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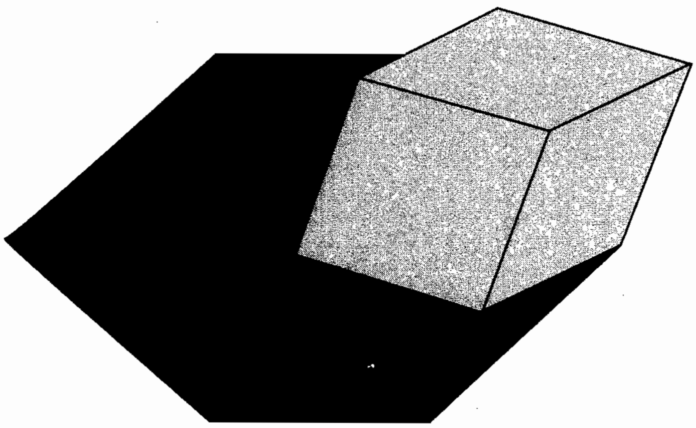

FORUM

THE SUBHYPERGROUPS OF THE FORTIFIED JOIN HYPERGROUP

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Abstract. The Fortified Join Hypergroup, which is a hypercompositional structure directly connected with computer theory, has certain special properties that give birth to several types of subhypergroups. In this paper those subhypergroups are being introduced along with a study of their properties and an analysis of the relations that are being developed between them.

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1. Introduction.

The *Fortified Join Hypergroup* (FJH) is a hypergroup that derived from the study of the theory of Languages and Automata with the use of methods of the hypercompositional algebra [5],[8]. As it is being indicated by its name, the FJH is a join hypergroup $(H,+)$ [i.e. a commutative hypergroup in which every quadruple of its elements satisfies the axiom:

(J) $$(a:b)\cap(c:d)\neq\emptyset \Longrightarrow (a+d)\cap(c+b)\neq\emptyset$$

where ":" is the induced from "+" hypercomposition and $a:b=\{x\in H \mid a\in b+x\}$] that also satisfies the axioms:

FJ_1 There exists a unique neutral element, denoted by 0, the zero element of H, such that for every $x\in H: x\in x+0$ and $0+0=0$ and

FJ_2 For every $x\in H\backslash\{0\}$ there exists only one element $x'\in H\backslash\{0\}$, the opposite or symmetrical of x, denoted by $-x$, such that: $0\in x+x'$. Also $-0=0$.

So the FJH, satisfying those axioms, places itself between the join and the canonical hypergroup, since, as it is known, if a join hypergroup has a scalar neutral element, it is a canonic one [1],[2].

Example 1.1. Let $(H,+)$ be a canonic hypergroup [10]. If a new hyperoperation † is defined in H as follows:

$$x\dagger y=(x+y)\cup\{x,y\},\ \text{for every } x,y\in H$$

then the $(H,\dagger)$ is a FJH.

Example 1.2. Let $(K, +)$ be a canonic hypergroup [10] and let a be an element which does not belong to K. The set $H = K \cup \{a\}$ with hypercomposition:

$$\begin{aligned}
&x\dagger y = (x+y)\cup\{x,y\}, && \text{if } x,y\in K \text{ and } x\neq -y,0\\
&y\dagger(-y) = H, && \text{for every } y\in K\\
&x\dagger 0 = 0\dagger x = x, && \text{for every } x\in K\\
&a\dagger a = \{0,a\}\\
&a\dagger 0 = \{0,a\}
\end{aligned}$$

becomes a FJH.

Example 1.3. Let H be a set totally ordered and symmetrical around a center denoted by $0 \in H$. With regard to this center a partition

$$h = H_1 \cup \{0\} \cup H_2 = H^- \cup \{0\} \cup H^+$$

can be defined, such that, for every $x \in H^-$ and $y \in H^+$ it is $x < 0 < y$, and, $x \leq y \Longrightarrow -y \leq -x$ for every $x, y \in H$ (where $-x$ is the symmetrical of x as for 0). Then H, with hypercomposition:

$$\begin{aligned}
&x\dot{+}y = \{x,y\}, \text{ if } y\neq -x \text{ and}\\
&x\dot{+}(-x) = [0,|x|]\cup\{-|x|\}
\end{aligned}$$

becomes a FJH. Besides, if the hypercomposition is:

$$\begin{aligned}
&x\underset{\cdot}{+}y = \{x,y\}, \text{ if } y\neq -x \text{ and}\\
&x\underset{\cdot}{+}(-x) = [-|x|,|x|]
\end{aligned}$$

then $(H, \underset{\cdot}{+})$ is again a FJH.

