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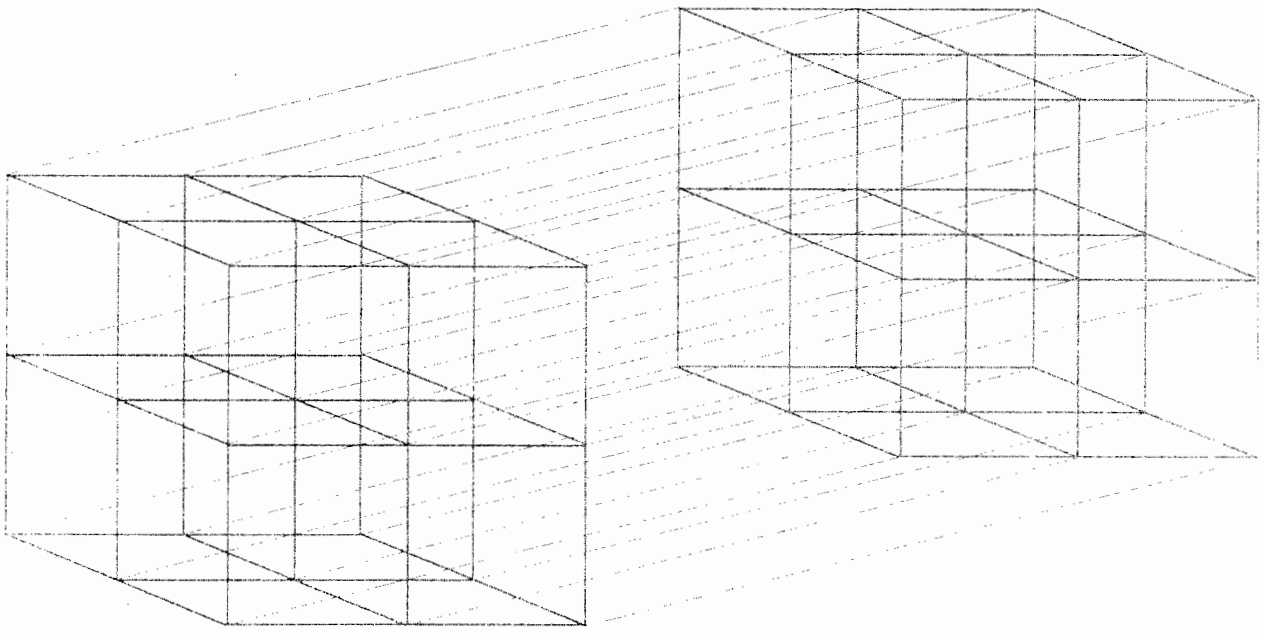

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The Hyperringoid

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This is an introductory paper for the hyperringoid, a new hypercompositional structure, derived from the theory of languages which led to the general consideration of such a structure, and was a generative source of certain types of hyperringoids. We study its fundamental properties, analyze certain subhyperringoids and give some of their properties.

Keywords: Theory of languages; hypergroup; composition; hypercomposition

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1. INTRODUCTION

The theory of languages, viewed from the standpoint of hypercompositional algebra, led to new hypercompositional structures. The hypercompositional algebra was founded in 1934 by F. Marty [5] with the introduction of the hypercomposition and the hypergroup. Let us recall some notions and definitions [1, 11]. First of all, a *hypercomposition* "$\cdot$" in a non void set H is a mapping with domain $H \times H$ whose range is the power set of H (*i.e.*, $x \cdot y \subseteq H$, for all $x, y \in H$). Next a *hypergroup* $(H, \cdot)$ is a non void set H equipped with a hypercomposition "$\cdot$", which satisfies the following axioms:

i. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for every $x, y, z \in H$ (the associative axiom)
ii. $x \cdot H = H \cdot x = H$ for every $x \in H$ (the reproductive axiom)

We should state already here that if "$\cdot$" is an internal composition in a set H and X, Y are subsets of H, then $X \cdot Y$ is the set of all $x \cdot y$ such that $(x, y) \in X \times Y$. Moreover, if "$\cdot$" is a hypercomposition in H, then $X \cdot Y$ signifies the union $\cup_{(x,y) \in X \times Y} x \cdot y$. In both cases $X \cdot y$ and $x \cdot Y$ will have the same meaning as $X \cdot \{y\}$ and $\{x\} \cdot Y$ respectively. Generally, when there is no danger of confusion, we make no distinction between an element x and the corresponding singleton $\{x\}$.

