Advances in Abstract Algebra

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ B & \xrightarrow[\varphi_n]{} & O \end{array}$$

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ON HYPERGROUPS WITH OPERATORS AND HYPEROPERATORS

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ABSTRACT. The application of external operations and hyperoperations on hypergroups leads to structures that are called hypermodules, supermodules, hypermoduloids and supermoduloids. These structures give interesting applications in the theory of graphs and in geometries.

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1. INTRODUCTION

A non void set Y endowed with a composition "·" and a hypercomposition "+" is called a **hyperringoid** [7] if:

i) (Y, +) is a hypergroup
ii) (Y, ·) is a semigroup
iii) the composition is bilaterally distributive to the hypercomposition.

If (Y,+) is a transposition hypergroup [1], then the hyperringoid is called transposition hyperringoid, while if (Y,+) is a canonical hypergroup [8], then the structure (Y,+,·) was named hyperring by M. Krasner [2].

The notions of the set of operators and hyperoperators from a hyperringoid Y, over an arbitrary non void set M, were introduced (in [5]), in order to describe the action of the state transition function in the theory of Automata. Y is a set of **operators** over M, if there exists an external operation from M×Y to M, such that $(s\kappa)\lambda = s(\kappa\lambda)$, for all $s\in M$ and $\kappa,\lambda\in Y$ and moreover $s1=s$ for all $s\in M$, when Y is a unitary hyperringoig. If there exists an external hyperoperation from M×Y to P(M) which satisfies the above axiom with the variation that $s\in s1$, when Y is a unitary hyperringoid, then Y is a set of **hyperoperators** over M.

If M is a hypergroup and Y a hyperringoid of operators over M such that, for each $\kappa,\lambda\in Y$ and $s,t\in M$, the axioms:

$$\text{(i)}\ \ (s + t)\lambda = s\lambda + t\lambda, \quad \text{(ii)}\ \ s(\kappa + \lambda) \subseteq s\kappa + s\lambda$$

hold, then M is called **right hypermoduloid** over Y. If Y is a set of hyperoperators, then M is called **right supermoduloid**. If the second of the above axioms holds as an equality, then the hypermoduloid is called **strongly distributive**. There is a similar definition of the **left hypermoduloid** and the **left supermoduloid** over Y in which the elements of Y operate from the left side. When M is both right and left hypermoduloid (resp. supermoduloid) over Y it is called **Y-hypermoduloid** (resp. **Y-supermoduloid**) [6]. If M is a canonical hypergroup, the set of operators Y is a hyperring, and $s1=s$, $s0=0$ for all $s\in M$, then M is named **right hypermodule**, while it is named **right supermodule** if Y is a set of hyperoperators [4, 9].

2. HYPERMODULOIDS

The set of the operators over a non empty set M, can define in M a hypercomposition and when the set of the operators is a unitary hyperringoid, M enriched with this hypercomposition, becomes a hypergroup. Before proceeding to this construction, it is necessary to give the following definition:

Definition 2.1. An element s_2 of M is called **connected** with an element s_1 of M, if there exists an element λ of Y such that $s_2=s_1\lambda$, when Y is a set of operators over M, or $s_2\in s_1\lambda$, when Y is a set of hyperoperators over M.

It must be mentioned that s_2 being connected to s_1, does not necessarily imply that s_1 is connected to s_2.

With the use of the notion of the connected elements, a hypercomposition can be defined in M, as follows:

$$(2.1) \qquad s_1+s_2 = \begin{cases} \{s\in M \mid s=s_1\kappa \text{ and } s_2=s\lambda, \text{ with } \kappa,\lambda\in Y\}, & \text{if } s_2 \text{ is connected to } s_1 \\ \{s_1, s_2\}, & \text{if } s_2 \text{ is not connected to } s_1 \end{cases}$$

Proposition 2.1. *If the set of the operators Y over a non void set M is a unitary hyperringoid, then M endowed with the hypercomposition (2.1) becomes a hypergroup.*

