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On The Hypergroup Theory

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ABSTRACT
This survey article presents elements of the Theory of Hypergroups, Algebraic Structures for which the result of the "composition" of two elements is a subset of the structure. It deals with the several kinds of elements and types of sub-hypergroups that exist in a hypergroup, as well as the homomorphisms. There also appear some special categories of hypergroups.

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1. General

In a congress of Scandinavian Mathematicians held in Stokholm, in 1934, F. Marty introduced a new mathematical structure, which he named **hypergroup** [15]. The axioms on which the hypergroup has been founded in a non empty set H that has a composition law "*", are:

i) The result of $a*b$ is a subset of H, for every $a,b\in H$

ii) $(a*b)*c=a*(b*c)$ for every $a,b,c\in H$ (associative axiom)

iii) $a*H=H*a=H$ for every $a\in H$ (reproductive axiom)

These axioms are not artificial products of some imaginary process. On the contrary they derived clearly as the result of the real mathematical necessity. Several problems in the non commutative algebra led, in a natural way, to the introduction of these axioms. Let's mention, for instance, the co-sets in the groups, that are produced from non invariant subgroups. (e.g. see [16], [37]).

If only the first of the above axioms is valid in H, then (H,*) is called **hypergroupoid**, while if the first two axioms are valid, then it is called **semi-hypergroup**. This composition law, which is a multivalued operation, is called **hypercomposition** and it has been proved that result of the hypercomposition in a hypergroup is always different from the empty set. One can easily see that every group is a hypergroup. The study of the properties of this new structure has become the subject of the research of many mathematicians ever since.

Next there appear some of the most important properties of the hypergroup and definitions of the fundamental notions which derive from the axioms of this structure. For the sake of simplicity we make no distinction between the elements and their corresponding singletons, when nothing opposes it.

The hypercomposition defined in the hypergroup, which, for the simplicity of the notation will be written as xy, instead of $x*y$, introduces two new hypercompositions, the **induced hypercompositions**, defined as follows [15]:

$$x{:}y=\{\omega\in H \mid x\in\omega y\} \text{ and } x..y=\{\omega\in H \mid x\in y\omega\}$$

If the hypercomposition is commutative, then $x{:}y=x..y$. It has been proved that the reproductive axiom is equivalent to the relations [18]:

$$x{:}y\neq\phi,\ x..y\neq\phi \text{ for every } x,y\in H$$

The interaction of the hypercomposition with the induced hypercompositions is given from the **mixed associative law** [19]:

$$\text{i) } (a{:}b){:}c=a{:}(cb) \text{ and ii) } (a..b)..c=a..(bc)$$

Lastly it is worth mentioning that the reproductivity holds for the induced hypercompositions as well, i.e. [19]:

$$H=h{:}a=a{:}H=H..a=a..H$$

An element of a hypergroup is called **neutral** (or **unit**, or **zero**) if the result of its hypercomposition with any other element $x\in H$, contains x.

More precisely:

■ A **left unit** element is an element e_l such that $a\in e_l\, a$,, for every $a\in H$.
A **right unit** element e_r is being defined in an analogous way.
■ If $ae=ea=a$ for every $a\in H$, then e is a **scalar unit**, or **absolute unit**.

An element of H is called **scalar** if the result of the hypercomposition of this element with any other element from H is a singleton. That is:
■ A **left scalar** element c_l is an element such that $c_l\, a\in H, \forall a\in H$.

Next, regarding these elements, we have the Theorems [11]:

Theorem 1.1. *Let the hypergroup* H *contain a left (right) scalar unit* e. *If there exists any right (left) unit in* H, *then it is unique and equal to* e *and* e *the only left (right) scalar unit of* H. *If* H *contains an absolute unit, then there are no other units.*

Theorem 1.2. *If a hypergroup contains a bilaterally scalar element, then it contains an absolute unit element* e, *and the set of all the bilaterally scalars is a group having* e *as uni element. The group of scalars is called the* ***nucleus*** *of* H.

Theorem 1.3. *If* H *is a finite hypergroup which contains both a right and a left scalar, then all scalars are two sided.*

For the definition of the **symmetrical** (or **inverse** or **opposite**) elements of an element x, we should keep in mind that a hypergroup may have more than one unit elements. So we speak for the left (resp. right) symmetricals of an element x only with regard to a neutral element e. Thus the set of the left symmetrical of x as for the neutral e, denoted by $S_l(x,e)$, is the set $S_l(x,e)=\{u\in H \mid e\in ux\}$, (resp. $S_r(x,e)=\{u\in H \mid e\in xu\}$).