F. Marty [5] also introduced the notion of the *induced hypercompositions*. So, when the hypercomposition is denoted multiplicatively, the induced hypercompositions are the *right division* and the *left division* of two elements. Thus:

$$\frac{x}{y|} = \{z \in H \mid x \in y \cdot z\} \text{ and } \frac{x}{|y} = \{z \in H \mid x \in z \cdot y\}$$

It can be proved that in a hypergroup the result of the hypercomposition as well as the results of the induced hypercompositions are non void [3, 6]. If the hypergroup is commutative, then the above two induced hypercompositions coincide. In this case the induced hypercomposition is denoted by "/" or by ":" (division) in the multiplicative case and by "$|$" in the additive. For his study of geometry through a hypercompositional algebra, *W.* Prenowitz introduced a hypercompositional structure, which he named *join space* [17]. The *join space*, also *join hypergroup* [6], is a commutative hypergroup H which, for all $a,b,c,d \in H$ satisfies:

$$a/b \cap c/d \neq \emptyset \Rightarrow a \cdot d \cap b \cdot c \neq \emptyset$$

Join hypergroups also appear in the theory of languages and automata [7]. Indeed, the definition of the regular expressions over an alphabet A requires the consideration of subsets $\{x, y\}$ of the free monoid A^* generated by A. This leads to the definition of the hypercomposition $x+y = \{x, y\}$ in A^* that endows A^* with a join hypergroup structure, which we named ***B-hypergroup***. Moreover, the empty set of words and its properties in the theory of the regular expressions lead to the following extension. Let $0 \notin A^*$. On the set $\underline{A}^* = A^* \cup \{0\}$ define a hypercomposition as follows:

$$\begin{aligned} x + y &= \{x, y\} \quad \text{if } x, y \in \underline{A}^* \text{ and } x \neq y \\ x + x &= \{x, 0\} \quad \text{for all } x \in \underline{A}^* \end{aligned}$$

This structure is called ***dilated B-hypergroup*** and it has led to the definition of a new class of hypergroups, the ***fortified join hypergroups***.

DEFINITION 1.1 A **fortified join hypergroup** (FJH) is a join hypergroup $(H,+)$ with a unique element 0, called the zero element of H, such that $0+0=0$, $x\in x+0$ for every $x\in H$ and for every $x\in H\backslash\{0\}$ there exists one and only one element $-x\in H\backslash\{0\}$, called the opposite of x, such that $0\in x+(-x)$. We denote $x+(-y)$ by $x-y$.

Already, it has been proved that every FJH consists of two types of elements, the ***canonical*** (***c-elements***) and the ***attractive*** (***a-elements***) [8, 12]. An element x is called a c-element if $0+x$ is the singleton $\{x\}$, while it is called an a-element if $0+x$ is the set $\{0,x\}$. We denote with A the set of the a-elements and with C the set of the c-elements. For the hypercomposition between these elements, hold certain significant properties which are necessary for the following:

(i) *the sum of two a-elements is a subset of* $A\cup\{0\}$ *and it always contains the two addends.*
(ii) *the sum of two non opposite c-elements consists of c-elements, while the sum of two opposite c-elements contains all the a-elements.*
(iii) *the sum of an a-element with a non zero c-element is the c-element.*

The proofs of these properties as well as other elements of the theory of the FJHs can be found in [12].

Moreover, another distinction between the elements of the FJH stems from the fact that the equality $-(x-x)=x-x$ is not always valid. The elements that satisfy the above equality are called ***normal***, while the others are called ***abnormal*** [8, 12].

A subset h of a hypergroup H is called *semisubhypergroup* if $x+y\subseteq h$, for every x, y in h while it is called *subhypergroup* of H if for every element x of h holds $x+h=h+x=h$. Moreover a subhypergroup h of H is called *closed from the right* (in H), (resp. *from the left*) if for every $x\in H\backslash h$ holds $(x+h)\cap h=\emptyset$ (resp. $(h+x)\cap h=\emptyset$). h is called *closed* if it is closed from the right and from the left [1, 11]. It has been proved [6] that a subhypergroup is closed if and only if it is stable under the induced hypercomposition. Regarding the subhypergroups of the FJH [14], we remark that since a FJH is a join hypergroup, it has subhypergroups that are ***join*** *i.e.*, subhypergroups

that are join hypergroups themselves. It has been proved that the join subhypergroups of the FJH are the closed ones [14]. The join subhypergroups belong to the class of the symmetrical subhypergroups. A subhypergroup h of a FJH is ***symmetrical***, if $-x \in h$ for every $x \in h$. Of course, in a FJH there also exist non symmetrical subhypergroups.

Furthermore the binary operation of the word concatenation in the free monoid A^* is bilaterally distributive over the hyperoperation of the B-hypergroup and so, generally:

DEFINITION 1.2 A **hyperringoid** is a non empty set Y equipped with an operation "$\cdot$" and a hyperoperation "$+$" such that:

(i) $(Y, +)$ is a hypergroup

(ii) $(Y, \cdot)$ is a semigroup

(iii) the operation "$\cdot$" distributes on both sides over the hyperoperation "$+$".

M. Krasner was the first one who introduced and studied hypercompositional structures with an operation and a hyperoperation. So, among other structures, he defined the *hyperrings* [2, 4]. I. Mittas, later, introduced the *superrings*, in which both, the addition and the multiplication are hypercompositions [16]. The new hypercompositional structures arising from the theory of languages and automata have been named according to the terminology by Krasner and Mittas. Thus, provided that $(Y, +)$ is a join hypergroup, $(Y, +, \cdot)$ is called a ***join hyperringoid***. The join hyperringoid that comes from a B-hypergroup is called ***B-hyperringoid*** and the special B-hyperringoid that appears in the theory of languages is the ***linguistic hyperringoid***. It must be pointed out that not every B-hyperringoid is isomorphic to a linguistic one. Indeed, every element (word) of the linguistic hyperringoid has a unique factorization into irreducible elements which are the elements of the alphabet (letters). So the linguistic hyperringoid contains a finite set of prime (initial and irreducible) elements, such that each of its elements has a unique factorization with factors from its prime subset. In this sense it has a property similar to the Gaussian rings.

