Journal of
Discrete Mathematical Sciences & Cryptography

VOLUME 6 NUMBERS 2-3 AUGUST-DECEMBER 2003 ISSN 0972-0529

Special Issue on

Algebraic Hyperstructures and Applications

Edited by

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Published by
TARU PUBLICATIONS, NEW DELHI

On the attached hypergroups of the order of an automation

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Abstract

The set of the states of an automation, enriched with properly defined hypercompositions gets the structure of a hypergroup. The hypergroups that derive in this way are called attached hypergroups to the automation. Here appears a study of the attached hypergroups of the order and through them the notion of the valuation is being introduced into the set of the states of the automation.

1. Introduction

In [1] it has been shown that the set of the states of an automation, equipped with different hypercompositions, can be endowed with the structure of the hypergroup. The hypergroups that have derived in this way, were named *attached hypergroups* to the automation. Up to this point we have introduced several kinds of attached hypergroups in order to describe the structure and the operation of the Automation with the use of tools from the Hypercompositional Algebra. Among them there are :

- the *attached hypergroups of the order* , and
- the *attached hypergroups of the grade* .

These two kinds of hypergroups have also been used for the minimization of the automata.

Moreover, in [2] another hypergroup, having derived through a different consideration of the hypercomposition, has been attached to the set of the states of an automation. Due to its definition, this hypergroup was named *attached hypergroup of the paths* and it has led to a new proof of Kleene's theorem.

Journal of Discrete Mathematical Sciences & Cryptography

Vol. 6 (2003), Nos. 2–3, pp. 207–215

Finally, in [3], one more hypergroup has been attached to the automation, the *attached hypergroup of the operation*. Apart from the other results, this hypergroup can indicated all the states in which an automation can be found after the t-clock pulse.

This paper deals with the attached hypergroups of the order of an automation. The fundamental notion for the definition of these hypergroups is the notion of the *order of a state* in an automation.

Let $\mathcal{A} = (A, S, s_0, \delta, F)$ be an automation, deterministic or not.

Definition 1.1. We call *order of a state* $s \in S$, denoted by $\operatorname{ord} s$, the natural number l, which is equal to the minimum of the lengths of all the words that lead from the start state s_0 to s.

Obviously $\operatorname{ord} s_0 = 0$. It must be mentioned though that in an automation it is possible to have states which can not be reached from the start state. Since such states have no influence to the operation of the automation, it can be assumed that the order is not defined for them. Therefore, only automata for which the order is defined for all of their states, are considered in the rest of this paper.

With the use of the notion of the order in the set S of the states of an automation, there can be defined the order equivalence, as follows [1] :

$$s_1 O s_2, \text{ if } \operatorname{ord} s_1 = \operatorname{ord} s_2, \text{ where } s_1, s_2 \in S.$$

This equivalence relation has the following properties

(i) The set of the classes is isomorphic to a subset of the set of the natural number N.

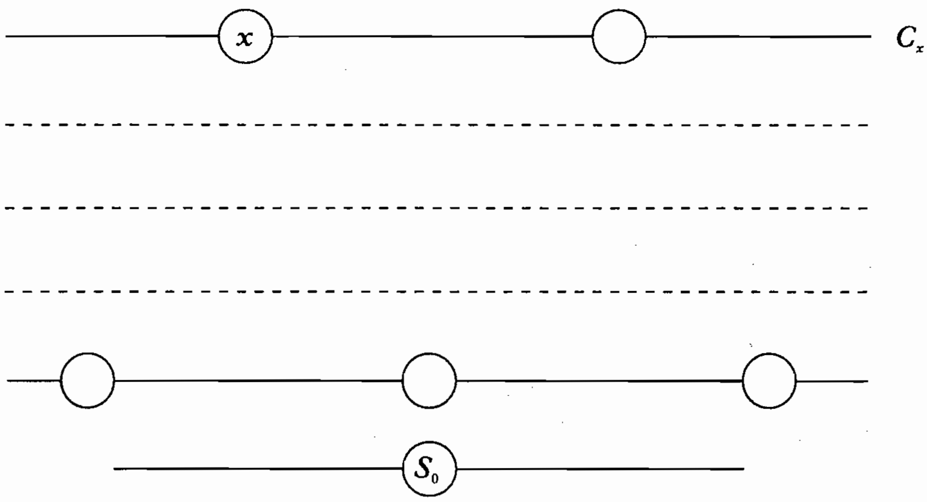
(ii) The set of the classes $\operatorname{mod} O$ is totally ordered.