The relevant study of the properties of the FJH has revealed that when 0 is being added to another element, it gives as a result, either that element, or the biset which consists of both the 0 and the participating element. The elements of the FJH that belong to the first category have been named *canonical* or *c-elements*, since they act exactly as the elements of the canonical hypergroup, while the others have been named *attractive* or *a-elements*, because they "attract" 0 into the result of the hyperoperation. Furthermore 0 is considered as a canonical element, since the result of $0 + 0$ is the singleton $\{0\}$. The set of the canonical and the set of the attractive elements are being denoted by C and A respectively.

Proposition 1.1. *Let $(C, \dot{+})$ be a canonical hypergroup and let $(A, \underset{\cdot}{+})$ be a* FJH, *every non zero element of which is attractive. Also let's assume that both* *hypergroups have the same neutral element. Then the set $H = C \cup A$ becomes a* FJH, *if the hypercomposition is defined as follows:*

$$\begin{array}{ll} x+y = x\underset{\cdot}{+}y & \text{if } x,y \in A \\ x+y = x\dot{+}y & \text{if } x,y \in C \text{ and } x \neq -y \\ x\underset{\cdot}{+}(-x) = (x\dot{+}(-x)) \cup A & \text{if } x \in C\backslash\{0\} \\ x+y = y & \text{if } x \in A,\ y \in C\backslash\{0\}. \end{array}$$

Example 1.4. Let H be a totally ordered set and symmetrical around a center $0 \in H$ and let's use the symbolisms of Example 1.3. It is known that the set $H^+ \cup \{0\}$, with hyperoperation:

$$x\dot{+}y = \left[\begin{array}{ll} \max\{x,y\} & \text{if } x \neq y \\ [0,x] & \text{if } x = y \end{array}\right.$$

becomes a canonical hypergroup [11].

Moreover, defining in the set $H^- \cup \{0\}$ the following hyperoperation "+":

$$x\underset{\cdot}{+}y = \left[\begin{array}{ll} \{x,y\} & \text{if } x \neq y \\ [x,0] & \text{if } x = y \end{array}\right.$$

we get a FJH which has only a–elements.

Now if we define in H the hyperoperation "+":

$$x+y = \left[\begin{array}{ll} \max\{x,y\} & \text{if } x \neq y \text{ and either } x \in H^+ \text{ or } y \in H^+ \\ \{x,y\} & \text{if } x \neq y \text{ and } x,y \in H^- \\ \{x,0\} & \text{if } y = 0 \text{ and } x \in H^- \\ x & \text{if } y = 0 \text{ and } x \in H^+ \\ [x,0] & \text{if } x = y \in H^- \cup \{0\} \\ [0,x] \cup H^- & \text{if } x = y \in H^+ \end{array}\right.$$

then $(H,+)$ is again a FJH, which has selfopposite elements. The elements of H which belong to $H^+ \cup \{0\}$ are c–elements, while all the elements that belong to H^- are a–elements.

For the c and a–elements we have the following fundamental properties:

Properties 1.1. *In a* FJH *the following are valid:*

(i) *If x is a c–element, then $-x$ is also a c–element.*

(ii) *If x is a a–element, then $-x$ is also an a–element.*

(iii) *The sum of two a–elements consists only of a–elements (and the 0, if they are opposite) and also it always contains the two addents.*

(iv) *The sum of two non opposite c–elements consists of c–elements, while the sum of two opposite c–elements contains all the a–elements.*

(v) *The sum of an a–element with a non zero c–element is the c–element.*

For the relevant proofs of the above see [6].