DEFINITION 1.3 A **fortified join hyperringoid** or **join hyperring** is a hyperringoid whose additive part is a fortified join hypergroup and

whose zero element is bilaterally absorbing with respect to the multiplication.

2. GENERAL PROPERTIES OF THE HYPERRINGOIDS

The class of the hyperringoids is very extensive and so, apart from the B-hyperringoids mentioned above, it also contains many other hyperringoids. Example 2.1. gives a hyperringoid in which (contrarily to the B-hyperringoid) the two participating elements are not contained into the result of their hyperaddition:

Example 2.1 Let $\leq$ be a linear order (also called a total order or chain) on Y *i.e.*, a binary reflexive and transitive relation such that for all $y, y' \in Y$, $y \neq y'$ exactly one of $y \leq y'$ and $y' \leq y$ holds. For $y, y' \in Y$, $y < y'$ set $[y, y'] := \{z \in Y : y \leq z \leq y'\}$ and $]y, y'[:= \{z \in Y : y < z < y'\}$. The order is dense if no $]y, y'[$ is void. Suppose that $(Y, \cdot, \leq)$ is a totally ordered group, *i.e.*, $(Y, \cdot)$ is a group such that for all $y \leq y'$ and $x \in Y$, $x \cdot y \leq x \cdot y'$, $y \cdot x \leq y' \cdot x$. If the order is dense then the set Y can be equipped with the hypercomposition:

$$x + y = \begin{cases} x & \text{if } x = y \\ ]\min\{x, y\}, \max\{x, y\}[& \text{if } x \neq y \end{cases}$$

and the triplet $(Y, +, \cdot)$ becomes a join hyperringoid. Indeed, since the equalities $x + y =]\min\{x, y\}, \max\{x, y\}[= y + x$ and $(x + y) + z =]\min\{x, y, z\}, \max\{x, y, z\}[= x + (y + z)$ are valid for every $x, y, z \in Y$, the commutativity and the associativity hold in Y. Also

$$x \mid y = \begin{cases} x & \text{if } x = y \\ \{t \in Y : x < t\} & \text{if } y < x \\ \{t \in Y : t < x\} & \text{if } x < y \end{cases}$$

Thus in any case that the intersection $(x \mid y) \cap (z \mid w)$ is non void, the intersection $(x + w) \cap (z + y)$ is also non void. Moreover

$$x\cdot(y+z)=$$

$$=\begin{cases} x\cdot y=x\cdot y+x\cdot z \text{ if } y=z \\ x\cdot]y,z[=x\cdot\bigcup_{y\langle t\langle z}\{t\}=\bigcup_{y\langle t\langle z}\{x\cdot t\}=\{xt: y<t<z\}=xy+xz \text{ if } y\neq z \end{cases}$$

It is worth mentioning that the hypercomposition $x + y = [\min\{x,y\}, \max\{x,y\}]$, for every $x, y \in Y$, endows $(Y, \cdot)$ with a join hyperringoid structure as well.

The class of the join hyperrings is also very extensive. The following proposition indicates how a join hyperring can be derived from Krasner's hypering [4], *i.e.*, a hyperringoid $(Y, +, \cdot)$ such that $(Y, +)$ is a canonical hypergroup [15] whose zero element is multiplicatively bilaterally absorbing.

PROPOSITION 2.1 *Let* $(P, +, \cdot)$ *be a hyperring. In* P *define a hypercomposition* "$\boldsymbol{+}$" *by setting for all* $a, b \in P$

$$a\boldsymbol{+}b = (a + b) \cup \{a\} \cup \{b\},$$

Then $(P, \boldsymbol{+}, \cdot)$ *is a join hyperring whose zero is the zero of* $(P, +, \cdot)$.

Proof Obviously $a + b = b + a$. Also the associativity holds. Indeed

$$\begin{aligned}(a\boldsymbol{+}b)\boldsymbol{+}c&=[(a+b)\cup\{a,b\}]\boldsymbol{+}c\\ &=\bigcup_{x\in a+b}(x+c)\cup(a+b)\cup\{c\}\cup(a+c)\cup\{a,c\}\cup(b+c)\cup\{b,c\}=\\ &=[a+(b+c)]\cup(a+b)\cup(a+c)\cup(b+c)\cup\{a,b,c\}=\\ &=a\boldsymbol{+}(b\boldsymbol{+}c)\end{aligned}$$

Next suppose that $a|b\cap c|d\neq\emptyset$. Then $[(a-b)\cup\{a\}]\cap[(c-d)\cup\{c\}]\neq\emptyset$.

