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# TRANSPOSITION POLYSYMMETRICAL HYPERGROUPS WITH STRONG IDENTITY

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Abstract. Transposition Polysymmetrical Hypergroups appeared during the study of the theory of Languages and Automata from the point of view of the hypercompositional structures theory. This paper presents examples and properties, of Transposition Polysymmetrical Hypergroups that have a strong identity, i.e. an element e with the property $x \in ex = xe \subseteq \{x, e\}$, for all the elements $x$ of the hypergroup.

## 1. **Introduction**

A transposition hypergroup (see [2]) is a hypergroup which satisfies a postulated property of transposition i.e. $(b \setminus a) \cap (c/d) \neq \varnothing \Rightarrow (ad) \cap (bc) \neq \varnothing$ where $a/b = \{x \in H | a \in xb\}$ and $b \setminus a = \{x \in H | a \in bx\}$ are the induced hypercompositions. A join space [15], also join hypergroup, is a commutative transposition hypergroup (for some recent interesting examples see [1]). A **Fortified Transposition Hypergroup** is a transposition hypergroup $H$, having an identity or neutral element $e$ such that $ee = e, x \in ex = xe$, for all $x \in H$ and also for every $x \in H - \{e\}$ there exists a unique element $x' \in H - \{e\}$ such that $e \in xx'$, and, furthermore, $x'$ satisfies $e \in x'x$ [4]. If $H$ is also commutative, then it is a **Fortified Join Hypergroup** [12, 13]. This last hypergroup resulted from the theory of Languages [9, 10, 11]. Moreover, from the theory of automata resulted the **Transposition Polysymmetrical Hypergroup (TPH)** [9, 10, 11], i.e. a transposition hypergroup $H$, having an identity or neutral element $e$ such that $ee = e, x \in ex = xe$, for all $x \in H$ and also for every $x \in H - \{e\}$ there exists at least one element $x' \in H - \{e\}$, a **symmetric** (or two sided inverse) of $x$, such that $e \in xx'$, and $e \in x'x$. The set of the symmetric elements of $x$ is denoted by $S(x)$ and is called the **symmetric set** of $x$. A commutative transposition polysymmetrical hypergroup is called a **Join Polysymmetrical Hypergroup**. An element $e$ of a hypergroup $H$ is a scalar identity if $ex = xe = x$ for each $x$ in $H$. If a scalar identity exists in $H$, then it is unique. An element $e$ of a hypergroup $H$ is a **strong identity** if

$$x \in ex = xe \subseteq \{x, e\},\ \textit{for all } x \in H.$$

Strong identity need not be unique [4]. The set $E$ of the strong identities is a central subhypergroup of $H$ [4]. An element $x$ of a transposition polysymmetrical hypergroup $H$, will be called **attractive** if $e \in xe$, while a non identity element $x$ will be called **non attractive** if $e \notin xe$. We denote by $A$ the set of the attractive elements and by $C$ the set of non attractive elements. Then $H = A \cup C$ and $A \cap C = \varnothing$.

In [4] it has been proved that identity of a fortified transposition hypergroup is strong and unique. Yet the examples in [14] show that in polysymmetrical transposition hypergroups identity need not be unique. But if a polysymmetrical transposition hypergroup has a strong identity, then the Proposition holds:

**Proposition 1.1.** *If a polysymmetrical transposition hypergroup has a strong identity e, then it is unique.*

*Proof.* Assume that $u$ is identity distinct from $e$ and let $S_u(e)$ be the set of the symmetric elements of $e$ with regard to $u$. Then, there would exist an element $e' \in S_u(e)$, distinct from $u$, such that $u \in ee'$. But $ee' = \{e, e'\}$. Thus, $u \in \{e, e'\}$, which contradicts the assumption. $\square$

## 2. Examples of TPH with strong identity

It is known, that in a transposition hypergroup with a scalar identity $e$, each element has a unique inverse. However this does not happen in a transposition hypergroup with a strong identity. The following example shows that in a transposition hypergroup with a strong identity each element may have not a unique inverse.

**Example 2.1.** *Let $H$ be a totally ordered, and symmetric set around a center, denoted by $0 \in H$. A hypercomposition is defined on $H$ by*

$$x + y = \{x, y\} \text{ if } 0 \notin [x, y]$$

$$x + y = \{x, y, 0\} \text{ if } 0 \in [x, y].$$

*Then $(H, +)$ is a transposition hypergroup with strong identity being the element $0$. Obviously, if $x$ belongs to the positive cone, then every element of the negative cone of $H$ is opposite of $x$ and similarly, if $x$ belongs to the negative cone, then $S(x)$ is the positive cone.*

In the next examples, starting from other hypergroups, two transposition polysymmetrical hypergroups with strong identity are constructed.