(iii) There exists a minimal class.

(iv) Every class "covers" another, except the one of s_0 which does not cover other class.

So, if we consider a set S and an equivalence relation R in it, which satisfies the above properties, then, with the use of R and of the property (i), we can define the order of the elements $x \in S$, to be the order of the corresponding class of R, which is the corresponding natural number due to the considered isomorphism. Moreover, we assume that the minimal class of S, with regard to the equivalence relation, is a singleton.

Therefore, the set S can be represented by the following diagram where C_x denotes the class of the element x of S.



So, through this notion of the order and with the use of properly defined hyperoperations, it is possible to introduce hypercompositional structures into the set S, in different ways. In [4] there appear several hypergroups that can possibly be defined, through the proper hyperoperations into the set S. There follows an analysis of the attached canonical hypergroups [6] of the order of an automation :

2. The attached canonical hypergroups of the order

Let S be the set of the states of an automation. If for every $x, y \in S$ we define

1st.

$$x + y = \begin{cases} y & \text{if } \operatorname{ord} x < \operatorname{ord} y \\ \bigcup\limits_{\operatorname{ord} z \leq \operatorname{ord} x} C_z & \text{if } \operatorname{ord} x = \operatorname{ord} y \end{cases}$$

then the deriving structure $(S, +)$ is a canonical polysymmetrical hypergroup [9] with neutral element the minimal element of S, which is the start state of the automation. In this hypergroup, the symmetrical set $S(x)$ of every element $x \in S$, is the equivalence class C_x of x and so x is selfopposite. Next, if in the above definition the strict order " $<$ " is used instead of the " $\leq$ ", i.e., if

$$x + x' = \bigcup_{\text{ord}\, z < \text{ord}\, x} C_z \quad \text{for every } x' \in C_x$$

then the structure $(S, +)$ is a hypergroup only when S is totally ordered. Indeed, in order to have the validity of the associativity in the case :

$$(x + x) + x' = x + (x + x')$$

where $x' \in C_x$, the equality :

$$\bigcup_{\text{ord}\, w < \text{ord}\, x} C_w + x' = x + \bigcup_{\text{ord}\, z < \text{ord}\, x} C_z$$

must hold. Thus x must be equal to x' and so $C_x = \{x\}$. Therefore we have the superiorly canonical hypergroup [7], [8] with

$$x + y = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ [s_0, x[& \text{if } x = y\,. \end{cases}$$

This happens when the states of the automation are totally ordered.

We remind [7], [8] that a canonical hypergroup $(H, +)$ is called superiorly canonical if it is strongly canonical and it also satisfies the additional conditions :

S_1. For every $x, y, z, w \in H$ such that $0 \notin x + y$ and $z, w \in x + y$, it holds : $z - z = w - w$.

S_2. If $x \in z - z$ and $y \notin z - z$, then $x - x \subseteq y - y$.

And it is called strongly canonical when :

F_1. For every $x, y, z, w \in H$ such that $(x + y) \cap (z + w) \neq \emptyset$, it holds that either $x + y \subseteq z + w$ or $z + w \subseteq x + y$

F_2. If for $x, y \in H$ holds $x \in x + y$, then $x + y = x$

2^{nd}.

$$x + y = \begin{cases} y & \text{if ord}\, x < \text{ord}\, y \\ \bigcup_{s_0 \neq \text{ord}\, z < \text{ord}\, x} C_z & \text{if ord}\, x = \text{ord}\, y \text{ and } s_0 \neq x \neq y \neq s_0 \\ \bigcup_{\text{ord}\, z \leq \text{ord}\, x} C_z & \text{if } x = y \end{cases}$$

then we have a canonical hypergroup with selfopposite elements.

3^{rd}.

$$x+y=\begin{cases} C_y & \text{if } s_0 \neq \operatorname{ord} x < \operatorname{ord} y \\ \displaystyle\bigcup_{s_0 \neq \operatorname{ord} z \leq \operatorname{ord} x} C_z & \text{if } \operatorname{ord} x = \operatorname{ord} y \text{ and } s_0 \neq x \neq y \neq s_0 \\ \displaystyle\bigcup_{\operatorname{ord} z \leq \operatorname{ord} x} C_z & \text{if } s_0 \neq x = y \end{cases}$$

and $s_0 + x = x + s_0 = x$ for every $x \in S$, then the hypergroup is again a canonical one with selfopposite elements.