Also the elements of the FJH are being separated into *normal* and *abnormal.* This distinction of the elements derives from the fact that even though in a FJH the equality $-(x+y) = -x-y$ holds when $x \neq y$, the equality $-(x-x) = x-x$ is generally not valid. So those elements that satisfy this equality are called *normal* while the others, that don't, are called *abnormal.* It has been proved that all the c–elements are normal as well as that the equality $-(x:y) = (-x):(-y)$ holds if y is normal, or if $x \notin y-y$. Moreover, it has been proved that in the FJHs the reversibility holds under conditions. More precisely, $z \in x+y \Longrightarrow y \in z-x$, except if $z = x \neq y$ where $x \in x+y \Longrightarrow x \in x-y$, while generally $y \notin x-x$. This gives as a result that for every $x \neq y$ holds $x-y = (x+y) \cup (-y) \cup (-y):(-x)$, while $x-x \subseteq (x:x)$. If one of the x, y is a c–element then $x-y = x:y = (-y):(-x)$ [6].

For the study of its subhypergroups and since the FJH is a join hypergroup, the relevant theory of the subhypergroups of the join hypergroup must be taken under consideration. Very significant in the join hypergroup are the intersections $x:y \cap z:w$ which appear in the first part of the join axiom. Thus if h is a subhypergroup of a join hypergroup H and if $x, y, z, w \in h$, then the following can happen:

(i) $[(x:y) \cap (z:w)] \cap (H \backslash h) \neq \emptyset$

(ii) $[(x:y) \cap (z:w)] \subseteq h$.

So let's start with the B–hypergroup, which is a hypergroup that appears in the theory of the Languages. The hypercomposition in a B–hypergroup $(Y,+)$ is being defined as follows:

$$x+y = \{x, y\}, \text{ for every } x, y \in Y.$$

It can be proved that the B–hypergroup is a join one, and because of the definition of the hypercomposition, every one of its subsets is a subhypergroup. In the subhypergroups of the B–hypergroup and according to the choice of the x, y, z, w, (ii) from the above is not always valid, since $x:x = Y$, for every $x \in Y$.

Next let's see another hypergroup, which is constructed on the set Z of the integers. Indeed, if we consider the partition $Z_1 \cup Z_2$ of Z, where Z_1 the set of the odd numbers and Z_2 the set of the even (the zero included), then Z becomes a join hypergroup with hypercomposition:

$$x \dagger y = \left[\begin{array}{ll} Z_2 & \text{if } x = y(\text{mod}2) \\ Z_1 & \text{if } x \neq y(\text{mod}2). \end{array} \right.$$

Z_2 is a subhypergroup of Z, in which the (ii) from the above is always valid. In this case we say that the join axiom is valid inside the subhypergroup. Such a subhypergroup, inside of which the join axiom is being verified (i.e. it is a join hypergroup itself) will be called *join subhypergroup*. But a subhypergroup h, in order to verify the join axiom inside it, must be stable (closed) with regard to the induced hypercomposition, i.e. for every $x, y \in h$ it must be $x : y \subseteq h$. Now, if a subhypergroup of a hypergroup H is stable with regard to the induced hypercompositions, then it is a closed subhypergroup of H [2],[3]. Therefore, the join subhypergroups of a join hypergroup are its closed subhypergroups. Analogously, in the case of the FJHs we have subhypergroups that are join and others that are not join. And since every closed subhypergroup of any hypergroup contains all its neutral elements, every join subhypergroups h of a FJH H will contain 0. Furthermore, since the closed subhypergroups are stable with regard to the induced hyprcomposition, it will be $0 : x \subseteq h$. But $-x \in 0 : x$ and therefore $-x \in h$ for every $x \in h$. So we have the Proposition:

Proposition 1.2. *Every join subhypergroup of a* FJH *is a* FJH *itself, with the same zero.*

Moreover it is known [10] that if a subhypergroup of a canonic hypergroup contains its zero element, then it is a canonical subhypergroup. The same thing does not hold in the FJHs. Let's see it with an example, again in the set Z of the integers. This set becomes a FJH with the hypercomposition: $x \dagger y = \{x, y, x + y\}$. Here Z_2 is again a subhypergroup of $(Z, \dagger)$, which is not join, since the induced hypercomposition is not stable in Z_2. (It is enough to observe that for every $x \in Z_2$ holds $x : x = Z$). We observe though that in Z_2 hold:

(a) $0 \in Z_2$

(b) if $x \in Z_2$ then $-x \in Z_2$

and obviously (b) alone give (a). Thus there derives a wider from the join subhypergroups category of subhypergroups of the FJH, the symmetrical ones. We call *symmetrical subhypergroup* of a FJH H a subhypergroup h of H for which $-x \in h$ for every $x \in h$. Direct result from the definition is that $0 \in h$, as well as:

Proposition 1.3. *Every join subhypergroup of a* FJH *is also symmetrical.*

Thus, according to the above we have the following tree types of subhypergroups of a FJH:

(i) join subhypergroups

(ii) symmetrical subhypergroups that are not join

(iii) subhypergroups that are not symmetrical and therefore not join.

2. Join subhypergroups

Proposition 2.1. *Necessary and sufficient condition for a non empty subset h of H to be a join subhypergroup of H is that $x : y \subseteq h$ and $x - y \subseteq h$ for every $x, y \in h$.*

Proof. Obviously the condition is necessary. Conversely, if $x \in h$ then $x - x \subseteq h$, so $0 \in h$ and consequently $0 - x \subseteq h$, from where $-x \in h$. Thus $x + h = x - (-h) = x - h \subseteq h$. Moreover, for every $y \in h$ we have $y - x \subseteq h$, so $y - x + x = y + (x - x) \subseteq x + h$ from where $y \in x + h$, i.e., $h \subseteq x + h$. Thus $x + h = h$ and h is a hypergroup, which, is also closed, since it is stable under the induced hypercomposition, and therefore join.

Corollary 2.1. *A subset h of H is a join subhypergroup of H if and only if it is stable with regard to the hypercomposition and the induced hypercomposition of H and also it contains the opposite of every $x \in h$.*

Using Property 1.1 (iv), we derive to:

Proposition 2.2. *No (proper) join subhypergroup of a* FJH

(i) *has only c–elements*

(ii) *is a canonical subhypergroup* $(\neq \{0\})$.

Also for the join subhypergroups of the FJH we have:

Proposition 2.3. *If $h \neq \{0\}$ is a join subhypergroup of H and $h \cap C \backslash \{0\} = \emptyset$ then $h = A \cup \{0\}$.*

Proof. Obviously $h \subseteq A \cup \{0\}$. Moreover, for an arbitrary $x \in A \cup \{0\}$, it will be:

$$(x + h) \cap h \supseteq (x + 0) \cap h = \{x, 0\} \cap h \ni 0.$$

So $(x + h) \cap h \neq \emptyset$ and since h, as a join subhypergroup is closed, it derives that $x \in h$ and therefore $A \cup \{0\} \subseteq h$.

Corollary 2.2. *The* FJHs *which consist only of a–elements have no proper join subhypergroups.*

Corollary 2.3. *The* FJHs *which consist only of a–elements have no invertivle subhypergroups.* (Since every invertible subhypergroup is also closed.)

Proposition 2.4. *The $A\hat{} = A \cup \{0\}$ is a join subhypergroup of H.*

Proof. The sum of two a-elements consists only of a-elements and 0 if they are opposite [Property 1.1 (iii)]. Thus if x is an a-element then $x + A\hat{}\subseteq A\hat{}$. Now, for the proof of the converse we make use of the fact that H has a partition in the two disjoint sets $A\hat{}$ and $C\backslash\{0\}$ and that the sum of an a-element with a non zero c-element is the c-element [Property 1.1 (v)]. So:

$$A\hat{}\subseteq H = x+H = x+[A\hat{}\cup(C\backslash\{0\})] = (x+A\hat{})\cup[x+(C\backslash\{0\})] = (x+A\hat{})\cup(C\backslash\{0\})$$

Consequently $A\hat{}\subseteq x + A\hat{}$. Thus $A\hat{}$ is a subhypergroup of H. Next let z be an element of the set $H\backslash A\hat{}$, i.e., a non zero c-element. Then $z + A\hat{} = z$ [Property 1.1 (v)]. Thus $(z + A\hat{}) \cap A\hat{} = \emptyset$ for every $z \in H\backslash A\hat{}$ and so $A\hat{}$ is closed and therefore join subhypergroup of H.