**Example 2.2.** *Let the hypergroups $(Ai, \cdot), i \in I$. We consider the union $T = \cup_{i \in I} A_i$. In this set we define the following hyperoperation:*

$$a \circ b = ab \text{ if } a, b \text{ are elements of the same hypergroup } A_i$$

$$a \circ b = A_i \cup A_j \text{ if } a \in A_i, b \in A_j \text{ and } i \neq j$$

*It is known* [6] *that $(T, \circ)$ is a hypergroup and that, if $A_i, i \in I$ are transposition hypergroups, then $(T, \circ)$ is also transposition. Now let $A_i, i \in I$ be a family of fortified transposition hypergroups which consist only of attractive elements and assume that the hypergroups $A_i, i \in I$ have their identity e, common. Ithe hypercomposition is defined as follows:*

$$a \odot b = ab \text{ if } a, b \text{ are elements of the same hypergroup } A_i$$

$$a \odot b = A_i \cup A_j \text{ if } a \in A_i - \{e\}, b \in A_j - \{e\} \text{ and } i \neq j$$

*then $(T, \odot)$ becomes a transposition polysymmetrical hypergroup, which has e as its strong identity. Obviously if $a \in A_i$, then $S(a) = (T - Ai) \cup \{a'\}$, where $a'$ is the inverse of $a$ in $A_i$.*

**Example 2.3.** *Let $(A_i, \cdot), i \in I$, be a family of hypergroups, which have the property that the two elements which participate to the hypercomposition are always included in the result of the hypercomposition i.e. $\{x, y\} \subseteq xy$ for all $x, y$ in $A_i$. Next the union $T = \cup_{i \in I} A_i$ is equipped with a hypercomposition " $\bullet$ " defined as follows:*

$$a \bullet b = ab \text{ if } a, b \text{ are elements of the same hypergroup } A_i$$

$$a \bullet b = \{a, b\} \text{ if } a \in A_i, b \in A_j \text{ and } i \neq j$$

*Then $(T, \bullet)$ is a hypergroup and moreover, if $A_i, i \in I$ are transposition hypergroups, then $(T, \bullet)$ is also transposition. Now let $A_i, i \in I$ be a family of fortified transposition hypergroups which consist only of attractive elements and suppose that the hypergroups $A_i, i \in I$ have their identity e, common. Then $(T, \bullet)$ is also a fortified transposition hypergroup. Yet, if the hypercompositions " $\bullet$ " is modified slightly in the following way:*

$$a \odot b = ab \text{ if } a, b \text{ are elements of the same hypergroup } A_i$$

$$a \odot b = \{a, e, b\} \text{ if } a \in A_i - \{e\}, b \in A_j - \{e\} \text{ and } i \neq j$$

*then, $(T, \bullet)$ becomes a transposition polysymmetrical hypergroup, which has e as its strong identity. In order to verify the axioms, the property of the fortified transposition hypergroups must be used, according to which, in the result of the hypercomposition of two attractive elements, these two elements are contained* [4, 12]. *Obviously, if $a \in A_i$, then $S(a) = (T - A_i) \cup \{a'\}$, where $a'$ is the inverse of $a$ in $A_i$. Note that if $A_i, i \in I$ are transposition polysymmetrical hypergroups, then $(T, \bullet)$ is also a transposition polysymmetrical hypergroup.*

From the above Examples, the following interesting remarks derive:

($i$) the non existence of $e$ in $ab$ does not necessarily imply that $e$ does not also belong to $S(a)S(b)$, or to $S(b)S(a)$,

($ii$) the non void intersection $S(a) \cap S(b)$ does not imply that $S(a)$ is equal to $S(b)$.

## 3. Some properties of TPH with strong identity

Jantosciak, in [2], shows that a principle of duality holds for both the theory of hypergroups and the theory of transposition hypergroups. This principle is condensed as follows: *Given a theorem, the dual statement, which results from the interchanging of the order of the hypercomposition is also a theorem.* Since we are working in transposition hypergroups this principle is used throughout. Let $H$ be a transposition polysymmetrical hypergroup with strong identity $e$, and suppose that each element of $H$ is attractive. Then:

**Proposition 3.1.** *Let $x \neq e$ be an element of $H$, then*

($i$) $e/x = eS(x) = \{e\} \cup S(x) = S(x)e = x \setminus e$

($ii$) $x/e = e \setminus x = x$.

*Proof.* ($i$) Since $e$ is strong identity it is straight forward that the equalities $eS(x) = \{e\} \cup S(x) = S(x)e$ hold. Next, $t \in e/x$ if and only if $e \in tx$, which means that either $t$ equals to $e$ or $t$ belongs to $S(x)$ and so $e/x = \{e\} \cup S(x)$. Duality yields the rest.