This implies that one of the following might happen:

$(a-b)\cap(c-d)\neq\emptyset$, $(a-b)\cap\{c\}\neq\emptyset$, $\{a\}\cap(c-d)\neq\emptyset$ or a = c, but all of them lead to the relation: $(a + d)\cap(c + b)\neq\emptyset$.

Finally $a\boldsymbol{+}0 = (a + 0)\cup\{a, 0\} = \{a, 0\}$ and $a\boldsymbol{+}(-a) = (a-a)\cup\{a\}\cup\{-a\}\ni 0$, for all $a\in P$. So $(P, \boldsymbol{+})$ is a FJH. Also the distributivity holds, for example:

$r(a\boldsymbol{+}b) = r(a + b)\cup r\{a\}\cup r\{b\} = (ra + rb)\cup\{ra\}\cup\{rb\} = ra\boldsymbol{+}rb$ and so the Proposition. ■

COROLLARY 2.1 *Let* $(R, +, \cdot)$ *be a ring. If in* R *we define the hypercomposition*:

$$a+b = \{a, b, a+b\}, \quad \text{for all } a, b \in R$$

then $(R, +, \cdot)$ *is a join hyperring.*

According to the definition, the multiplication in a hyperringoid distributes over hyperaddition. This is not the same though with the induced hypercomposition, which satisfies a weak form of distributivity. In the following $xa|xb$ stands for $(xa)|(xb)$.

PROPOSITION 2.2 *If* x, a, b *are elements of a hyperringoid* Y, *then*:

(i) $x(a|b) \subseteq xa|xb$,
(ii) $(a|b)x \subseteq ax|bx$.

Proof Let $w \in x(a|b)$. Then there exists $y \in a|b$ such that $w = xy$. Also, since $y \in a|b$, the element belongs to $y + b$, thus $xa \in xy + xb$ from which $xy \in xa|xb$, so $w \in xa|xb$ and therefore (i). (ii) can be proved in a similar way. ■

COROLLARY 2.2 *If* A, B *are subsets of* Y *and* $x \in Y$, *then*:

(i) $x(A|B) \subseteq xA|xB$
(ii) $(A|B)x \subseteq Ax|Bx$

In a hyperringoid the multiplication has the following properties with regard to the hypercompositions:

PROPERTIES 2.1 *Let* $(Y, +, \cdot)$ *be a hyperringoid. Then for all* $a, b, c, d, x \in Y$;

- $(a + b)(c + d) \subseteq ac + bc + ad + bd$,
- $x[(a + b)|(c + d)] \subseteq (xa + xb)|(xc + xd)$,
- $[(a + b)|(c + d)]x \subseteq (ax + bx)|(cx + dx)$,
- $(a|b)(c|d) \subseteq (ac|bc)|(ad|bd)$,
- $(a + b)(c|d) \subseteq (ac + bc)|(ad + bd)$,
- $(a|b)(c + d) \subseteq (ac|bc) + (ad|bd)$,
- $x[(a|b)|c] \subseteq xa|(xb + xc)$,
- $x[(a + b)|c] \subseteq (xa + xb)|xc$,
- $x[a + (b|c)] \subseteq (xa + xb)|xc$.

The proof of these properties is reached with the use of the above Propositions and the relations between the hyperoperation and the induced hyperoperation. For instance we give the proof of the fourth and seventh one. Indeed let's consider an element t which belongs to $(a/b)(c/d)$. Then there exists $s \in a/b$ such that $t \in s(c/d)$. But $s(c/d) \subseteq sc/sd$ (Prop. 2.2.i.). Thus t belongs to $[(a/b)c]/[(a/b)d]$, which is a subset of $(ac/bc)/(ad/bd)$ [Prop. 2.2.ii]. Next we shall prove the seventh of the above properties. In the hypergroups holds the mixed associativity, *i.e.*, the equality $(a/b)/c = a/(c + b)$ [6]. Thus $x[(a/b)/c] = x[a/(b + c)] \subseteq xa/x(b + c) = xa/(xb + xc)$.

It is worth mentioning that all the above properties hold in the case of the B-hyperringoids with equality instead of inclusion.

3. JOIN HYPERRINGS

The join hyperring has properties that do not appear in other relevant structures. Indeed, the distinction of the elements of its additive hypergroup into c and a-elements gives certain special properties to the multiplication. We remark that a join hyperring having only c-elements is a Krasner's hyperring. A *proper join hyperring*, is a join hyperring which is not Krasner's hyperring.

PROPOSITION 3.1 *Let $(Y, +, \cdot)$ be a join hyperring. Then $C^2 \subseteq C$ and $CA = AC = \{0\}$.*

Proof Let x, y be two c-elements. Then, $xy + 0 = xy + 0y = (x + 0)y = \{xy\}$, so xy is a c-element. Next let x be a c-element and w an a-element. Then, $xw + 0 = xw + 0w = (x + 0)$. Moreover, $xw + 0 = = xw + x0 = x(w + 0) = x\{w, 0\} = \{xw, x0\} = \{xw, 0\}$. Thus $\{xw\}$ must be equal to $\{xw, 0\}$, and so $xw = 0$. ■

As in the case of the commutative rings and hyperrings, two elements x, y of a commutative join hyperring Y are zero divisors in Y if $x \neq 0$, $y \neq 0$ and $xy = 0$. An integral join hyperring is a non-trivial commutative join hyperring with no zero divisors.