P r o o f. Since Y is a unitary hyperringoid, the result of the hypercomposition (2.1) always contains the two participating elements, thus $s+M=M+s=M$ for all $s\in M$ and so the reproductive axiom is valid. Moreover, the associativity holds. Indeed, if s_1, s_2 and s_3 are not connected to each other, then

$$s_1+(s_2+s_3) = (s_1+s_2)+s_3 = \{s_1, s_2, s_3\}$$

Suppose next that s_2 and s_3 are connected to s_1. Also let s_3 be connected to s_2. Then:

$$\begin{aligned}(s_1+s_2)+s_3 &= \{t\in M \mid t=s_1\kappa \text{ and } s_2=(s_1\kappa)\lambda, \text{ with } \kappa,\lambda\in Y\} + s_3 = \\ &= \{s\in M \mid s=(s_1\kappa)\mu,\ s_2=(s_1\kappa)\lambda, \text{ and } s_3=(s_1\kappa\mu)\nu \text{ with } \kappa,\lambda,\mu,\nu\in Y\} = \\ &= s_1+s_3\end{aligned}$$

and

$$\begin{aligned}s_1+(s_2+s_3) &= s_1 + \{t\in M \mid t=s_2\kappa \text{ and } s_3=(s_2\kappa)\lambda, \text{ with } \kappa,\lambda\in Y\} = \\ &= \{s\in M \mid s=s_1\mu \text{ and } (s_1\mu)\lambda=s_2\kappa,\ (s_2\kappa)\nu=s_3 \text{ or } (s_1\mu)\rho=s_3, \text{ with } \kappa,\lambda,\mu,\nu\in Y\} =\end{aligned}$$

$= s_1+s_3$

Similar is the proof of all the other cases and so the proposition.

Corollary 2.1. *The set of vertices of a directed graph, is endowed with the structure of the hypergroup, if the result of the hypercomposition of two vertices v_i and v_j is the set of the vertices which appear in all the possible paths that connect v_i to v_j, or the biset $\{v_i, v_j\}$, if there do not exist any connecting paths from vertex v_i to vertex v_j.*

Proposition 2.2. *If M_1, M_2 are two right Y-hypermoduloids, then $M=M_1\times M_2$ becomes a right Y-hypermoduloid, if M is endowed with the hypercomposition:*

$$(s_1,t_1) + (s_2,t_2) = \{ (s,t) \mid s \in s_1+s_2,\ t \in t_1+t_2 \}$$

and the external operation from $M\times Y$ to M:

$$(s,t)\lambda = (s\lambda, t\lambda)$$

M is not strongly distributive, even when M_1 and M_2 are strongly distributive.

P r o o f. M is obviously a hypergroup and moreover M is a transposition hypergroup when M_1 and M_2 are transposition hypergroups. Next for the axioms of the external operation it holds:

$$[(s_1,t_1) + (s_2,t_2)]\lambda = [\bigcup_{\substack{s\in s_1+s_2\\ t\in t_1+t_2}} (s,t)]\lambda = \bigcup_{\substack{s\in s_1+s_2\\ t\in t_1+t_2}} (s,t)\lambda = \bigcup_{\substack{s\in s_1+s_2\\ t\in t_1+t_2}} (s\lambda,t\lambda) =$$

$$= (s_1\lambda, t_1\lambda) + (s_2\lambda, t_2\lambda) = (s_1, t_1)\lambda + (s_2, t_2)\lambda$$

and

$$(s,t)(\kappa+\lambda) = \bigcup_{\mu\in\kappa+\lambda} (s,t)\mu = \bigcup_{\mu\in\kappa+\lambda} (s\mu,t\mu) \subseteq \bigcup_{\mu,\nu\in\kappa+\lambda} (s\mu,t\nu) = (s\kappa,t\kappa) + (s\lambda,t\lambda) =$$

$$= (s,t)\kappa + (s,t)\lambda$$

Let H and H′ be two hypergroups and let $R \subseteq H \times H'$ be a binary relation from H to H′.