($ii$) $t \in x/e$ if and only if $x \in te \subseteq \{t, e\}$. Since $x \neq e$, clearly $t = x$. So $x/e = x$. The rest follows by duality. □

**Corollary 3.2.** *If $X$ is a non empty subset of $H$ and $e \notin X$, then*

$$e/X = eS(X) = \{e\} \cup S(X) = S(X)e = X \setminus e.$$

Since $H$ is a transposition hypergroup with strong identity, the algebraic results of section 2.3, of [4], must hold. So:

**Proposition 3.3.** *If $x, y$ are elements of $H$, then*

($i$) $\{x, y\} \subseteq xy$,

($ii$) $x \in x/y$ *and* $x \in y \setminus x$,

($iii$) $x/x = x \setminus x = H$.

**Proposition 3.4.** *If $e \notin aS(b)$, then*

$$aS(b) = a/b \cup S(b) \text{ and } S(b)a = b \setminus a \cup S(b).$$

*Proof.* Since $e \in bS(b)$ it derives that $b \in e/S(b)$. Thus $a/b \subseteq a/(e/S(b))$. So we have

$$a/b \cup S(b) \subseteq a/(e/S(b)) \cup S(b) \subseteq aS(b)/e \cup S(b) = aS(b) \cup S(b) = aS(b).$$

On the other hand, since $e/b = \{e\} \cup S(b)$ we have:

$$aS(b) \subseteq a(e/b) \subseteq ae/b = \{a, e\}/b = a/b \cup e/b = a/b \cup \{e\} \cup S(b).$$

Since $e \notin aS(b)$ it derives that $aS(b) \subseteq a/b \cup S(B)$. Thus $aS(b) = a/b \cup S(B)$. Dually, $S(b)a = b \setminus a \cup S(b)$ □

**Corollary 3.5.** *If $B_1, B_2$ are non empty subsets of $H$ and $e \notin B_1S(B_2)$, then*

$$B_1S(B_2) = B_1/B_2 \cup S(B_2) \text{ and } S(B_2)B_1 = B_2 \setminus B_1 \cup S(B_2).$$

**Proposition 3.6.** *If $e \notin ab$, then $eS(ab) = eS(b)S(a)$.*

*Proof.* Because of Corollary 3.2 $e/ab = S(ab) \cup \{e\} = eS(ab)$. Also because of the mixed associativity [3, 7] $e/ab = (e/b)/a$. Thus we have: $S(ab) \cup e = e/ab = (e/b)/a = [S(b) \cup \{e\}]/a = S(b)/a \cup e/a = S(b)/a \cup S(a) \cup \{e\} = S(b)S(a) \cup \{e\}$. If we assume that $e \notin S(b)S(a)$ and since $e \notin ab$ implies that $e \notin S(ab)$, then from $S(ab) \cup \{e\} = S(b)S(a) \cup \{e\}$ it derives that $S(ab) = S(b)S(a)$. □

**Corollary 3.7.** *If $e \notin ab$ and $e \notin S(b)S(a)$, then $S(ab) = S(b)S(a)$.*

**Remark 3.8.** It has been proved (see [12]) that, in the case of fortified join hypergroups, if $b = a^{-1}$, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ may not be valid. Since the fortified join hypergroup is a partial case of the transposition polysymmertical hypergroup, it derives that if $e \in ab$, then the equality $S(ab) = S(b)S(a)$ may not be valid. Also, from the above Examples 2.2 and 2.3, it becomes evident that this equality fails to hold when $e \in S(b)S(a)$.

**Corollary 3.9.** *Let $B_1, B_2$ be two non empty subsets of $H$, then*

($i$) *$eS(B_1B_2) = eS(B_2)S(B_1)$, if $e \notin B_1B_2$,*
($ii$) *$S(B_1B_2) = S(B_2)S(B_1)$, if $e \notin B_1B_2$ and $e \notin S(B_2)S(B_1)$.*

**Proposition 3.10.** *If $x, y, z \in H$ and $z \in xy$, then:*

($i$) *$x'z \cap ey \neq \varnothing$ for every $x' \in S(x)$,*
($ii$) *$zy' \cap ex \neq \varnothing$ for every $y' \in S(y)$*

*Proof.* $z \in xy$ implies that $x \in z/y$ and $y \in x \setminus z$. If $x', y'$ are arbitrary elements of $S(x)$ and $S(y)$ respectively, then $e \in x'x$ and $e \in yy'$ yields that $x \in x' \setminus e$ and $y \in e/y'$. Thus $x' \setminus e \cap z/y \neq \varnothing$ and $x \setminus z \cap e/y' \neq \varnothing$ , and the proposition follows by the property of transposition. □