The **cancellation law** holds for a left scalar element w if any relation $wx=wy$ implies that $x=y$.

Theorem 1.4. *Let* H *be a hypergroup in which the cancellation law holds for left scalars. Then:*
i) The existence of a left scalar x, *implies the existence of a left scalar unit* e_x.
ii) To each left scalar x *there exists a unique left scalar inverse* x^{-1}, *such that* $xx^{-1}=x^{-1}x=e_x$.
iii) When xy *is a left scalar, then both* x *and* y *are left scalars.*

Besides the study on the hypergroups, an extensive study on the hypergroupoids has been carried out. For instance M. De Salvo [7], [8], [9] has studied the finite hypergroupoids of constant length (i.e. the ones in which all the hyperproducts xy have the same size) and in [10] he has studied the natural correspondence between the hypergroupoids and the combinatorial structures. Furthermore, R. Migliorato has defined a geometric structure for all hypergroupoids and has presented relevant properties.

2. On the sub-hypergroups

A non empty subset h of H is a **semi-subhypergroup** if $xy\subseteq h$ for every $x,y\in h$. Moreover if $xh=hx=h$ for every $x\in h$, then h is a **sub-hypergroup** of H.
A **sub-hypergroup** h of H is **right closed** (resp. left closed) in H if $ah \cap h=\phi$ (resp. $ha \cap h=\phi$) for every $a\in H\setminus h$.

Theorem 2.1. *The following are equivalent:*

- h *is right (resp. left) closed.*
- $(H\backslash h)h = H\backslash h$ [resp. $h(H\backslash h) = H\backslash h$] *(after [43])*
- $a{:}b \subseteq h$ (resp. $a{..}c \subseteq h$) *for every* $a,b \in h$ *(after [19])*

Theorem 2.2. *Let* A *be a non empty subset of a hypergroup* H *and let's use the notation:*

$$A_0 = A \cup AA \cup A{:}A \cup A{..}A$$

....

....

$$A_{n+1} = A_n \cup A_nA_n \cup A_n{:}A_n \cup A_n{..}A_n$$

....

Then $\langle A\rangle = \bigcup_{n\geq 0} A_n$ *is the least closed sub-hypergroup of* H, *which contains* A *[6].*

Theorem 2.3. *The intersection of two sub-hypergroups, if it is not void, it is a semi-subhypergroup, while the non void intersection of two closed sub-hypergroups is a closed sub-hypergroup.*

Remark The case of the hypergroups is different from the case of the groups, where the intersection of two subgroups is always a subgroup.

Another category of sub-hypergroups is the invertible **sub-hypergroups**, i.e. sub-hypergroups for which:

- if $c \in H$, $c' \in H$ and $ch \neq c'h$, then $ch \cap c'h = \emptyset$ (right invertible)
- if $c \in H, c' \in H$ and $hc \neq hc'$, then $hc \cap hc' = \emptyset$ (left invertible)

From their definition it derives that every invertible sub-hypergroup is also closed, but the opposite is not valid.

Theorem 2.4 *If* h *is right invertible in* H, *then [14]*

i) $c \in ch$, *for every* $c \in H$

ii) $a \in bh \Leftrightarrow b \in ah \Leftrightarrow ah = bh$ *for every* $a,b \in H$

iii) $(ch)_{c \in H}$ *is a partition of* H, *denoted* H/h *[resp.* $H\backslash h$ *when* h *is left invertible].*

If h and h′ are two sub-hypergroups, then the set $\Pi(h,h')$ of all the elements that are contained in some product made of factors from h and h′ is multiplicatively closed, but usually, is not a sub-hypergroup even when h and h′ are closed. We denote by $[h,h']$ the least sub-hypergroup which contains the elements of $h \cup h'$. Then:

Theorem 2.5. *If* h *and* h′ *are invertible in* H, *then* $\Pi(h,h')=[h,h']$ *and* $[h,h']$ *is an invertible sub-hypergroup.*

Y. Sureau in his theses [43] introduced two other types of sub-hypergroups, the ultra-closed and the conjugable. A sub-hypergroup h of H is called **right ultra-closed** (resp. left) in H, if for every $x \in H$ holds:

$$xh \cap x(H \setminus h) = \phi \text{ (resp. } hx \cap (H \setminus h)x = \phi).$$

If a sub-hypergroup is right and left ultra-closed, then it is called **ultra-closed** in H.
The way I see it, the ultra-closed sub-hypergroup most of the times behaves analogously to a bubble moving around, in a liquid which is at rest.
Imagine that the hypergroup H is the entire mass (liquid+gass), and the ultra-closed sub-hypergroup h is the gass somewhere in it. The hyperproduct xh causes the bubble to move to another place, while its previous place is now being occupied by liquid. Of course the bubble may expand as it moves towards the surface or shrink as it moves towards the bottom, but the entire mass remains the same.