From the above Proposition and Corollary 2.2 derives the Proposition:

Proposition 2.5. *The subhypergroup $A\hat{}$ is the minimum, in the sense of inclusion, join subhypergroup of H.*

Next, for every x, y from $A\hat{}$ holds $x \in x + y$ [Property 1.1 (iii)] and so $y \in x : x$. Thus $A\hat{}\subseteq x : x$. Moreover if z is a non zero c-element, $z \notin x : x$ because $x + z = z$ [Property 1.1 (v)]. Consequently $x : x = A\hat{}$ and so:

Proposition 2.6. *The minimum join subhypergroup $A\hat{}$ equals to $x : x$, for every $x \in A\hat{}$.*

Proposition 2.7. *Every join subhypergroup of $(H, +)$ is invertible and conversely.*

Proof. We shall prove that if the relation $(x + h) \cap (y + h) \neq \emptyset$ is valid, then $x+h = y+h$. Initially let $x \in h$ and $w \in (x+h)\cap(y+h) = h\cap(y+h)$. Then there exists $z \in h$ such that $w \in y+x$, from where $y \in w : z$. But h is a join subhypergroup and so $w : z\subseteq h$, thus $y \in h$ and therefore $x+h = y+h$. Next let $x, y \notin h$ and assume that there exist $x, w \in h$ such that $t \in (x + z) \cap (y + w)$. t does not belong to h, because if it did, then it would be (for instance) $y \in t : w\subseteq h$, which is absurd, and therefore $t \neq z, w$. Thus in the relation $t \in x + z$ the reversibility of z is valid and consequently $x \in t - z$ or $x \in y + w - z$ or $x \in y + h$. Therefore $x + h\subseteq y + h$. Similarly $y + h\subseteq x + h$. Thus $x + h = y + h$. And since every invertible subhypergroup is closed, it derives that it is also join, and so the Proposition.

If h is a join subhypergroup of H, then it defines an equivalence relation $\equiv_j$, as follows [2]:

$$x \equiv_j y \Longleftrightarrow h : x = h : y$$

But, according to the above Proposition, h is also invertible and so it defines another equivalence relation $\equiv_i$, as follows:

$$x \equiv_i y \Longleftrightarrow x + h = y + h$$

the classes C_x of which are equal to $x+h$ for every $x \in H$. This equivalence relation is normal. It is worth mentioning that according to Proposition 2.5, h contains all the a-elements of H. Thus if $C_x \in H/h$, then x is a c-element and so for C_0 we have:

$$C_0 + C_x = (0+h) + (x+h) = (0+x) + (h+h) = x+h = C_x$$

Let's consider now the class $h : x$ with $x \notin h$. Since $-x \in 0 : x$ it derives that $-x \in h : x$. Then, with the use of Corollary 2.1 from [2] we have:

$$-x \in h : x \Longrightarrow -x + h \subseteq (h : x) \subseteq (h+h) : x = h : x$$

Thus

(i) $$-x + h \subseteq h : x$$

Next let's consider an element w from the class $h : x$. Then $(w+x) \cap h \neq \emptyset$. Consequently there exists $y \in h$ such that $y \in w+x$ and the reverisibility of x holds, because $y \neq x$, since $x \notin h$. Therefore it is $w \in -x+y$ or $w \in -x+h$. Thus

(ii) $$h : x \subseteq -x + h$$

From (i) and (ii) derives that $h : x = -x + h$ and so:

$$h : x = h : y \Longleftrightarrow -x + h = -y + h \Longleftrightarrow x + h = y + h$$

Thus the Proposition holds:

Proposition 2.8. *The equivalence relations $\equiv_i$ and $\equiv_j$ coincide.*

Also the following Proposition is valid:

Proposition 2.9. *The set H/h with hypercomposition:*

$$C_x \dagger C_y = \{C_w \in H/h \mid w \in x+y\}$$

is a canonical hypergroup with neutral element $C_0 (= h)$.