**Proposition 3.11.** *Let $x, y, z \in H - \{e\}$ and $z \in xy$:*

($i$) *if $S(x) \cap S(z) = \varnothing$, then $y \in x'z$ for every $x' \in S(x)$,*
($ii$) *if $S(y) \cap S(z) = \varnothing$, then $x \in zy'$ for every $y' \in S(y)$.*

*Proof.* By the previous proposition $z \in xy$ implies that $x'z \cap ey \neq \varnothing$ for every $x' \in S(x)$. So $x'z \cap \{e, y\} \neq \varnothing$. Thus, since $S(x) \cap S(z) = \varnothing$ is given, it follows that $e \notin x'z$ and therefore $y \in x'z$. Similarly ($ii$) is established. □

**Remark 3.12.** $e \in xx'$ implies $x \in ex$ and $x' \in ex'$. Also $e \in ex$ implies $e \in ex'$ while $x \notin ee$. Thus, from these observations and Proposition 3.11, it is clear that in transposition polysymmetrical hypergroups with strong identity the property of reversibility is valid under conditions.

Next a few Propositions regarding the subhypergroups of these hypergoups are given. First let us recall that a subhypergroup $h$ of a transposition polysymmetrical hypergroup is symmetric, if $x \in h$ implies $S(x) \subseteq h$, while a subhypergroup of a hypergoup is called closed if and only if $x/y \subseteq h$ and $y \setminus x \subseteq h$, for every $x, y \in h$.

**Proposition 3.13.** *$H$ does not have non empty proper closed subhypergroups.*

*Proof.* Since $x/x = x \setminus x = H$, for every $x \in H$, $H$ does not have nonempty proper closed subhypergroups. □

**Proposition 3.14.** *Let $h$ be a symmetric subhypergroup of $H$. If $x \notin h$, then*

(*i*) $x/h \cap h = \varnothing$ *and* $h \setminus x \cap h = \varnothing$,
(*ii*) $xh = x/h \cup h$ *and* $hx = h \setminus x \cup h$,
(*iii*) $h/x = hS(x)$ *and* $x \setminus h = S(x)h$.

*Proof.* (*i*) Let $x \notin h$. If $x/y \cap h \neq \varnothing$ for some $y \in h$, then $x \in hy \subseteq h$, which contradicts the assumption. Thus $x/h \cap h = \varnothing$.

(*ii*) Since $h$ is symmetric, it derives that $S(h) = h$. Also $e \notin xh$, since $x \notin h$. So, according to Corollary 3.5 we have $xh = xS(h) = x/h \cup S(h) = x/h \cup h$.

(*iii*) According to Proposition 3.1, $S(x) \subseteq e/x$. Obviously $e/x$ is a subset of $h/x$, so $S(x) \subseteq h/x$. Since $x \notin h$ we have $e \notin hS(x)$. Thus Corollary 3.5 implies that $hS(x) = h/x \cup S(x) = h/x$. The rest in each of (*i*), (*ii*) and (*iii*) follows by duality. □

**Proposition 3.15.** *Let $h$ be a symmetric subhypergroup of $H$. If $x \notin h$, then*

$$(x/h)h = xh \text{ and } h(h \setminus x) = hx$$

*Proof.* Since $x \in x/h$, it derives that $xh \subseteq (x/h)h$. Also, because of Proposition 3.14 (*ii*) it holds: $x/h \subseteq xh$. Thus $xh \subseteq (x/h)h \subseteq (xh)h = xh$. Duality yields the other part. □

**Proposition 3.16.** *Let $h$ be a symmetric subhypergroup of $H$. If $x, y \notin h$, then*

(*i*) $x/h \cap y/h \neq \varnothing$ *implies* $x/h = y/h$,
(*ii*) $h \setminus x \cap h \setminus y \neq \varnothing$ *implies* $h \setminus x = h \setminus y$,
(*iii*) $h \setminus (x/h) \cap h \setminus (y/h) \neq \varnothing$ *implies* $h \setminus (x/h) = h \setminus (y/h)$

*Proof.* (*i*) $x/h \cap y/h \neq \varnothing$ implies that $x \in (y/h)h$. Since $y \notin h$, from Propositions 3.15 and 3.14(*ii*) follows the equality: $(y/h)h = yh = y/h \cup h$. Thus $x \in y/h \cup h$. Since $x \notin h$, it derives that $x \in y/h$. So $x/h \subseteq (y/h)/h = y/hh = y/h$. By symmetry $y/h \subseteq x/h$. Hence $x/h = y/h$. Duality gives (*ii*).