Proposition 2.1 *The right (resp. left) ultra-closed sub-hypergroups in* H *are right (resp. left) invertible and closed sub-hypergroups in* H.

Theorem 2.6 *Let* H *be a hypergroup and* I_p *the set of partial identities, i.e.* $I_p=\{e \mid \exists\, x : x \in ex\} \cup \{e' \mid \exists\, y : y \in ye'\}$
If h *is a sub-hypergroup of* H, *then* h *is ultra-closed if and only if it is closed and contains* I_p.

Now if H is a hypergroup and h is a closed sub-hypergroup of H, h is called **conjugable** if and only if for every $x \in H$ there exists $x' \in H$ such that $x'x \subseteq h$. Y. Sureau has proved that this is equivalent to the fact that for every $x \in H$ there exists $x'' \in H$ such that $x'' \subseteq h$ (and also to the fact that for every $x,y \in H$ the hyperproduct xy is contained either in h or in the complemetary subset $H \setminus h$ of h in H). Then, for every $x \in H$, the set xhx' (resp. $x''hx$) is a conjugable sub-hypergroup of h, called a **conjugate** of h, and for every y satisfying the conditionyx $\subseteq$ h holds $y \in hx' = hy$, hence $xhx' = xhy$. If h is conjugable the it is also ultra-closed, hence invertible (the same holds for xhx' and $x''hx$). Moreover, two essential results that Y. Sureau has presented are:

i. for any equivalence relation R,H/R is a group if and only if R is a relation modulo a conjugable and invariant sub-hypergroup h of H (i.e. $H/R \approx H/h$)

ii. the heart of H,ω_H is the intersection of all conjugable sub-hypergroups of H

Lastly another category of sub-hypergroups is the invariant sub-hypergroups. A sub-hypergroup h of H is called **right semi-invariant**, if $xh \in hx$ for every $x \in h$, while it is called **invariant** if it is both right and left semi-invariant, i.e. $xh = hx$, for every $x \in h$. If h is invertible and H/h is a group, then h is called **strongly invariant**.

Theorem 2.7.

i) If h *is strongly invariant, then it is invariant as well.*

ii) Let G *be a group and* G',g *subgroups with* $g \in G'$ *and also let the hypergroups* $H = G/g$ *and* $h = G'/g$. *If* h *is invariant, then it is strongly invariant in* H. *[14]*

Some other important subsets of a hypergroup are the complete parts. The notion of the complete parts has been introduces by Koskas [12]. In the bibliography there appear many papers that analyse the subject from different points of view. P. Corsini [14] and Y. Sureau [43] have been mainly studying it withing the context of the general theory of hypergroups.
M. De Salvo has approached the subject in such a way that the combinatorial aspect of the theory arises.

Let A be a part of a semi-hypergroup H. A is called **complete** if the following is valid:

$$\forall h \in N, \forall (x_1,\ldots,x_n) \in H^n, \prod_{i=1}^{h} x_i \cap A \neq \phi \Rightarrow \prod_{i=1}^{h} x_i \in A$$

Complete closure C(A) of A in H is the intersection of all complete parts of H that contain A. A semi-hypergroup H is called **complete** if for every $x,y \in H$ $C(xy) = xy$. On this subject, R. Migliorato has introduced and studied the n-complete semi-hypergroups, which are a generalisation of the complete hypergroups. He has also shown that the complete hypergroups H are totally characterised by the structure of the group H/β and he has **calculated** the number of all the complete hypergroups of a given order.

Theorem 2.8. *Let* A *be a non empty part of* H. *If we put* $K_1(A)=A$, *and* $K_{n+1}(A)=\{x\in H \mid \exists\, p\in N, \exists\, (x_1,\ldots x_p)\in H_p : x\in\prod_{i=1}^{p} x_i, \prod_{i=1}^{p} x_i \cap K_n\neq\phi\}$ *then the complete closure* C(A) *of* A *is the union* $K(A)=\bigcup_{n\geq 1} K_n(A)$.

Theorem 2.9. *The relation:*

$$xKy \Leftrightarrow x\in C(\{y\})$$

is an equivalence relation, and the quotient H/k *is a group.*

3. On the homomorphisms

Let H and H′ be two hypergroupoids and P(H′) the power set of H′.