COROLLARY 3.1 *If a proper commutative join hyperring has no divisors of zero, then 0 is the single c-element.*

PROPOSITION 3.2 *In a join hyperring which contains a non zero c-element, the product of two a-elements equals to zero.*

Proof Let x be a c-element and z, w be a-elements. Then (Prop. 3.1.) $xz = 0$. Thus, $wz \in 0 + wz = xz + wz = (x + w)z$. But, the sum of a canonical with an attractive element gives as a result the canonical element [12]. So $x + w = x$ and therefore $(x + w)z = xz = 0$, (Prop. 3.1.) which implies that $wz = 0$. ■

It is known [4] that in the hyperrings, as in the rings, holds:

$$\begin{aligned} &\text{i) } x(-y) = (-x)y = -xy \\ &\text{ii) } (-x)(-y) = xy \\ &\text{iii) } w(x-y) = wx - wy, (x-y)w = xw - yw \end{aligned} \tag{1}$$

These properties can be proved with the help of the addition. In the hyperringoids though these properties are not generally valid, as it can be seen in the following example:

Example 3.1 Let S be a multiplicative semigroup having a bilaterally absorbing element 0. Consider the set:

$$P = (\{0\} \times S) \cup (S \times \{0\})$$

With the use of the hypercomposition "+":

$$\begin{aligned} &(x, 0) + (y, 0) = \{(x, 0), (y, 0)\} \\ &(0, x) + (0, y) = \{(0, x), (0, y)\} \\ &(x, 0) + (0, y) = (0, y) + (x, 0) = \{(x, 0), (0, y)\} \quad \text{for } x \neq y \\ &(x, 0) + (0, x) = (0, x) + (x, 0) = \{(x, 0), (0, x), (0, 0)\} \end{aligned}$$

P becomes a fortified join hypergroup with neutral element (0, 0). Moreover, the opposite of every element $(x, 0)$, is the element $(0, x)$. Obviously this hypergroup has no c-elements. Now let's introduce in P a multiplication defined as follows:

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1 y_2)$$

This multiplication makes $(P, +, \cdot)$ a join hyperring, in which $(x, 0)(y, 0) = (xy, 0)$, $(0, x)(0, y) = (0, xy)$ while $(x, 0)(0, y) = (0, 0)$

and $(0, x)(y, 0) = (0, 0)$. That is, if $x' = (x, 0)$, $y' = (y, 0)$, and $0' = (0, 0)$, it is $x'y' \neq 0'$, and so $-(x'y') \neq 0'$ while $x'(-y') = (-x')y' = 0'$. Furthermore $(0, x)(0, y) = (0, xy)$, which is the opposite of $(xy, 0)$, that is $(-x')(-y') = -(x'y')$.

Thus, this example is a case where (i) and (ii) [and consequently (iii)] of the equalities (1) are not valid. But those relations hold in the join hyperrings under certain conditions:

PROPOSITION 3.3 *In a join hyperring the equalities* (1) *hold if* $-x$, $-y$, x, y, w *are not divisors of zero.*

Proof

(i) $0 \in x(y-y) = xy + x(-y)$ and since xy and $x(-y)$ are different than 0, it derives that the one is the opposite of the other. Similarly $(-x)y = -xy$.

(ii) It is $0 \in x(y-y) = xy + x(-y)$ and $0 \in (x-x)(-y) = x(-y) + (-x)(-y)$ and since xy, $x(-y)$, $(-x)(-y)$ are different than 0, from the uniqueness of the opposite it derives that $xy = (-x)(-y)$.

(iii) It is $w(x-y) = wx + w(-y)$ and since (because of *i*) $w(-y) = -wy$ we have $w(x-y) = wx - wy$. ■

COROLLARY 3.2 *The equalities* (1) *hold in every join hyperring with no divisors of zero.*

Before proceeding with the next Proposition we remark that in a join hyperring, if x is an a-element, then $-x$ is also an a-element and if y is a c-element, then $-y$ is also a c-element. Indeed let x be an a-element, then $0 \in 0 + x$, whence $x \in 0 \mid 0$. But x belongs also to $0 \mid (-x)$, thus $(0 \mid 0) \cap [0 \mid (-x)] \neq \emptyset$, from where $(0+0) \cap (-x+0) \neq \emptyset$, thus $0 \in 0 + (-x)$, and so $-x \in A$. And since the opposite of every a-element is an a-element itself, if y is a c-element, then $-y$ belongs to C as well.