(*iii*) Algebra of hypergroups [3, 6] and Proposition 3.15 gives:
$h \setminus (x/h) \cap h \setminus (y/h) \neq \varnothing \Rightarrow (h \setminus x)/h \cap h \setminus (y/h) \neq \varnothing \Rightarrow h \setminus x \cap [h \setminus (y/h)]h \neq \varnothing \Rightarrow$
$h \setminus x \cap h \setminus [(y/h)h] \neq \varnothing \Rightarrow h \setminus x \cap h \setminus yh \neq \varnothing \Rightarrow x \in yh \Rightarrow y \in x/h \Rightarrow y/h \subseteq (x/h)/h \Rightarrow$
$y/h \subseteq x/(hh) \Rightarrow y/h \subseteq x/h \Rightarrow h \setminus (y/h) \subseteq h \setminus (x/h)$. By symmetry $h \setminus (x/h) \subseteq h \setminus (y/h)$, and so equality holds. □

## 4. **Cosets**

This paragraph refers to cosets defined from a nonempty symmetric subhypergroup in a transposition polysymmertical hypergroup $H$. It is assumed that $H$ has a strong identity and consists only of attractive elements. The definitions of the left, the right and the double coset are the same to these in [4].

**Definition 4.1.** Let $x \in H$ and let $h$ be a nonempty symmetric subhypergroup. Then $x_{\overleftarrow{h}}$, the left coset of $h$ determined by $x$, and dually, $x_{\overrightarrow{h}}$, the right coset of $h$ determined by $x$, are given by

$$x_{\overleftarrow{h}} = \begin{cases} h, & \text{if } x \in h \\ x/h, & \text{if } x \notin h. \end{cases} \quad \textit{and } x_{\overrightarrow{h}} = \begin{cases} h, & \text{if } x \in h \\ x \setminus h, & \text{if } x \notin h. \end{cases}$$

For $A \subseteq H$, $A_{\overleftarrow{h}}$ and $A_{\overrightarrow{h}}$ denote the unions $\cup\{x_{\overleftarrow{h}} | x \in A\}$ and $\cup\{x_{\overrightarrow{h}} | x \in A\}$ respectively.

Recalling that in any hypergroup the equality $(B \setminus A)/C = B \setminus (A/C)$ is valid, we have:

**Definition 4.2.** Let $x \in H$ and let $h$ be a nonempty symmetric subhypergroup. Then, $x_h$, the double coset of $h$ determined by $x$, is given by

$$x_h = \begin{cases} h, & \text{if } x \in h \\ h \setminus (x/h) = (h \setminus x)/h, & \text{if } x \notin h. \end{cases}$$

Following the above notation, if $A$ is a non void subset of $H$, then $A_h$ denotes the union $\cup\{x_h|x \in A\}$. Next, it is being shown that properties which are in value for the cosets in fortified transposition hypergroups [4] are also valid in these hypergroups.

**Proposition 4.3.** *Let $h$ be a symmetric subhypergroup of $H$. Then*

$(i)$ $x \in x_{\overleftarrow{h}}, x \in x_{\overrightarrow{h}}$ *and* $x \in x_h$

$(ii)$ $x_{\overleftarrow{h}} \subseteq x_h$ *and* $x_{\overrightarrow{h}} \subseteq x_h$

$(iii)$ $x_h = (x_{\overleftarrow{h}})_{\overrightarrow{h}} = (x_{\overrightarrow{h}})_{\overleftarrow{h}}$

Proposition 3.15 assures that distinct left cosets, dually, right cosets as well as double cosets are disjoint. Thus:

**Proposition 4.4.** *The families $H : \overleftarrow{h} = \{x_{\overleftarrow{h}}|x \in H\}$, $H : \overrightarrow{h} = \{x_{\overrightarrow{h}}|x \in H\}$ and $H : h = \{x_h|x \in H\}$ of left, right and double cosets are each one partitions of $H$.*

**Proposition 4.5.** *Let $h$ be a symmetric subhypergroup of $H$. Then*

$(i)$ $x_{\overleftarrow{h}}h = xh = x_{\overleftarrow{h}} \cup h$

$(ii)$ $hx_{\overrightarrow{h}} = hx = x_{\overrightarrow{h}} \cup h$

*Proof.* $(i)$ If $x \in h$, then the equalities are valid, since every part of each equality equals $h$. If $x \notin h$, then because of Proposition 3.15 $x_{\overleftarrow{h}}h = (x/h)h = xh$ and because of Proposition 3.14 $(ii)$ $xh = x/h \cup h = x_{\overleftarrow{h}} \cup h$. Duality gives $(ii)$. □

