi-hypergroup. Suppose that $(x \circ y)(z) \neq 0$ for every $x,y,z \in H$. Then, in this fuzzy semi-hypergroup, $a/b \neq 0_H$ and $a \backslash b \neq 0_H$ is valid for every $a,b \in H$. Yet, if $(x \circ y)(z) \neq 1$ for any $x,y,z \in H$, one can easily see that the reproduction is not verified.

These ideas lead to the introduction of the following definition:

Definition 4.1. *If $\circ: H \times H \to F(H)$ is a fuzzy hypercomposition, then H is called mimic fuzzy hypergroup ($fuzzy_M$-hypergroup), if the following two axioms are valid:*

    i. $(a \circ b) \circ c = a \circ (b \circ c)$ *for every $a,b,c \in H$ (associativity),*

    ii. $a/b \neq 0_H$ and $a \backslash b \neq 0_H$ *for every $a,b \in H$.*

If (ii) is only valid, then H is called mimic fuzzy quasi-hypergroup ($fuzzy_M$-quasi-hypergroup) while, if instead of (i) the weak associativity is valid, then H is called mimic fuzzy H_v- group ($fuzzy_{MHv}$-group).

Proposition 4.2. *In a $fuzzy_M$-hypergroup H, it holds that $(H \circ a)(x) \neq 0$ and $(a \circ H)(x) \neq 0$ for every $a,x \in H$.*

Proof. Since $x/a \neq 0_H$ for every $a,x \in H$, it follows that there exists $y \in H$ such that $(x/a)(y) \neq 0$. Hence, there exists $y \in H$ such that $(y \circ a)(x) \neq 0$, for any $a,x \in H$. Since

$(H \circ a)(x) = \vee_{y \in H} \left(\left[(y \circ a)(x) \right] \wedge X_H(y) \right)$, it follows that $(H \circ a)(x) \neq 0$ for every $a, x \in H$.
Per duality, $(a \circ H)(x) \neq 0$ for every $a, x \in H$.

Proposition 4.3. *In a fuzzy$_M$-hypergroup H, it holds that $a \circ b \neq 0_H$ for every $a, b \in H$.*

Proof. Suppose that there are $a, b \in H$ such that $a \circ b = 0_H$. Then $(a \circ b) \circ H = 0_H$. Per Lemma 3.1, $(a \circ b) \circ H = a \circ (b \circ H)$. Hence $a \circ (b \circ H) = 0_H$. But

$$\left[a \circ (b \circ H) \right](z) = \vee_{y \in H} \left[(a \circ y)(z) \wedge (b \circ H)(y) \right]$$

Per Proposition 4.2, $(b \circ H)(y) \neq 0$, for every $b, y \in H$. Therefore $(a \circ y)(z) = 0$ for every $z, y \in H$. Thus $(a \backslash z)(y) = 0$ for every $y \in H$, which is absurd.

It is obvious that if a fuzzy$_M$-hypergroup H is commutative, then $a \circ H = H \circ a$ for any $a \in H$. However, generally speaking, this equality is not valid. Hence, we have the following definition:

Definition 4.2. *A fuzzy$_M$-hypergroup H will be called commutable fuzzy$_M$-hypergroup, if $a \circ H = H \circ a$ for any $a \in H$.*

Example 4.2. Let $H = \{a, b\}$ and suppose that:

$(a \circ a)(a) = 0,1;\ (a \circ a)(b) = 0,2$
$(b \circ a)(a) = (a \circ b)(b) = (b \circ a)(b) = 0,2;\ (a \circ b)(a) = 0,5$
$(b \circ b)(a) = 0,7;\ (b \circ b)(b) = 0,9$

Then $(H, \circ)$ is a non-commutative fuzzy$_M$-hypergroup, since $(a \circ b)(a) \neq (b \circ a)(a)$. Furthermore $(H, \circ)$ is non-commutable. Next if we define:

$(a \circledast a)(b) = (b \circledast a)(a) = 0,1$
$(a \circledast a)(a) = (b \circledast b)(a) = (b \circledast b)(b) = (a \circledast b)(a) = (a \circledast b)(b) = (b \circledast a)(b) = 0,9$

then $(H, \circledast)$ is a non-commutative fuzzy$_M$-hypergroup, since $(a \circledast b)(a) \neq (b \circledast a)(a)$. However $(H, \circledast)$ is commutable. Moreover if we define:

$(a \cdot a)(a) = (b \cdot b)(a) = (a \cdot b)(a) = (b \cdot a)(a) = 0,1$
$(a \cdot a)(b) = (b \cdot b)(b) = (a \cdot b)(b) = (b \cdot a)(b) = 0,2$

then $(H, \cdot)$ is a commutative fuzzy$_M$-hypergroup.