The hypergroup of the 1^{st} case has a hypervaluation, since it is superiorly canonical [5], [8]. The values of the hypervaluation are from S itself, or, because of the isomorphism with a subset of N, they are from R, and thus, this is a hypergroup with valuation. More precisely, if $\|x\|$ is the valuation of $x \in S$, then $\|x\| = \operatorname{ord} x$.

For the cases 2 and 3 (as well as for 1), the mapping

$$d : S \times S \to R^+$$

such that

$$d(x,y)=\begin{cases} \max\{\operatorname{ord} x, \operatorname{ord} y\} & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases}$$

is an ultrametric distance in S, that is for d and for every $x, y, z \in S$ the following are valid [8] :

$$\begin{aligned} d(x,y) &= 0 \iff x = y \\ d(x,y) &= d(y,x) \\ d(x,y) &\leq \max\{d(x,z), d(z,y)\}. \end{aligned}$$

Now, in order to see if these hypergroups have a strong valuation, the following conditions must be verified :

h_1. For every $x, y \in S$, the hypersum $x + y$ is a circle of the space (S, d) with radius analogous to the $\max\{d(0,x), d(0,y)\}$.

h_2. For every $x, y, a \in S$ such that $(x+a) \cap (y+a) = \emptyset$ it holds :

$$d(x+a, y+a) = d(x,y)$$

where, if A and B are two subsets of S, then

$$d(A,B) = \{d(a,b) \mid (a,b) \in A \times B\}.$$

Also, in order to examine whether these hypergroups have a simple valuation or a hypervaluation, the following conditions must be verified :

h'_1: For every $x, y \in H$ the sum $x+y$ is a circle (H,d).
$h'_2 \equiv h_2$
$h'_3 \equiv F_2$

Therefore :

I. *For the* 2^{nd} *case :*

(a) For h_1 or h'_1:
In the beginning we are checking whether, for every $x, y \in S$ the sum $x+y$ is a circle of the ultrametric space (S,d).

Let $\operatorname{ord} x < \operatorname{ord} y$. Then it is $x+y = y = C(y,0)$, i.e. a circle with center y and with proper radius 0. Since $0 \notin x+y$ for every semireal radius r of $C(y,0)$ it will be :

$$r \leq d(0,y) = \operatorname{ord} y = \rho \max\{d(0,x), d(0,y)\}$$

and it will have its maximum value either for $\rho = 1$ or for $\rho = 1^-$. In the case that $\rho = 1$, the elements $z \in S$, with $d(y,z) = \operatorname{ord} y$, i.e. the elements for which $\operatorname{ord} z \leq \operatorname{ord} y$ (and so the 0 as well), belong to the circle $C(y, \operatorname{ord} y)$ and therefore $x+y$ is not this circle. In the case that $\rho = 1^-$, the elements $z \neq y$ with $\operatorname{ord} z \geq \operatorname{ord} y$, and so $d(y,z) = \operatorname{ord} z$, (if there exists any), do not belong to this circle. Neither those elements, for which $\operatorname{ord} z < \operatorname{ord} y$ belong to the above circle, since $d(y,z) = \operatorname{ord} y > (\operatorname{ord} y)^- = r$. Thus, for the case under consideration, if holds :

$$x+y = C(y, \rho \max\{d(0,x), d(0,y)\})$$

with $\rho = 1^-$.

Next, if $\operatorname{ord} x = \operatorname{ord} y$, then, since $x \in x+y$, for every $z \in C(x, \operatorname{ord} x)$, it is $d(x,z) \leq \operatorname{ord} x$, so $\operatorname{ord} z \leq \operatorname{ord} x$ and consequently

$$C(x, \operatorname{ord} x) = \{z \in S : \operatorname{ord} z \leq \operatorname{ord} x\} = x+y.$$

Therefore, in this case

$$x+y = C(z, \rho \max\{d(0,x), d(0,y)\})$$

with $\rho = 1$ and any $z \in x+y$. Therefore the condition h_1' is satisfied, i.e., $x+y$ is a circle of (S,d). But the condition h_1 is not satisfied, because, even through the hypersum $x+y$ is a circle $C(z, \rho \max\{d(0,x), d(0,y)\})$ with z arbitrary from $x+y$, it does not have one and the same coefficient ρ. Here ρ is either 1 or 1^-, depending on the considered case.