PROPOSITION 3.4 *In an arbitrary join hyperring the equalities* (1) *hold, provided that at least one of the participating elements is a c-element.*

Proof If $x \in C$, $y \in A$, then, $-x \in C$, $-y \in A$ and because of Proposition 3.1 the equalities (i) and (ii) are valid, since all the products are equal to 0. Also, if both x and y are c-elements then, $-x$, $-y$ are c-elements as well and so, with the use of Proposition 3.1, the

equalities (i) and (ii) can be proved in an analogous way to Proposition 3.3. For (iii) now we have:

a) if $w \in C$ and $x, y \in A$, then $x-y \subseteq AU\{0\}$ and so (Prop. 3.1) $w(x-y)=0=wx-wy$.
b) if $w \in C$ and one of the x, y is also a c-element while the other is an a-element (*e.g.*, $x \in C$, $y \in A$), then, since the sum of an a-element with a non zero c-element is the c-element [12], we have that $x-y=x$. Also since the product of a c-element with an a-element is 0 (Prop. 3.1) we have: $w(-y) = -wy = 0$. So, $w(x-y) = wx = = wx + 0 = wx-wy$.
c) if $w \in A$ and $x, y \in C$, then $wx=wy=0$. Moreover, if $x \neq y$, then $x-y \subseteq C$ and so $w(x-y) \subseteq wC=0$, while if $x=y$, then the difference $x-x$, besides any c-elements it contains, it also contains the whole set of the attractive elements. Now, since there exists even one c-element, then $wA=0$ (Prop. 3.2) and so $w(x-x) = 0$. Thus (iii) holds again.
d) The rest cases ($x, y, w \in C$ and $x \in C, y, w \in A$) can be proved analogously and thus (iii) has been proved from the left. Similarly it can be proved from the right and so the Proposition. ■

It also follows from Example 3.1. that in the join hyperrings the cancellation law is not generally valid. Relatively we have the Proposition:

PROPOSITION 3.5 *In every join hyperring with no zero divisors, the cancellation law (with non zero element) holds for the multiplication and conversely.*

A join hyperring in which the additive hypergroup consists only of normal elements, is called ***normal***. Relatively we have the following Proposition:

PROPOSITION 3.6 *Every join hyperring which has no divisors of zero is normal.*

Proof Let a, x be two elements of the join hyperring. Then:

$$\begin{aligned} a[-(x-x)] &= \{a(-y) \mid y \in x-x\} = \{(-a)y \mid y \in x-x\} = \\ &= (-a)(x-x) = (-a)x + (-a)(-x) = \\ &= a(-x) + ax = a(x-x) \end{aligned}$$

Thus, for every $z \in -(x-x)$ there exists $w \in x-x$, such that $az = aw$, and so, because of Proposition 3.5, $z = w$ and therefore $-(x-x) \subseteq x-x$. Similarly $x-x \subseteq -(x-x)$ and so $x-x = -(x-x)$. ■

Remark 3.1 The opposite of the above Proposition is not valid. For instance the join hyperring of Example 3.1., though normal, it also has zero divisors.

PROPOSITION 3.7 *A finite commutative join hyperring with no divisors of zero is a join hyperfield.*

4. SUBHYPERRINGOIDS

A non empty subset Y' of a hyperringoid $(Y, +, \cdot)$ is a ***subhyperringoid*** of Y if $(Y', +)$ is a subhypergroup of $(Y, +)$ and $(Y', \cdot)$ is a subsemigroup of $(Y, \cdot)$. When Y is a join hyperringoid, then Y' is called a ***join subhyperringoid*** of Y if $(Y', +)$ is a join subhypergroup of $(Y, +)$ and $(Y', \cdot)$ is a subsemigroup of $(Y, \cdot)$. Also in the case of the join hyperrings, $(Y', +)$ can be a specific subhypergroup, *i.e.*, a symmetrical or a join one (and consequently closed). Thus, besides the subhyperringoids, there also exist the ***symmetrical subhyperrings*** and the ***join subhyperrings***.

PROPOSITION 4.1 *A non empty subset Y' of any hyperringoid Y is a subhyperringoid of Y, if and only if*

$$x + Y' = Y' \text{ and } xy \in Y' \quad \text{for every } x, y \in Y'$$

while Y' is a join subhyperringoid of a join hyperringoid Y, if and only if

$$x + y \subseteq Y',\ x \mid y \subseteq Y' \text{ and } xy \in Y' \quad \text{for every } x, y \in Y'.$$

Next let's denote by S the set of the subhyperringoids of Y, S_J the set of its join subhyperringoids, S_I the set of its join hyperidealoids, P the set of the multiplicatively constant parts of Y and I the subset of P which consists of the multiplicatively absorbing subsets of Y. Moreover let's denote by H_M, H, H_J the sets of the semisubhypergroups, the subhypergroups and the join subhypergroups respectively of the additive subhypergroup of Y. Then $S = P \cap H$,

$S_J = P \cap H_J$, $S_I = I \cap H_J$ while $S_M = P \cap H_M$ will be the set of the semisubhyperringoids in the sense that the hypercompositional part of their structure is generally a semisubhypergroup. And since $H_J \subseteq H \subseteq H_M$ and $I \subseteq P$ it derives directly that $S_I \subseteq S_J \subseteq S \subseteq S_M$.