The join subhypergroups of a FJH are closed and according to Proposition 2.7, they are invertible. Also from Proposition 1.2 it derives that their intersection is always non valid. So:

Proposition 2.10. *The set of the join subhypergroups of H is a complete lattice which coincides with the complete lattices of its closed and of its invertible subhypergroups.*

It is known that the join subhypergroup generated by a non void subset X of a join hypergroup consists of the union of finite sum of elements of X and of the results of the induced hypercomposition between such sums [6]. Thus for the generated join subhypergroup from a non empty subset X of a FJH H we shall have:

a) it will contain all the a-elements of H, since it has been proved that $A^{\hat{}}$ is the minimum join subhypergroup of H

b) if X also contains c-elements, e.g. x_i, $i \in I$, then $x_i : y = x_i - y$, and, according to Proposition 8.1 of [10], it will also contain all the unions of finite sums of the elements of $-(X \backslash A) \cup (X \backslash A)$.

So

Proposition 2.11. *The join subhypergroup $\bar{X}$ which is generated from a non empty set X consists of all the a-elements as well as of the unions of all the finite sums of the c-elements that are contained in the union $-X \cup X$.*

3. Symmetrical subhypergroups

In the Introduction of this paper we have also refered to the B-hypergroup, which is a join hypergroup. In this hypergroup $(Y, +)$, for reasons dictated by the theory of Languages and Automata (see [7]), we have introduced a neutral element 0 and its hypercomposition has been redefined in the set $Y \cup \{0\}$ as follows:

$$x + y = \{x, y\} \text{ if } x \neq y \text{ and } x + x = \{x, 0\}$$

This new hypergroup, called *dilated B-hypergroup*, is a FJH in which holds:

Proposition 3.1. *The subhypergroups of the dilated B-hypergroup are symmetrical.*

The symmetrical subhypergroup of a FJH, being defined with weaker conditions than the ones of the join subhypergroup, presents corresponding properties to the ones of the join subhypergroup, with fewer conditions though. Here also appear relations that combine those two types of subhypergroups.

Now let x, y, with $x \neq y$ be two elements of a symmetical subhypergroup h of a FJH H. Then $x : y = \{z \in H \mid x \in z + y\}$ and since $x \neq y$ the implication $x \in z + y \Longrightarrow z \in x - y$ holds. We mention that the converse of this implication is not always valid, because if x and y are a-elements, then $-y \in x - y$, while, generally, $x \notin y - y$. Therefore:

$$\{z \in H \mid x \in z + y\} \subseteq \{z \in H \mid z \in x - y\} \subseteq h$$

So:

Proposition 3.2. *If h is a symmetrical subhypergroup, then $x : y \subseteq h$, for every $x, y \in h$ with $x \neq y$.*

Proposition 3.3. *A non empty subset h of H is a symmetrical subhypergroup of H if and only if $x - y \subseteq h$ for every $x, y \in h$.*

Proof. The above condition is obviously valid when h is a symmetrical subhypergroup of H. Conversely now, let $x \in h$. Then $x-x \subseteq h$ and therefore 0 belongs to h. Moreover $0-x \subseteq h$, thus $-x \in h$. Also, since h is a subset of H the only axiom that needs to be proved is the reproductive one. Let $x \in h$. Then: $x+h=x-(-h) \subseteq h$. Next let $t \in h$, then: $t-x \subseteq h \Longrightarrow t-x+x \subseteq x+h \Longrightarrow$ $\Longrightarrow t+0 \subseteq x+h \Longrightarrow t \in x+h$, that is $h \subseteq x+h$. So $x+h=h$.

Corollary 3.1. *A non empty subset h of H is a symmetrical subhypergroup of H if and only if it is stable with regard to the hypercomposition and if it also contains the opposite $-x$ for every element $x \in h$.*

(Indeed, if $x, y \in h$ then $x, -y \in h$ and therefore, since h is stable, $x-y \subseteq h$ and according to the above Proposition, h is a symmetrical subhypergroup.)