**Corollary 4.6.** *If $A$ is a non empty subset of $H$ and $h$ a symmetric subhypergroup of $H$. Then*

$$A_{\overleftarrow{h}}h = Ah = A_{\overleftarrow{h}} \cup h \text{ and } hA_{\overrightarrow{h}} = hA = A_{\overrightarrow{h}} \cup h.$$

**Proposition 4.7.** *Let $h$ be a symmetric subhypergroup of $H$. Then*

$$hx_h = hx_{\overleftarrow{h}} = x_h \cup h = hxh = x_{\overrightarrow{h}}h = x_h h$$

*Proof.* By Proposition 4.5$(i)$, it follows that:

$$hxh = h(x_{\overleftarrow{h}} \cup h) = hx_{\overleftarrow{h}} \cup h = hx_{\overleftarrow{h}}$$

and by duality it holds: $hxh = x_{\overrightarrow{h}}h$. Next, Proposition 4.3$(iii)$ and Corollary 4.6 gives the equalities: $hx_h = h(x_{\overleftarrow{h}})_{\overrightarrow{h}} = hx_{\overleftarrow{h}} = (x_{\overleftarrow{h}})_{\overrightarrow{h}} \cup h = x_h \cup h$. Duality gives the rest and so the Proposition holds. □

**Corollary 4.8.** *If $A$ is a non empty subset of $H$ and $h$ a symmetric subhypergroup of $H$. Then*

$$hA_h = hA_{\overleftarrow{h}} = A_h \cup h = hAh = A_{\overrightarrow{h}}h = A_h h.$$

**Proposition 4.9.** *Let $h$ be a symmetric subhypergroup of $H$. Then*

$(i)$ $(xy)_{\overleftarrow{h}} \subseteq x_{\overleftarrow{h}}y_{\overleftarrow{h}} \cup h$

$(ii)$ $(xy)_{\overrightarrow{h}} \subseteq x_{\overrightarrow{h}}y_{\overrightarrow{h}} \cup h$

*Proof.* $(i)$ Because of Corollary 4.6 $(xy)_{\overleftarrow{h}} \subseteq (xy)_{\overleftarrow{h}}h = xyh$ is valid. Next, because of Proposition 3.3, it holds: $xyh \subseteq (x/h)yh = x_{\overleftarrow{h}}yh$. Now, Proposition 4.4 gives:

$$x_{\overleftarrow{h}}yh = x_{\overleftarrow{h}}(y_{\overrightarrow{h}} \cup h) = x_{\overleftarrow{h}}y_{\overrightarrow{h}} \cup x_{\overleftarrow{h}}h = x_{\overleftarrow{h}}y_{\overrightarrow{h}} \cup x_{\overleftarrow{h}} \cup h.$$

Finally, because of Proposition 3.3, the equality $x_{\overleftarrow{h}}y_{\overrightarrow{h}} \cup x_{\overleftarrow{h}} \cup h = x_{\overleftarrow{h}}y_{\overrightarrow{h}} \cup h$, holds. Duality gives part $(ii)$. □

**Corollary 4.10.** *Let $A, B$ be non empty subsets of $H$ and $h$ a symmetric subhypergroup of $H$. Then*

$$(AB)_{\overleftarrow{h}} \subseteq A_{\overleftarrow{h}}B_{\overleftarrow{h}} \cup h \text{ and } (AB)_{\overrightarrow{h}} \subseteq A_{\overrightarrow{h}}B_{\overrightarrow{h}} \cup h$$

**Proposition 4.11.** *Let $h$ be a symmetric subhypergroup of $H$. Then*

$$(xy)_h \subseteq x_h y_h \cup h$$

*Proof.* Proposition 4.3 (iii) and Corollary 4.10 gives

$$(xy)_h = ((xy)_{\overleftarrow{h}})_{\overrightarrow{h}} \subseteq [x_{\overleftarrow{h}} y_{\overleftarrow{h}} \cup h]_{\overrightarrow{h}} = (x_{\overleftarrow{h}} y_{\overleftarrow{h}})_{\overrightarrow{h}} \cup h_{\overrightarrow{h}} \subseteq (x_{\overleftarrow{h}})_{\overrightarrow{h}} (y_{\overleftarrow{h}})_{\overrightarrow{h}} \cup h = x_h y_h \cup h$$

and so the Proposition. □

**Corollary 4.12.** *Let $A, B$ be non empty subsets of $H$ and $h$ a symmetric subhypergroup of $H$. Then*

$$(AB)_h \subseteq A_h B_h \cup h.$$

**Corollary 4.13.** *Let $A, B$ be non empty subsets of $H$ and $h$ a symmetric subhypergroup of $H$. Then*

(*i*) $h \cap A_h B_h \neq \varnothing$ *implies* $(A_h B_h)_h = A_h B_h \cup h$ *and*
(*ii*) $h \cap A_h B_h = \varnothing$ *implies* $(A_h B_h)_h = A_h B_h$