Definition 2.1. R is called **homomorphic relation**, if, for all (a_1,b_1), $(a_2,b_2) \in R$ it holds:

$(\forall x \in a_1+a_2)(\exists y \in b_1+b_2)\ [(x,y) \in R]$ and $(\forall y' \in b_1+b_2)(\exists x' \in a_1+a_2)\ [(x',y') \in R]$ (D_1)

or equivalently for all $x \in a_1+a_2$ and for all $y \in b_1+b_2$ it holds:

$[\{x\} \times (b_1+b_2)] \cap R \neq \varnothing$ and $[(a_1+a_2) \times \{y\}] \cap R \neq \varnothing$ (D_1')

Let Y and Y′ be two hyperringoids and let $R \subseteq Y \times Y'$ be a binary relation from Y to Y′.

Definition 2.2. R will be called **homomorphic relation**, if it satisfies the axioms of the Definition 2.1. and, moreover, if for every $(a_1,b_1) \in R$ and $(a_2,b_2) \in R$ it holds:

$$(a_1a_2,\ b_1b_2) \in R \qquad (D_2)$$

A homomorphic relation which is also an equivalence relation is named **congruence relation**.

Proposition 2.3. *If M is a strongly distributive hypermoduloid over a hyperringoid Y, then the relation*

$$T = \{ (k,k') \in Y\times Y \mid (\forall s \in M)\ sk = sk' \}$$

is a congruence relation.

P r o o f. It is obvious that T is reflexive and symmetric. Also if $sk_1=sk_2$ and $sk_2=sk_3$ then $sk_1=sk_3$, thus T is transitive as well. So T is an equivalence relation. Next suppose that $(k_1,k_2)\in T$ and $(k_3,k_4)\in T$. Then from $(k_1,k_2)\in T$ it derives that $sk_1=sk_2$ for each $s\in M$ and since $(k_3,k_4)\in T$ it holds that $(sk_1)k_3=(sk_2)k_4$ for each $s\in M$. Thus for each $s\in M$ the equality $s(k_1k_3)=s(k_2k_4)$ is valid and therefore $(k_1k_3, k_2k_4)\in T$. Moreover from the equalities $sk_1=sk_2$, for each $s\in M$ and $sk_3=sk_4$, for each $s\in M$, it derives that $sk_1+sk_3=sk_2+sk_4$, for each $s\in M$ or equivalently $s(k_1+k_3)=s(k_2+k_4)$, for each $s\in M$. Hence T is a homomorphic relation and therefore the Proposition.

It is easy to verify that if an equivalence relation R in a hyperringoid Y satisfies the property:

$$xRy \text{ and } w\in E \Rightarrow xwRyw \text{ and } wxRwy \qquad [D_2']$$

then it satisfies the axiom $[D_2]$ of the Definition 2.2. An equivalence relation whish satisfies $[D_2']$, is called **compatible** to the composition. It is possible though that an equivalence relation satisfies only one of the conditions of the second part of $[D_2']$. Such a relation is called **right** or, resp. **left compatible** to the composition.

Lemma 2.1. *Every congruence relation R in a hypergroup H is a normal equivalence relation and therefore the set H/R becomes a hypergroup under the hypercomposition*

$$C_x \dagger C_y = \{ C_z \mid z \in x+y \} \tag{2.2}$$

where C_x is the class of an arbitrary element $x\in H$.