According to Krasner's definition a function $f:H\to P(H')$ is called **homomorphism** if $f(xy)\subseteq f(x)f(y)$ (1) for every $x,y\in H$. f is called **strong homomorphism** if the above relation holds as an equality i.e. $f(xy)=f(x)f(y)$ (2). A function $f:H\to H'$ is called **strict homomorphism** if (1) is valid. $f:H\to H'$ is called **normal** or **good homomorphism** if (2) is valid. Moreover, P. Corsini has introduced many new types of homomorphisms and has studied relevant properties in depth [2].

Theorem 3.1. *If* $f:H\to H'$ *is a good homomorphism then* Imf *is a sub-hypergroup of* H′. *If* $f:H\to P(H')$ *is a strong homomorphism then* Imf *is, in general, a semi-subhypergroup of* H′.

The image of a sub-hypergroup under a homomorphism or a strict homomorphism is, in general, neither a sub-hypergroup nor a semi-subhypergroup.

Theorem 3.2. *Let* $f:H\to H'$ *be a strict homomorphism. Then the inverse image of a semi-subhypergroup of* H′ *is a semi-subhypergroup of* H *and the inverse of a sub-hypergroup is a closed sub-hypergroup.*

So if H′ has a scalar neutral element e, then $f^{-1}(e)$ is a closed sub-hypergroup of H.

Theorem 3.3. *Let* $f:H\to H'$ *be a strict homomorphism,* X *a non empty subset of* H *and* $\langle X\rangle$ *the closed sub-hypergroup generated by* X, *Then*

i) $f(\langle X\rangle)\subseteq\langle f(X)\rangle$

ii) $\langle f(X)\rangle=\langle f(\langle X\rangle)\rangle$

iii) If $f(<X>)$ *is a closed sub-hypergroup of* H', *then* $<f(X)>=f(<X>)$
iv) $f(C(x))\subseteq C(f(x))$, *for every* $x\in H$.

Theorem 3.4. *If* $f:H\to H'$ *is a good homomorphism and* H' *has a scalar neutral element* e, *then* $f^{-1}(e)$ *is an invertible sub-hypergroup and* $H/f^{-1}(e)$ *is isomorphic to* H'.

Theorem 3.5. *Let* H *be a semi-hypergroup,* K *the equivalence relation of Theorem 2.9 and* $\varphi:H\to H/K$ *the canonical projection. If* S *is a semigroup and* $f:H\to S$ *is a homomorphism, then a homomorphism* $g:H/K\to S$ *exists such that* $g\varphi_H=f$.

Another notion, which is important for the theory of the hypergroups, is the one of the heart or core of the hypergroup. If H is a hypergroup and G a group, **Kernel** of the homomorphism $f:H\to G$ is the set

$$\ker f = \{x\in H \mid f(x)=1\}$$

Then **heart** or **core** ω_H of H is the kernel of the canonical projection φ_H.

Theorem 3.6. *The heart of a hypergroup* H *is the intersection of all the sub-hypergroups of* H *that are complete parts.*

Now, if $f:H\to H'$ be a strict homomorphism of hypergroups, then the kernel of f is the set $\ker f=f^{-1}(\omega_{H'})$.

Theorem 3.7. *Let* $f:H\to H'$ *be a strict homomorphism of hypergroups. Then*
i) $\ker f$ *is a complete part, which is also a sub-hypergroup.*
ii) $\ker f$ *is an invariant sub-hypergroup.*

4. Types of hypergroups

The hypergroup can be considered as one of the most general structures of the abstract Algebra and very often it is being enriched with axioms which vary in strength, but that are always defined by a specific mathematical or technological need. Already F. Marty in [16]has introduced more axioms and he has spoken of the "hypergroupe normal".

These new axioms created a great number of different branches, each one of which deals with different kinds of difficulties, resulting thus to the development of new technics, methods and tools in order to continue the research. In this part we present some types of hypergroups:

4.1. Regular: is a hypergroup if it has at least one identity and each element has at least one inverse.

On the regular hypergroups P. Corsini has studied the structure of their heart and he has presented a series of papers on this subject (see [3], [4], [5]).

R. Migliorato has introduced and studied the **totally regular** hypergroups, that is the hypergroups in which $\forall(x,y)\in H^2, \{x,y\}\subseteq xy$. A special case of totally regular hypergroupoids are the Steiner Hypergroupoids (in particular Steiner Hypergroup).