Let " $\bullet$" be the induced hypercomposition on $H : h$. Using Proposition 3.1 of [3] it can be proved that associativity holds in $(H : h, \bullet)$ if and only if $((x_h y_h)_h z_h)_h = (x_h (y_h z_h)_h)_h$. In a similar way to Theorem 31 of [4] one can prove that the above equality holds in transposition polysymmetrical hypergroups of attractive elements and therefore

**Proposition 4.14.** *Let $h$ be a symmetric subhypergroup of $H$. Then $(H : h, \bullet)$ is a hypergroup.*

## 5. **Homomorphisms**

According to the terminology introduced by M. Krasner [5], if $H$ and $H'$ are two hypergroups, then a homomorphism from $H$ to $H'$ is a mapping $\phi : H \to \mathcal{P}(H')$ such that $\phi(xy) \subseteq \phi(x)\phi(y)$, for every $x, y \in H$. A homomorphism is called normal if $\phi$ is a mapping from $H$ to $H'$ and $\phi(xy) = \phi(x)\phi(y)$ for every $x, y \in H$.

**Proposition 5.1.** *If $\phi$ is a normal homomorphism from $H$ to $H'$, then*

$$\phi(b \setminus a) \subseteq \phi(b) \setminus \phi(a) \text{ and } \phi(a/b) \subseteq \phi(a)/\phi(b).$$

*Proof.* If $y \in \phi(b \setminus a)$, then $\phi(x) = y$ for some $x \in b \setminus a$, from where it follows that $a \in bx$. Thus $\phi(a) \in \phi(bx) = \phi(b)\phi(x)$ and consequently $\phi(x) \in \phi(b) \setminus \phi(a)$. Therefore, the first relation is established. The second relation follows by duality. □

Now let $H$ and $H'$ be two transposition polysymmetrical hypergroups with strong identities $e$ and $e'$ respectively and suppose that they consist only of attractive elements. As usual the kernel of $\phi$, denoted by $ker\phi$, is the subset $\phi^{-1}(\phi(e))$ of $H$. Also the homomorphic image $\phi(H)$ of $H$ is denoted by $Im\phi$.

**Proposition 5.2.** *If $\phi$ is a normal homomorphism from $H$ to $H'$, then*

(*i*) *$ker\phi$ is a semisubhypergroup of $H$,*
(*ii*) *$Im\phi$ is a subhypergroup of $H'$ which generally does not contain the strong identity of $H'$, but $\phi(e)$ is a neutral element in $Im\phi$.*

*Proof.* (*i*) If $x \in ker\phi$, then $\phi(xker\phi) = \phi(e)$. Thus $xker\phi \subseteq ker\phi$.

(*ii*) Let $x \in H$. Then $\phi(x)\phi(H) = \cup_{y \in H} \phi(xy) = \phi(xH) = \phi(H)$. Similarly $\phi(H)\phi(x) = \phi(H)$. Thus $Im\phi$ is a subhypergroup of $H'$. Yet, since $x \in ex = xe$ it holds $\phi(x) \in \phi(e)\phi(x) = \phi(x)\phi(e)$. □

**Proposition 5.3.** *Let $\phi$ be a normal epimorphism from $H$ to $H'$. Then $\phi(e) = e'$.*

*Proof.* Since $\phi$ is a normal epimorphism, there exists a subset $X$ of $H$ such that $\phi(X) = S(\phi(e))$. Next the relation $e' \in \phi(e)S(\phi(e))$ implies:

$$e' \in \phi(e)\phi(X) = \phi(eX) = \phi(\{e\} \cup X) = \{\phi(e)\} \cup \phi(X).$$

Consequently either $e' = \phi(e)$ or $e' \in \phi(X) = S(\phi(e))$. By the definition of the symmetric set, from $e' \in S(\phi(e))$ it derives that $e' = S(\phi(e))$. Thus $e' = \phi(e)$ and so the proposition. □

The notion of the complete homomorphism, which was introduced in [8], is defined as follows for the case of transposition polysymmetrical hypergroups.

**Definition 5.4.** A homomorphism will be called complete if for every $x \in ker\phi$ follows that $S(x) \subseteq ker\phi$.