As in the case of fuzzy hypergroups [1, 2, 15, 16], the transposition axiom can be

introduced in fuzzy$_M$-hypergroups as well. Thus, we have the definition:

   Definition 4.3. *A fuzzy$_M$-hypergroup H will be called transposition fuzzy$_M$-hypergroup, if for any a, b, c, d $\in$ H for which there exists $p \in (0,1]$ such that $b\backslash a \boxdot_p c/d$, there exists also $q \in (0,1]$ such that $a \circ d \boxdot_q b \circ c$. If $p = q$ for every a, b, c, d $\in$ H, then H will be called p-transposition fuzzy$_M$-hypergroup.*

REFERENCES

[1]   R. Ameri**,** *Fuzzy transposition hypergroups*, Ital. J. Pure Appl. Math. No. 18, 167-174, 2005

[2]   G. Chowdhury, *Fuzzy transposition hypergroups*, Iranian Journal of Fuzzy Systems Vol. 6, No. 3, pp. 37-52, 2009

[3]   P. Corsini, *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[4]   P. Corsini, *Join spaces, power sets, fuzzy sets*, Algebraic Hyperstructures and Applications, ed. M. Stefanescu, Hadronic Press, pp. 45-52, 1994.

[5]   P. Corsini, I. Tofan, *On fuzzy hypergroups*. Pure Math Appl 8, pp. 29-37, 1997.

[6]   P. Corsini, V. Leoreanu, *Applications of Hyperstructures Theory*, Kluwer Academic Publishers, 2003.

[7]   I. Cristea, *Hyperstructures and fuzzy sets endowed with two membership functions*, Fuzzy Sets and Systems 160 (8), 1114-1124, 2009.

[8]   I. Cristea, *About the fuzzy grade of the direct product of two hypergroupoids*, Iran. J. Fuzzy Syst. 7, No. 2, 95-108, 2010

[9]   I. Cristea, *On the Fuzzy Subhypergroups of Some Particular Complete Hypergroups (I)*, World Applied Sciences Journal 7 pp. 57-63, 2009

[10] I. Cristea, S. Hoskova, *Fuzzy Pseudotopological Hypergroupoids*, Iranian Journal of Fuzzy Systems Vol. 6, No. 4, pp. 11-19, 2009.

[11] B. Davvaz, *Fuzzy H$_V$ –groups*, Fuzzy Sets and Systems 101, 191-195, 1999

[12] A. Hasankhani, M.M. Zahedi, *Fuzzy sub-F-polygroups*, Ratio Mathematica 12, pp. 35-44, 1997.

[13] J. Jantosciak, Transposition hypergroups, Noncommutative Join Spaces, Journal of Algebra, 187, pp. 97-119, 1997.

[14] J. Jantosciak, Ch. G. Massouros, Strong Identities and fortification in Transposition hypergroups, Journal of Discrete Mathematical Sciences & Cryptography 6, no 2-3, pp. 169-193, 2003.

[15] Ath. Kehagias, *An example of L-fuzzy Join Space*, Rendiconti del Circolo Matematico del Palermo, vol.51, pp.503-526, 2002.

[16] Ath. Kehagias, *L-fuzzy Meet and Join Hyperoperations*, Proc. of 8th Conference on Algebraic Hyperstructures and Applications, Samothraki, 2002.

[17] Ath. Kehagias, K. Serafimidis, *The L-Fuzzy Nakano Hypergroup*, Information Sciences, vol. 169, pp.305-327, 2005.

[18] Ath. Kehagias, *The Construction of Fuzzy-valued t-norms and t-conorms*, Studies in Computational Intelligence (SCI) 67, pp. 361-370, 2007.

[19] M. Krasner, *La loi de Jordan - Holder dans les hypergroupes et les suites generatrices des corps de nombres P - adiqes*. (I) Duke Math. J. 6, pp. 120-140, 1940, (II) Duke Math. J. 7, pp. 121-135, 1940.

[20] M. Krasner, *La characterisation de hypergroupes de classes et le probleme de Schreier dans ces hypergroupes*, C. R. Acad. Sci. (Paris), 212, 1941, pp. 948-950, 218, pp. 483-484 et pp. 542-544, 1944.

[21] J. Kuntzmann, *Opérations multiformes. Hypergroupes*, Comptes Rendus Acad. Sci., vol 204, pp. 1787-1788 (Paris, 1937)

[22] V. Leoreanu, *About hyperstructures associated with fuzzy sets of type 2*, Italian J. of Pure and Appl. Math., 17, 127-136, 2005.

[23] V. Leoreanu-Fotea, *Fuzzy join n-ary spaces and fuzzy canonical n-ary hypergroups*, Fuzzy Sets and Systems 161, pp. 3166–3173, 2010.

[24] F. Marty, *Sur un généralisation de la notion de groupe*. Huitième Congrès des mathématiciens Scand., Stockholm 1934, pp. 45-49.

[25] F. Marty, *Rôle de la notion de hypergroupe dans l' étude de groupes non abéliens*, C.R. Acad. Sci. (Paris) 201, pp. 636-638, 1935.

[26] F. Marty, *Sur les groupes et hypergroupes attachés à une fraction rationelle*, Annales de l' école normale, 3 sér., vol. 53, pp. 83-123, 1936.

[27] Ch. G. Massouros, *On the theory of hyperrings and hyperfields*, Algebra i Logika 24, No 6, pp. 728-742, 1985.