Besides it is known that if the intersection of two semigroups is non void, then it is a semigroup, and that the non void intersection of two closed subhypergroups is a closed subhypergroup [1, 3, 11]. Thus the intersection of two join subhyperringoids, if it is non void, it is a join subhyperringoid. Moreover, since the (non void) intersection of two hypergroups is generally a semihypergroup [1, 11], the non void intersection of two subhyperringoids is a semisubhyperringoid. So:

PROPOSITION 4.2 *The sets S_J, S_I and S_M of the join subhyperringoids, the join hyperidealoids and the semisubhyperringoids that contain a non empty subset X of the join hyperringoid form a complete lattice.*

Furthermore, since the intersection of two symmetrical subhypergroups of a FJH is a symmetrical subhypergroup [12, 14], the intersection of two symmetrical subhyperrings or two symmetrical hyperideals of a join hyperring is a symmetrical subhyperring or a symmetrical subhyperideal respectively.

PROPOSITION 4.3 *In a join hyperring, the set of its symmetrical subhyperrings and the set of its symmetrical hyperideals form a complete lattice.*

It derives from Proposition 4.2. that to a given subset X of an arbitrary join hyperringoid Y, corresponds (through the closure mapping) the minimum, in the sense of inclusion, join subhyperringoid $X^{\wedge}$ of Y, which contains X. $X^{\wedge}$ is the ***join subhyperringoid*** of Y which is ***generated from*** X and contains it. In a similar way, $X^{\sim}$ is the ***semisubhyperringoid*** of Y which is ***generated from*** X and contains it. Analogously we consider the symmetrical subhyperring of a join hyperring which is generated from X. Thus:

PROPOSITION 4.4 *The semisubhyperringoid $X^{\sim}$ of a hyperringoid Y generated by a set X consists of the union of finite sums of products of the type*:

$$\prod_{i=1}^{k} \varepsilon_i \quad \text{where } \varepsilon_i \in X, \quad i = 1, \ldots, k$$

If the above sums produce a subhypergroup of $(Y, +)$, then $X^{\sim}$ is a subhyperringoid. Relatively we have:

PROPOSITION 4.5 *If for the hypercomposition of a hyperringoid Y holds*:

$$a, b \in a + b, \quad \text{for all } a, b \in Y$$

then every semisubhyperringoid of Y is a subhyperringoid.

Proof Let Y' be a semisubhyperringoid of Y and $a \in Y'$. Then $a + Y' \subseteq Y'$. Moreover $a + Y' = \cup_{b \in Y'} (a + b) \supseteq \cup_{b \in Y'} \{a, b\} = Y'$. Thus $Y' \subseteq a + Y'$, and so the Proposition. ■

PROPOSITION 4.6 *The join subhyperringoid of a join hyperringoid Y generated by a set X, consists of the union of finite sums of products of the type*:

$$\prod_{i=1}^{k} \varepsilon_i \quad \text{where } \varepsilon_i \in X, \ i = 1, \ldots, k$$

and also of the results of the induced hypercompositions between such sums.

Especially, regarding the subhyperring which is generated from the a-elements of a join hyperring, it holds:

PROPOSITION 4.7 *In a join hyperring Y, the union $A^{\wedge} = A \cup \{0\}$ of the a-elements with the zero element is the minimal bilateral join hyperideal of Y and furthermore it is the minimal join subhyperring of Y.*

Proof Firstly, it will be proved that $A^{\wedge}$ is a subhypergroup of Y. Indeed, it is known [12], that the sum of two attractive elements consists only of attractive elements (and also the zero element, if the two participating elements are opposite). Consequently, for every x from $A^{\wedge}$ holds $x + A^{\wedge} \subseteq A^{\wedge}$. Regarding the proof of the opposite, *i.e.*, $A^{\wedge} \subseteq x + A^{\wedge}$ for every $x \in A^{\wedge}$, let's consider an arbitrary element y of $A^{\wedge}$. Then, for every x from $A^{\wedge}$ holds $0 + y \subseteq (x - x) + y$, from where $\{0, y\} \subseteq x + (-x + y)$ and therefore $y \in x + (-x + y)$. But, as we have seen above, if an element is attractive, then its opposite is attractive as well. Thus $-x + y$ is a subset of $A^{\wedge}$ and therefore there

exists z from $-x + y$ such that $y \in x + z \subseteq x + A^{\wedge}$. So $A^{\wedge} \subseteq x + A^{\wedge}$ and consequently $x + A^{\wedge} = A^{\wedge}$. Hence $A^{\wedge}$ is a subhypergroup of Y, which is symmetrical as well. Next we shall prove that $A^{\wedge}$ is a closed subhypergroup in Y. Indeed, let z be an element of $H \backslash A^{\wedge}$. Then z is a non zero canonical element. But the sum of a non zero canonical element with an attractive one is the canonical element [12], thus $z + A^{\wedge} = z$ and so $(z + A^{\wedge}) \cap A^{\wedge} = \{z\} \cap A^{\wedge} = \varnothing$, *i.e.*, $A^{\wedge}$ is closed in Y.