**Proposition 5.5.** *If $\phi$ is a complete and normal homomorphism, then $ker\phi$ is a symmetric subhypergroup of $H$.*

*Proof.* $x \in ker\phi$ implies $xker\phi \subseteq ker\phi$, since $ker\phi$ is a semisubhypergroup of $H$. Next let $y \in ker\phi$ and $x' \in S(x)$. Then $y \in (xx')y = x(x'y) \subseteq xker\phi$. Thus $ker\phi \subseteq xker\phi$ and so $ker\phi = xker\phi$. Similarly $(ker\phi)x = ker\phi$, and therefore $ker\phi$ is a subhypergroup of $H$. In addition $ker\phi$ is a symmetric subhypergroup of $H$, since $x \in ker\phi$ implies $S(x) \subseteq ker\phi$. □

**Proposition 5.6.** *Let $\phi$ be a complete and normal homomorphism for which $\phi(e) = e'$ is valid. Then $\phi(S(x)) \subseteq S(\phi(x))$.*

*Proof.* $e' \in Im\phi$, since $\phi(e) = e'$. Next let $y \in Im\phi$. Then $y = \phi(x)$ for a $x$ in $H$. Let $x' \in S(x)$. Then $e' = \phi(e) \in \phi(xx') = \phi(x)\phi(x')$. If $\phi(x) \neq e'$, then $\phi(x') \neq e'$, since $\phi$ is complete. Thus $e' \in \phi(x)\phi(x')$ implies that $\phi(x') \in S(\phi(x))$. Consequently $\phi(S(x)) \subseteq S(\phi(x))$. □

**Corollary 5.7.** *Let $\phi$ be a normal homomorphism for which it holds $\phi(e) = e'$ and $\phi(S(x)) = S(\phi(x))$ for every $x \in H$. Then*

(*i*) *$Im\phi$ is a symmetric subhypergroup of $H'$,*

(*ii*) *The homomorphic image of every symmetric subhypergroup of $H$ is a symmetric subhypergroup of $H'$.*

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## References

[1] S. HOSKOVA, Abelization of join spaces of affine transformation of ordered field with proximity, *Appl. Gener. Topology* **6**, No 1(2005), 57-65.
[2] J. JANTOSCIAK, Homomorphisms, equivalences and reductions in hypergroups, *Riv. Mat. Pura Appl.* **9**(1991), 23-47.
[3] J. JANTOSCIAK, Cvransposition hypergroups, Noncommutative Join Spaces, *Journal of Algebra* **187**(1997), 97-119.
[4] J. JANTOSCIAK, CH. G. MASSOUROS, Strong Identities and fortification in Transposition hypergroups, *Journal of Discrete Mathematical Sciences and Cryptography* **6**, No 2-3(2003), 169-193.
[5] M. KRASNER, Cour sur la theorie de groupes, *Notre Dame University*, USA 1952.
[6] CH. G. MASSOUROS, Hypergroups and convexity, *Riv. di Mat. pura ed applicata* **4**, 1989.
[7] CH. G. MASSOUROS, On the semi-subhypergroups of a hypergroup, *Internat. J. Math. and Math. Sci.* **14**, No 2(1991), 293-304.

[8] CH. G. MASSOUROS, Normal homomorphisms of Fortified Join Hypergroups, *Proceedings of the 5th Internat. Cong. on Algebraic Hyperstructures and Applications, Hadronic Press 1994*, Iasi(1993) 133-142.

[9] G.G. MASSOUROS, J. MITTAS, Languages - Automata and hypercompositional structures, *Proceedings of the 4th Internat. Cong. in Algebraic Hyper-structures and Applications, Xanthi, Greece*,(1990), 137-147.

[10] G.G. MASSOUROS, Hypercompositional Structures in the Theory of the Languages and Automata, *An. stiintifice Univ. Al. I. Cuza, Iasi, Informatica, t. iii*,(1994), 65-73.

[11] G. G. MASSOUROS, A new approach to the theory of Languages and Automata, *Proceedings of the 26th Annual Iranian Mathematics Conference, Kerman, Iran,* **2**,(1995) 237-239.

[12] G. G. MASSOUROS, CH. G. MASSOUROS, I. D. MITTAS, Fortified Join Hyper-groups, *Ann. Matematiques Blaise Pascal,* **3**, No 2,(1996), 155-169.

[13] G. G. MASSOUROS, The subhypergroups of the Fortified Join Hypergroup, *Italian Journal of Pure and Applied Mathematics, no 2*,(1997) 51-63.

[14] G. G. MASSOUROS, F. A. ZAFIROPOULOS, CH. G. MASSOUROS, Transposition Polysymmetrical Hypergroups, *Proceedings of the 8th Internat. Cong. in Algebraic Hyperstructures and Applications*,(1990), 191-202, Samothraki, Greece. Spanidis Press.

[15] W. PRENOWITZ, J. JANTOSCIAK, Geometries and Join Spaces, *J. Reine Angew. Math.* **257**(1972), 100-128.

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