Proposition 3.4. *If a symmetrical subhypergroup h of H contains a c-element different from 0, then it is join.*

Proof. If x is a c-element of h then $-x \in h$ and therefore $x-x \subseteq h$. But, according to Property 1.1 (iv), $x-x$ contains all the a-elements of H, that is $A^{\hat{}} \subseteq h$ and so $x : y \subseteq h$ for every $x, y \in A^{\hat{}}$, due to Proposition 2.4. Moreover, if x, y are c-elements, then $x : y = x - y \subseteq h$ and thus, again $x : y \subseteq h$. Lastly, if x is a c-element and y an a-element, then, $x : y = \{w \mid x \in w + y\}$ and since $x \neq y$ it derives that $x : y = \{w \mid w \in x - y\}$, so $x : y \subseteq h$. Similarly $y : x \subseteq h$ and consequently, for every $x, y \in h$ we have $x : y \subseteq h$. Thus h is join.

Corollary 3.2. *The symmetrical subhypergroups that are not join consist only of a-elements.*

For the intersection of symmetrical subhypergroups of a FJH hold similar properties to the ones regarding the intersection of join subhypergroups. So:

Proposition 3.5. *The intersection of two symmetrical subhypergroups of symmetrical subhypergroup of H.*

Proof. Let h_1, h_2 be two symmetrical subhypergroups of H. Then $0 \in h_1 \cap h_2$ and if $x \in h_1 \cap h_2$ with $x \neq 0$, also $-x \in h_1 \cap h_2$. Next let x be an element from the intersection $h_1 \cap h_2$. Then

$$x + (h_1 \cap h_2) \subseteq x + h_1 = h_1 \text{ and } x + (h_1 \cap h_2) \subseteq x + h_2 = h_2$$

Therefore

$$x + (h_1 \cap h_2) \subseteq h_1 \cap h_2$$

Now let $y \in h_1 \cap h_2$, then

$$y \in y + 0 \subseteq y + (x - x) = x + (y - x) \subseteq x + (h_1 \cap h_2)$$

so $h_1 \cap h_2 \subseteq x + (h_1 \cap h_2)$, thus $x + (h_1 \cap h_2) = h_1 \cap h_2$, and so the Proposition.

Now, from the above Proposition and since the intersection of two symmetrical subhypergroups is always non empty (it always contains the 0) derives the:

Proposition 3.6. *The set of the symmetrical subhypergroups of a* FJH *consists a complete lattice.*

And since every join subhypergroup of H is also symmetrical, for their lattices holds:

Proposition 3.7. *The lattice of the join subhypergroups of a* FJH *is sublattice of the lattice of the symmetrical ones.*

Now, let's deal with the symmetrical subhypergroup that is generated from a subset X of H, considering only the case where H is normal. Let $\widetilde{X}$ be the set of all the elements $x \in H$ which belong to sums of the type $\sum_{i=1}^{k} x_i$ where the x_i belong to $-X \cup X$. If y_1, y_2 are two elements of X then $y_1 \in \sum_{i=1}^{n} w_i$, $y_2 \in \sum_{j=1}^{m} z_j$ and so, since H is normal, we have:

$$y_1 - y_2 \subseteq \sum_{i=1}^{n} w_i - \sum_{j=1}^{m} z_j = \sum_{k=1}^{n+m} x_k \subseteq X$$

Thus $\widetilde{X}$ is a symmetrical subhypergroup of H which contains $-X \cup X$. So, if $h(X)$ is the generated from X symmetrical subhypergroup, then $h(X) \subseteq \widetilde{X}$. But $h(X)$ contains all the sums of elements from $-X \cup X$, and therefore $\widetilde{X} \subseteq h(X)$. Consequently we have the Proposition:

Proposition 3.8. *The symmetrical subhypergroup of a normal* FJH *which is generated from a non empty set X consists of the unions of all the finite sums of the elements that are contained in the union $-X \cup X$.*

In the following appear certain symmetrical subhypergroups that generally exist in every FJH. So let H be an arbitrary FJH and let X be a subset of H. Let $\Omega(X)$ be the union of the sums $(x_1 - x_1) + \cdots + (x_n - x_n)$ where n arbitrary non negative integer and x_i, $i = 1, \ldots, n$ takes all the possible values from X. Also suppose that X consists only of normal elements. Then:

Lemma 3.1. *If $w \in (x_1 - x_1) + \cdots + (x_n - x_n)$, then*

$$-w \in (x_1 - x_1) + \cdots + (x_n - x_n), \textit{ for every } x_1, x_2, \ldots, x_n \in X.$$

Proof. This Lemma will be proved by induction. So for $n = 1$ it is: $w \in x_1 - x_1 \Longrightarrow -w \in -(x_1 - x_1) = x_1 - x_1$, since x_1 is normal. Now suppose that it is true for $n = k$ and let $w \in (x_1 - x_1) + \cdots + (x_k - x_k) + (x_{k+1} - x_{k+1})$. Then there exists $s \in (x_1 - x_1) + \cdots + (x_k - x_k)$ such that $w \in s + (x_{k+1} - x_{k+1})$ or $w \in (s + x_{k+1}) - x_{x+1}$. Therefore there exists $t \in s + x_{k+1}$ such that

$w \in t - x_{k+1}$. So $-w \in -(t - x_{k+1})$. Now if $t = x_{k+1}$, then t is a normal element and so $-(t - x_{k+1}) = -t + x_{k+1}$. The last equality is always valid if $t \neq x_{k+1}$. Thus $-w \in -t + x_{k+1} \subseteq -(s + x_{k+1}) - x_{x+1} = -s + (x_{k+1} - x_{k+1})$, which, due to the induction, gives:

$$-w \in (x_1 - x_1) + \cdots + (x_k - x_k) + (x_{k+1} - x_{k+1}).$$

Proposition 3.9. *$\Omega(X)$ is a symmetrical subhypergroup of H.*

Proof. Apparently $0 \in \Omega(X)$ and because of the Lemma $-x \in \Omega(X)$ for every $x \in \Omega(X)$. Next if $x \in \Omega(X)$ then $x \in (x_1 - x_1) + \cdots + (x_n - x_n)$ for some $x_1, ..., x_n$ from X. So:

$$x + \Omega(X) \subseteq (x_1 - x_1) + \cdots + (x_n - x_n) + \Omega(X) \subseteq \Omega(X)$$

Thus $\Omega(X)$ is stable with regard to the hypercomposition and it also contains the opposite of every one of its elements. Therefore, according to Corollary 3.1 it is a symmetrical subhypergroup of H.

Proposition 3.10. *If X is the set of the a-elements A, or if X contains at least one non zero c-element, then $\Omega(X)$ is a join subhypergroup of H, and therefore invertible.*

Proof. Suppose that $X = A$. Then, because of Property 1.1 (iii), $x \in x - x$ for every $x \in A$. So $A\hat{} \subseteq \Omega(X)$. Also, according to the same Property, the sum of a-elements consists only of a-elements and 0. Thus $\Omega(A) \subseteq A\hat{}$ and therefore $\Omega(A) = A\hat{}$. Now if X is a subset of H that contains a nonzero c-element z, then $\Omega(X)$ is a symmetrical subhypergroup which contains the difference $z - z$. If in $\Omega(X)$ exists at least one c-element, then, because of Proposition 3.4, $\Omega(X)$ is join. Otherwise $\Omega(X) = A\hat{}$, since $A\hat{} \subseteq z - z$ [Property 1.1 (iv)].

Corollary 3.3. *If H does not contain non zero c-elements, then $\Omega(H) = H$.*

Proposition 3.11. *If $Q(X)$ is the union of the sums $(x_1 : x_1) + \cdots + (x_n : x_n)$, where n arbitrary non negative integer and x_i, $i = 1, ..., n$ are elements from an arbitrary subset X of H, then $Q(X)$ is a join subhypergroup of H.*

Proof. If $X \subseteq A\hat{}$, then $Q(X) = A\hat{}$ (Proposition 2.6). Moreover, if X contains a non zero c–element y then $y : y = y - y$ contains $A\hat{}$ and $Q(X) = \Omega(X)$, which is join because of Proposition 3.10.

The field for research on the FJHs is very extensive. The behaviour of the subhypergroups under the homomorphisms [4], the monogene subhypergroups and other relevant topics [9], which have already been studied do not lessen the multitude of the problems that still remain open.

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For the terms and Properties of the general Hypergroup Theory that have been used in this paper without special reference see: CORSINI, P., *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

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