P r o o f. Since R is a homomorphic relation, for each $x,y\in H$ it holds:

$$z'\in C_x + C_y \Rightarrow$$
$$\Rightarrow (\exists (x',y')\in C_x\times C_y)[z'\in x'+y'] \Rightarrow (\exists z\in x+y)[z'R z] \Rightarrow z'\in C_z \Rightarrow$$
$$\Rightarrow C_x + C_y \subseteq \bigcup_{z\in x+y} C_z$$

Conversely now:

$$z''\in \bigcup_{z\in x+y} C_z \Rightarrow$$
$$\Rightarrow (\exists z\in x+y)[z''R z] \Rightarrow$$
$$\Rightarrow (\exists (x'',y'')\in H^2)[x''Rx \wedge y''Ry \wedge z''\in x''+y''] \Rightarrow$$
$$\Rightarrow z''\in C_x + C_y \Rightarrow \bigcup_{z\in x+y} C_z \subseteq C_x + C_y$$

Thus $C_x+C_y = \bigcup_{z\in x+y} C_z$, and so the quotient set H/R, enriched with the hypercomposition $C_x \dagger C_y = \{ C_z \mid z\in x+y \}$ is a hypergroup.

Lemma 2.2. *If the hypergroup H is transposition, then H/R is also a transposition hypergroup.*

P r o o f. Suppose that for some elements C_x, C_y, C_z, C_w, of the quotient set H/R it holds: $C_y\backslash C_x \cap C_z/C_w \neq \varnothing$. Then there exists elements x', y', z', w' belonging to

C_x, C_y, C_z, C_w respectively, such that $y'\backslash x' \cap z'/w' \neq \varnothing$. Since the transposition axiom is valid in H, it derives that $x'+w' \cap y'+z' \neq \varnothing$. Therefore $C_x+C_w \cap C_y+C_z \neq \varnothing$ and so the Lemma.

Proposition 2.4. *Let R be a congruence relation in a hyperringoid Y right compatible to the multiplication. Then the quotient set Y/R becomes a right hypermoduloid over Y.*

Proof. In Y/R an external composition from Y/R×Y to Y/R is defined as follows:

$$[x]k = [xk], \text{ for each } [x] \in Y/R \text{ and } k \in Y$$

According to Lemma 2.1, R is a normal equivalence relation and therefore the quotient Y/R endowed with the hypercomposition (2.2) becomes a hypergroup. On the other hand it holds:

$$[x]k + [y]k = [xk] + [yk] = \bigcup_{t \in xk+yk} [t] = \bigcup_{t \in (x+y)k} [t] = \bigcup_{s \in x+y} [sk] = (\bigcup_{s \in x+y} [s])k =$$
$$= ([x] + [y])k$$

and

$$[x]k + [x]m = [xk] + [xm] = \bigcup_{t \in xk+xm} [t] = \bigcup_{t \in x(k+m)} [t] = \bigcup_{n \in k+m} [xn] = \bigcup_{n \in k+m} [x]n =$$
$$= [x](k + m)$$

Hence Y/R is a right hypermoduloid over Y. In an analogous way Y/R can become a left hypermoduloid over Y, or a bilateral one, when Y is commutative.

From Propositions 2.3.and 2.4 it derives the

Corollary 2.2. *If M is a finite strongly distributive hypermoduloid over a hyperringoid Y, then the hypermoduloid Y/T is also finite.*

3. HYPERMODULES

Suppose that M is a module over a unitary ring P and let G be a subgroup of the multiplicative semigroup $P^*=P\backslash\{0\}$ of P, which satisfies the condition $xG \cdot yG = xyG$, for each $x,y \in P$. The above equality is equivalent to the normality of G in P* only when P* is a group, that is, when P is a division ring [3]. G defines in P a partition, the equivalence classes of which are the cosets xG, $x \in P$. The quotient set of this partition is denoted by P/G and it becomes a hyperring if it is enriched with the following composition and hypercomposition:

$$xG \cdot yG = xyG$$
$$xG \dagger yG = \{(xp+yq)G \mid p,q \in G\}$$

for each xG, $yG \in P/G$. This hyperring, which was constructed by M. Krasner, was named quotient hyperring [2]. Furthermore, this construction is extended to the hypermodules through the introduction of a relation g in the module M, in the following way:

$$(x,y) \in g \Leftrightarrow x=qy, \ q \in G$$

It can easily be proved that g is an equivalence relation. Let x_g signifies the equivalence class of an arbitrary element x and let M_g be the set of the equivalence classes modulo g. M_g becomes a canonical hypergroup, if it is endowed with the hypercomposition:

$$x_g \underset{\bullet}{+} y_g = \{ z_g \in M_g \mid z_g \subseteq x_g + y_g \}$$

i.e. if $x_g \underset{\bullet}{+} y_g$ consists of all the classes $z_g \in M_g$ which are contained in the setwise sum of x_g, y_g. Now let P_G be the quotient hyperring of P by G. Then:

Proposition 3.1. *M_g becomes a strongly distributive hypermodule over P_G, if an external operation from $P_G \times M_g$ to M_g is defined as follows:*

$$k_G\, x_g = (kx)_g, \quad \textit{for each } k_G \in P_G,\ x_g \in M_g$$

It is worth mentioning that the elements of this hypermodule are selfopposite, i.e. $x_g \underset{\bullet}{+} x_g = \{0, x_g\}$, when $-1 \in G$.

In accordance to the above, suppose that V is a vector space over an ordered field F and suppose that F^+ is the positive cone of F. Since F^+ is a multiplicative subgroup of F*, there exists the quotient hyperfield $F/F^+=\{F^-,0,F^+\}$ of F by F^+. Next let $\overline{V}$ be the hypermodule (vector hyperspace) over F/F^+, which derives from V, using the above described construction. Then the set $\overline{V}$ is exactly what is called "ray join space" in [10] and [11]. Next, consider a hypersphere S of V centered at 0. The map $\overline{x} \to x$; of $\overline{V}$ onto $S \cup \{0\}$ is one to one and the elements of the hypersum $\overline{x} + \overline{y}$, $\overline{x} \neq \overline{y}$ are mapped to the points of the minor arc which has end points x, y and lies on the great circle of the hypersphere that passes through x, y. In this case, the two end points x and y do not belong to the minor arc xy, since $\overline{x}, \overline{y} \notin \overline{x} + \overline{y}$, while $\overline{x}+(-\overline{x})=\{-\overline{x},0,\overline{x}\}$.

Proposition 3.2. *Let M be a strongly distributive hypermodule over a division hyperring (D,+,·). A new commutative hypercomposition is introduced in M, which is defined as follows:*

$$x \dagger y = \begin{cases} x+y \cup \{x, y\}, & \textit{if } x, y \neq 0 \textit{ and } x \neq -y \\ M, & \textit{if } x = -y \\ x, & \textit{if } y=0 \end{cases}$$

and a similar one is introduced in D, that is:

$$m \dagger n = \begin{cases} m+n \cup \{m, n\}, & \textit{if } m, n \neq 0 \textit{ and } m \neq -n \\ D, & \textit{if } x = -y \\ m, & \textit{if } n=0 \end{cases}$$

Then (D,†,·) is a division hyperring and M endowed with the hypercomposition "†" becomes a hypermodule over (D,†,·), which is not strongly distributive.

P r o o f. The verification of the axioms is a rather extensive work and it will not be presented here. The only axiom which will be proved is the one which shows that the hypermodule is not strongly distributive. Indeed:
let $n \neq -m$ be two elements of the division hyperring and $x \in M$, then
$(m \dagger n)x = [(m+n) \cup \{m, n\}]x = (m+n)x \cup \{m, n\}x = (mx+nx) \cup \{mx, nx\} = mx \dagger nx$
But if $n=-m$, then $(m-m)x = Dx \subseteq M = mx-mx$.

Let M be a module over a non commutative field K and let the equivalence relation g be defined as follows:

$$(x,y) \in g \Leftrightarrow x=qy, \ q \in K^*$$

then, according to Proposition 3.1, M_g becomes a strongly distributive hypermodule over the quotient hyperfield $K/K^*=\{0,K^*\}$. If the construction which is presented in Proposition 3.2 is applied to this hypermodule and also, if the elements of $M_g \setminus \{0\}$ are defined as points and the result of the hypercomposition $x_g \dagger y_g = (x_g \dot{+} y_g) \cup \{x_g, y_g\}$ of any two points x_g, y_g with $x_g \neq y_g$, are defined as lines, then an analytic projective geometry is formed. Moreover all analytic projective geometries can derive using this method (see also [10]).