Since every closed subhypergroup of any hypergroup contains all its neutral elements [1, 3] and since it is also stable under the induced hypercomposition [6], the zero element belongs to all closed subhypergroups of Y and $0|0$ is contained in all of them. But $0|0 = A^{\wedge}$. Indeed, $0 \in 0 + t$ for every $t \in A^{\wedge}$, so $t \in 0|0$ and therefore $A^{\wedge} \subseteq 0|0$. Also $s \notin 0|0$, for every $s \in C\{0\}$, since $0 \notin 0 + s = s$. Thus $0|0 = A^{\wedge}$ and so $A^{\wedge}$ does not contain a closed (and consequently join) subhypergroup of Y. Moreover a join subhypergroup h of Y, different from $A^{\wedge}$ must contain a c-element. But the difference of a c-element from itself contains all the a-elements [12]. Thus $A^{\wedge}$ is contained in h. Therefore $A^{\wedge}$ is the minimal join subhypergroup of Y. Also in Propositions 3.1 and 3.2 it has been proved that the product of two a-elements, if it is not zero, it is an a-element, while the product of a c-element with an a-element is always zero. Thus $A^{\wedge}$ is both, absorbing and multiplicatively closed and so the Proposition. ■

COROLLARY 4.1 *Every join hyperring which consists only of a-elements has no proper join subhyperrings.*

PROPOSITION 4.8 *Every join subhyperring Y' of a join hyperring is a symmetrical subhyperring.*

Proof Since Y' is a join subhyperring, its additive part is a subhypergroup closed in Y. As it is known, every closed subhypergroup of any hypergroup contains all its neutral elements [1, 3], and it is stable under the induced hypercompositions [6]. Thus $0 \in Y'$ and $0|x \subseteq Y'$ for every $x \in Y'$. But $-x \in 0|x$, because $0 \in x - x$. Therefore $-x \in Y'$ and so the Proposition. ■

PROPOSITION 4.9 *If a symmetrical subhyperring Y' of a join hyperring Y contains a non zero c-element, then it is join.*

Proof Let z be a c-element belonging to Y'. Then $A^{\wedge} \subseteq z-z$ and therefore all the elements of $Y \setminus Y'$ are c-elements. Now suppose that x is a non zero c-element. Then the set $x + Y'$ consists only of c-elements, because the sum of a non zero c-element with a-elements is the c-element. Therefore if there exists a c-element y in Y' such that $y \in x + w$, $w \in Y'$, then x belongs to $y-w$ which is a subset of Y'. Thus $(x + Y') \cap Y' = \varnothing$ for every $x \in Y \setminus Y'$. So $(Y', +)$ is a closed subhypergroup in $(Y, +)$ and therefore Y' is a join subhyperrring of Y. ■

COROLLARY 4.2 *Every symmetrical subhyperring which is not a join one, is a subset of the minimal join subhyperring.*

Furthermore, an important symmetrical subhypergroup of the FJHs is $\Omega(X)$ which consists of the unions of sums of the type:

$$(x_1 - x_1) + \cdots + (x_n - x_n)$$

where x_i, $i = 1,\ldots,n$ belong to a set of normal elements X. Assuming that $x, y \in \Omega(X)$, then (with regard to the multiplication) we have:

$$xy \in [(x_1 - x_1) + \cdots + (x_n - x_n)]\,[(y_1 - y_1) + \cdots + (y_m - y_m)] \subseteq$$
$$\subseteq \sum_{i,j=1}^{n} (x_i - x_i)(y_j - y_j) \subseteq \sum_{i,j=1}^{n} [x_i y_j + (-x_i) y_j + x_i (-y_j) + (-x_i)(-y_j)]$$

Now if the elements of $-X \cup X$ are not divisors of zero, the above hypersum becomes (Prop. 3.3):

$$\sum_{i,j=1}^{n} [(x_i y_j - x_i y_j) + (x_i y_j - x_i y_j)]$$

This hypersum is a subset of $\Omega(X)$ only when X is multiplicatively closed. Thus:

PROPOSITION 4.10 *Let X be a non empty subset of a join hyperring Y, which*

(i) *is multiplicatively closed*
(ii) *consists of normal elements*
(iii) *the elements of $-X \cup X$ are not divisors of zero.*

Then $\Omega(X)$ *is a symmetrical subhyperring of* Y. *If* X *is also multiplicatively absorbing, then* $\Omega(X)$ *is a symmetrical hyperideal.*

COROLLARY 4.3 *If* E *is an integral join hyperring, then* $\Omega(E)$ *is a symmetrical hyperideal.*

Also, since a proper integral join hyperring consists only of a-elements (Cor. 3.1) and since the join hyperrings which consist only of a-elements have no proper join subhyperrings (Cor. 4.1), the Proposition holds:

PROPOSITION 4.11 *The proper integral join hyperrings have no proper join subhyperrings.*

In addition to the above fundamental properties and the basic analysis of the hyperringoids, a study has been carried out on more specific topics such as the congruence relations, the equation and system solving [13] *etc.* Other hypercompositional structures such as the hypermoduloid and the supermoduloid have also been developed, based on them. All the above, when applied to the linguistic hyperringoid, give interesting results in the theory of Languages and Automata such as the theorems which have as corollaries the theorems of Myhill-Nerode and Kleene [8, 9, 10]. The hyperringoid, being a new mathematical structure, also reveals a whole new area for developing research of pure mathematical interest.

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