[28] Ch. G. Massouros, *Free and cyclic hypermodules*, Annali Di Mathematica Pura ed Applicata, Vol. CL. pp. 153-166, 1988.

[29] Ch. G. Massouros, *On the Hypercompositional Structures*, Fuzzy Systems & A.I.-Reports and Letters, Academia Romana, Vol. 3, no. 3, pp. 15-27, 1994.

[30] Ch. G. Massouros, *Canonical and Join Hypergroups*, An. Stiintifice Univ. "Al. I. Cuza", Iasi, Tom. XLII, Matematica, fasc.1, pp. 175-186, 1996.

[31] Ch. G. Massouros, G. G. Massouros, *Transposition Polysymmetrical Hypergroups with Strong Identity*, Journal of Basic Science Vol. 4, No. 1, pp. 85-93, 2008.

[32] Ch. G. Massouros, G. G. Massouros, *On Join Hyperrings.* Proceedings of the 10th Internat. Cong. on Algebraic Hyperstructures and Applications. pp. 203-216, Brno, Czech Republic 2008

[33] Ch. G. Massouros, A. Dramalidis, *Transposition H_v-groups,* Ars Combinatoria, Vol. 106, 2012. pp.143-160

[34] G. G. Massouros, *Automata and Hypermoduloids*, Proceedings of the 5th Internat. Cong. on Algebraic Hyperstructures and Applications, pp. 251-266, Iasi 1993, Hadronic Press 1994.

[35] G. G. Massouros, *On the Hypergroup Theory*, Fuzzy Systems & A.I. - Reports and Letters, Academia Romana, IV, no. 2/3, pp. 13-25, 1995.

[36] G. G. Massouros, *The Hyperringoid*, Multiple Valued Logic, Vol. 3, pp. 217-234, 1998.

[37] G. G. Massouros, *Hypercompositional Structures from the Computer Theory*, Ratio Matematica, no 13, pp. 37-42, 1999.

[38] G. G. Massouros, F. A. Zafiropoulos and Ch. G. Massouros, *Transposition Polysymmetrical Hypergroups*, Proceedings of the 8th Internat. Cong. on Algebraic Hyperstructures and Applications, Samothraki 2002, Spanidis Press 2003, pp. 191-202.

[39] A. Maturo, *Conditional events, conditional probabilities and hyperstructures*, in Proceedings EUFIT '97, September 8-11, 1997, pp. 1509/1513, Aachen, Germany

[40] A. Maturo, *On a non-standard algebraic hyperstructure and its applications to the coherent probability assessments*, Italian Journal of Pure and Applied Mathematics, n. 7, pp. 33-50, 2000.

[41] A. Maturo, *Finitely additive conditional probability with value on a hyperstructure*, J. of Information & Optimization Sciences ,vol. 22, 1, 2001

[42] A. Maturo, *Hypergroups and Generalized probabilities*, Advances in Generalized Structures, Approssimate Reasoning and Applications, pp. 13-29, Performantica Press, Iasi, Romania, 2001

[43] J. Mittas, *Sur certaines classes de structures hypercompositionnelles*. Proceedings of the Academy of Athens, Vol. 48, pp. 298 - 318, Athens 1973.

[44] W. Prenowitz, A Contemporary Approach to Classical Geometry, Amer. Math. Month. vol. 68, no 1, part II, pp. 1-67, 1961.

[45] W. Prenowitz, J. Jantosciak, Geometries and Join Spaces. J. Reine Angew. Math. 257, pp. 100-128, 1972.

[46] A. Rosenfeld, Fuzzy Groups, J. Math. Anal. Appl. 35, pp. 512-517, 1971.

[47] M. K. Sen, R. Ameri, G. Chowdhury, *Fuzzy hypersemigroups*, Soft Comput. 12 pp. 891-900, 2008.

[48] K. Serafimidis, Ath. Kehagias, M. Konstantinidou, *The L-Fuzzy Corsini Join Hyperoperation*, Italian Journal of Pure and Applied Mathematics, vol.12, pp.83-90, 2002.

[49] M. Stefanescu, I. Cristea, *On the fuzzy grade of hypergroups*, Fuzzy Sets and Systems 159 (9), pp. 1097-1106, 2008.

[50] I. Tofan, A. C. Volf, *On some connections between hyperstructures and fuzzy sets*, Italian J. of Pure and Appl. Math., **7**, 63-68, 2000.

[51] I. Tofan, A. C. Volf, *Algebraic aspects of information organization*, Systematic Organisation of Information in Fuzzy Systems, P. Melo-Pinto et al. (Eds) IOS Press, pp. 71-87, 2003.

[52] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*. Proc. of the 4th AHA, Xanthi, Greece 1990, World Scientific 1991, pp. 209-217.

[53] L. A. Zadeh, *Fuzzy sets*, Inform. Control, 8, 338-353, 1965.

[54] M. M. Zahedi, A. Hasankhani, *F-Polygroups*, Int. J. Fuzzy Math., vol. 4, pp.533 -548, 1996.

[55] M. M. Zahedi, A. Hasankhani, *F-Polygroups (II)*, Inf. Sciences, vol.89, pp.225-243, 1996.