(b) For h_2:

This condition is also satisfied, since from the relation $(x+a) \cap (y+a) = \emptyset$ it derives that it can neither be $x = y$ nor $\max\{\operatorname{ord} x, \operatorname{ord} y\} \leq \operatorname{ord} a$. Thus, the only possible cases are the apparent ones :

$$\operatorname{ord} a < \operatorname{ord} x = \operatorname{ord} y (x \neq y) \quad \text{and} \quad \operatorname{ord} a < \operatorname{ord} x < \operatorname{ord} y,$$

(which derive from the definition of the hypercomposition), as well as

$$\operatorname{ord} x < \operatorname{ord} a < \operatorname{ord} y$$

and the deriving ones with alteration of x and y, for all of which the condition h_2 is valid (because of the definition of the hyperdistance and the properties of the triangles of the ultermetric space).

So it derives that the canonical hypergroup $(S,+)$ has a valuation. The correlated hypervaluation to the ultrametric distance d is :

$$|x| = d(0,x) = \operatorname{ord} x, \quad \text{for every } x \in S$$

and it holds that

$$0 \notin x+y \quad \Longrightarrow \quad d(x,y) = |x+y|$$

(where, obviously, if $A \subseteq S$, $|A| = \{|z| \in R : z \in A\}$).

II. *For the* 3^{rd} *case :*

We firstly remark that when $\operatorname{ord} x < \operatorname{ord} y$, the sum $x+y = C_y$ is not a circle except for the case that $C_y = \{y\}$. Since if it were a circle, its semireal radius would either be $\operatorname{ord} y$, which is impossible for the same reason as above, or $(\operatorname{ord} y)^-$, which implies that there does not exist any $z \neq y$ in the circle $C(y, (\operatorname{ord} y)^-)$. Therefore it is $C(y, (\operatorname{ord} y)^-) = \{y\}$.

It derives thus that from the three cases in which the structure $(S, +)$ is a canonical hypergroup, in the 1^{st} and in the 2^{nd} we can define a valuation and therefore we are led to the following propositions for the theory of the automata :

Proposition 2.1. *If the set S of the states of an automation is totally ordered, then, this set, equipped with the hypercomposition :*

$$s_1 + s_2 = \begin{cases} \max\{s_1, s_2\} & \text{if } s_1 \neq s_2 \\ [0, s[& \text{if } s_1 = s_2 = s \end{cases}$$

is a canonical hypergroup with strong valuation.

Proposition 2.2. *The set S of the states of every automation becomes a canonical hypergroup with valuation if it is equipped with the hypercomposition :*

$$s_1 + s_2 = \begin{cases} s_2 & \text{if } \operatorname{ord} s_1 < \operatorname{ord} s_2 \\ \bigcup\limits_{s_0 \neq \operatorname{ord} s < \operatorname{ord} s_1} C_s & \text{if } \operatorname{ord} s_1 = \operatorname{ord} s_2 \text{ and } s_0 \neq s_1 \neq s_2 \neq s_0 \\ \bigcup\limits_{\operatorname{ord} s \leq \operatorname{ord} s_1} C_s & \text{if } s_1 = s_2 . \end{cases}$$

Proposition 2.3. *In both of the above cases the ultrametric distance in S is the mapping*

$$d : S \times S \to R^+$$

with

$$d(s_1, s_2) = \begin{cases} \max\{\operatorname{ord} s_1, \operatorname{ord} s_2\} & \text{if } s_1 \neq s_2 \\ 0 & \text{if } s_1 = s_2 \end{cases}$$

and the correlated to it hypervaluation

$$| \quad | : S \to R^+$$

with $|s| = d(0, s) = \operatorname{ord} s$, *for every* $s \in S$.

Also it holds that

$$d(s_1, s_2) = |s_1 - s_2| = |s_1 + s_2|.$$

In both cases of the above canonical hypergroups, the neutral element is the start state (probably the conventional one) and every state is selfopposite.

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