4.2. Reversible: is a regular hypergroup which also the following conditions:

i. If $y\in ax$, there exists an inverse a' of a, such that $x\in a'y$
ii. If $y\in xa$, there exists an inverse a'' of a, such that $x\in ya''$

4.3. Canonical: is a hypergroup $(H,.)$, that for every $x,y,z\in H$ satisfies the axioms:

i. $xy=yx$
ii. $x(yz)=(xy)z$
iii. there exists an element $1\in H$ for which $1x=x$
iv. for every $x\in H$ there exists one and only one $x'\in H$ shuch that $1\in xx'$.
v. $z\in xy \Rightarrow x\in zy'$

The name canonical has been given to this hypergroup by J. Mittas, who is the first one that studied it [22], [29]. The canonical hypergroup is strongly related to the structure of the hyperfield [13], the additive part of which led to the introduction of its axioms [21], [30].

Moreover, J. Mittas, having under consideration the definition of the valuated hyperfield [13], [28], has introduced the **valuated ultrametric** or more general, the **hypervaluated** or **hyper-ultrametric** canonical hypergroup and has proved that necessary and sufficient condition for canonical hypergroup to be such, is the validity of certain additional properties of purely algebraic type (i.e. properties that are expressed without the intervention of the valuation or resp. the hypervaluation). There derived thus three new types of canonical hypergroups:

the **strongly** canonical, which also satisfies the axioms:

S_1: $\forall x,a\in H$, $x\in x+a \Rightarrow x+a=x$

S_2: $\forall x,y,z,w\in H$ such that $(x+y)\cap(z+w)\neq\phi$, either $x+y\subseteq z+w$, or $z+w\subseteq x+y$ is valid, the **almost strongly** canonical, which also satisfies S_2 and AS: $\forall x,y\in H$ with $x\neq y$, either $(x-x)\cap(y-x)=\phi$ or $(y-y)\cap(y-x)=\phi$ is valid, instead of S_1 and the **superiorly** canonical, which is a strongly canonical that also satisfies S_3: $\forall x,y,z,w\in H$ such that $0\notin x+y$ and $z,w\in x+y$ holds $z-z=w-w$

S_4: If $x\in z-z$ and $y\notin z-z$ then $x-x\subseteq y-y$.

J. Mittas has presented a very deep and extensive study on this area, with a great number and variety of results [23], [24], [26], [27], [31], [32], [36], among which we mention the following theorem from the theory of the hypervaluated canonical hypergroups:

Theorem 4.1. *Necessary and sufficient condition for a canonical hypergroup to be hypervaluatable (strictly), is to be superiorly canonical.*

Apart from the canonical hypergroups, . Mittas has introduced several types of **polysymmetrical hypergroups** [25], [34], the regenerative hypergroups [33] and with other collaborators other types of hypergroups, such as the reticuled canonical hypergroups [42], hypergroups defined from a linear space [35] etc.

Motivated by Mittas' strongly canonical hypergroup, P. Corsini has introduced the **i.p.s. hypergroup**, i.e. a canonical hypergroup which also satisfies the axiom S_1. P. Corsini has analysed thoroughly this hypergroup, having proved among others that 8 is the maximum integer k such that the class of i.p.s. hypergroups of order k coinsides with the one of the strongly canonicals. Moreover, a non commutative canonical hypergroup, which has been called **quasicanonical** by P. Corsini and P. Bonansinga and **polygroup** by S. Comer [1] has been studied by themselves as well as by Ch. Massouros and M. De Salvo.

4.4. Join hypergroup is a commutative hypergroup (H,.), which for every x,y,z,w $\in$H satisfies the axiom:

$$x{:}y \cap z{:}w \neq \phi \Rightarrow xw \cap yz \neq \phi$$

This axiom has been introduced by W. Prenowitz, who has used a special type of this hypergroup for a very important foundation of several types of Geometries through the theory of the hypergroups [38], [39], [40], [41]. (Also see [18], [17])

In the space between the canonical and the join hypergroup, there appeared the Fortified Join hypergroup, which came into being during the analysis of problems in the theory of Languages and Automata with methods and tools from the theory of the Hypercompositional Structures [20].

4.5. Fortified join hypergroup is a join hypergroup (H,$\dotplus$) which also satisfies the axioms:

FJ_1 There exists a unique neutral element, denoted by 0, -the zero element of H - such that for every x$\in$H:

$$x \in x \dotplus 0 \text{ and } 0 \dotplus 0 = 0$$

FJ_2 For every $x \in H \setminus \{0\}$ there exists one and only one element $x' \in H \setminus \{0\}$ - opposite or symmetrical of x - denoted by $-x$, such that:

$$0 \in x + x'$$

The theory of the hypercompositional structures, beyond its independent great development and the study of the problems created from inside it, has already been used in order to describe and solve problems not only in algebra, but in other brances of mathematics as well, such as harmonic analysis, geometry, graph theory, combinatorial analysis, probability, code theory etc. Recently the theory has extended its influence into computer science with very promising prospects.

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