Furthermore, applying the construction of Proposition 3.2 in the vector hyperspace $\overline{V}$, the two participating elements $\overline{x}, \overline{y}$ belong to their hypersum $\overline{x} \dagger \overline{y}$, giving thus closed arcs on the hypersphere S of V. Also $\overline{x} \dagger (-\overline{x}) = \overline{V}$, i.e. any two opposite points generate the whole hypersphere (which derives as the result of their hypercomposition). This construction is very natural, since two opposite points define infinitly many great circles that contain all the points of the sphere. Thus every Euclidian spherical geometry can be described algebraically as a quotient hypermodule.

Proposition 3.3. *Let R be a hyperring, then R^n is a hypermodule over R which is not strongly distributive.*
P r o o f. Verifying all the axioms is not so tricky, but it takes long. Yet, the interesting part is the proof that R^n is not strongly distributive. Indeed, let $(m_1,\ldots,m_n)$ be an element of R^n and $a,b \in R$, then:

$$(a+b)(m_1,\ldots,m_n) = \cup_{c \in a+b} c(m_1,\ldots,m_n) = \cup_{c \in a+b}(cm_1,\ldots,cm_n)$$

On the other hand

$$a(m_1,\ldots,m_n) + b(m_1,\ldots,m_n) = (am_1,\ldots,am_n) + (bm_1,\ldots,bm_n) = \\ = \{(c_1m_1,\ldots, c_nm_n), \ c_1,\ldots,c_n \in a+b\}$$

Thus $(a+b)(m_1,\ldots,m_n) \subseteq a(m_1,\ldots,m_n) + b(m_1,\ldots,m_n)$.

BIBLIOGRAPHY

[1] **J. JANTOSCIAK**: *Transposition hypergroups, Noncommutative Join Spaces.* Journal of Algebra, 187, pp. 97-119, 1997

[2] **M. KRASNER**: *A class of hyperrings and hyperfields.* Internat. J. Math. and Math. Sci. 6, No 2, pp. 307-312, 1983.

[3] **C.G. MASSOUROS**: *On the theory of hyperrings and hyperfields.* Algebra i Logika, 24, No 6, pp. 728-742, 1985.

[4] **C.G. MASSOUROS**: *Free and cyclic hypermodules.* Annali Di Mathematica Pura ed Applicata, Vol. CL. pp. 153-166, 1988.

[5] **G.G. MASSOUROS**: *Automata-Languages and hypercompositional structures.* Doctoral Thesis, Depart. of Electrical Engineering and Computer Engineering of the National Technical University of Athens, 1993.

[6] **G.G. MASSOUROS**: *Automata and Hypermoduloids.* Proceedings of the 5th Internat. Cong. in Algebraic Hyperstructures and Applications. pp. 251-266, Iasi 1993. Hadronic Press 1994.

[7] **G.G. MASSOUROS**: *The Hyperringoid.* Multiple Valued Logic, 3, pp. 217-234, 1998.

[8] **J. MITTAS**: *Hypergroupes canoniques.* Mathematica Balkanica, 2, pp. 165-179, 1972

[9] **J. MITTAS**: *Sur certaines classes de structures hypercompositionnelles.* Proceedings of the Academy of Athens, 48, pp. 298 - 318, Athens 1973.

[10] **W. PRENOWITZ - J. JANTOSCIAK**: *Geometries and Join Spaces.* J. Reine Angew. Math. 257, pp. 100-128, 1972.

[11] **W. PRENOWITZ - J. JANTOSCIAK**: *Join Geometries. A Theory of convex Sets and Linear Geometry.* Springer - Verlag, 1979.