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FORTIFIED JOIN HYPERGROUPS AND JOIN HYPERRINGS

BY

GERASIMOS G. MASSOUROS

1. Introduction. The *Fortified Join Hypergroup* (FJH) is a hypergroup which derived from the introduction of the Theory of Hypercompositional Structures into the Theory of Automata and Languages [4], [5]. The Join Hypergroup, as it is known, [9], [2], is a commutative hypergroup $(H, +)$ in which the join axiom is also valid:

$$(J) \qquad (a : b) \cap (c : d) \neq \emptyset \;\Rightarrow\; a + d \cap b + c \neq \emptyset$$

where ":" is the induced hypercomposition, i.e. $a : b = \{x \in H \mid a \in b + x\}$.

In the theory of languages the expression $x+y$ is used, with x, y being words over an alphabet A, to state "either x or y". Thus, beginning with the fact that $x + y$ is basically a biset, the set of the words A^* over an alphabet A, endowed with the hypercomposition $w_1 + w_2 = \{w_1, w_2\}$ becomes a join hypergroup [4], [5]. The special join hypergroup which derives in this way from the theory of languages, is named B-hypergroup. Furthermore, A^* is a semigroup with regard to the concatenation of the words and beyond that it has been proved [5] that this operation is bilaterally distributive as for the hyperoperation defined in the set of the words. Thus in a natural way, there derived the multiplicative–hyperadditive structure which was named *hyperringoid*.

A *hyperringoid* is a non void set Y with an operation "$\cdot$" and a hyperoperation "+" that satisfy the axioms:

i. *$(Y, +)$ is a hypergroup*

ii. *$(Y, \cdot)$ is a semigroup*

iii. *the operation is bilaterally distributive to the hyperoperation.*

If $(Y, +)$ is a join hypergroup, then the hyperringoid is called *join hyperringoid*. The hyperringoid of all the words is a special join hyperringoid, which was named B–hyperringoid.

Next, starting with the notion of the empty set of words from the theory of languages, the join hypergroup is being enriched with axioms. Actually, the use of the "*null word*" has led to the introduction of a non scalar neutral element in the join hypergroup with regard to which, every element has a unique opposite. Thys the *fortified join hypergroup* was defined. This new hypercompositional structure $(H, +)$ is a join hypergroup which also satisfies the axioms:

FJ$_1$ *There exists a u n i q u e neutral element, denoted by* 0*, the zero element of* H*, such that for every* $x \in H$*:* $x \in x + 0$ and $0 + 0 = 0$

FJ$_2$ *For every* $x \in H \setminus \{0\}$ *there exists one and only one element* $x' \in H \setminus \{0\}$*, the opposite or symmetrical of* x*, denoted by* $-x$*, such that* $0 \in x + x'$.

Especially for the case of languages, the fortified join hypergroup which corresponds to them and which motivated the development of this new structure is the *dilated B–hypergroup*. In this hypergroup very element is selfopposite. More precisely the hypercomposition of this structure has been defined in the following way:

$$x + y = \begin{cases} \{x, y\} & \text{if } x \neq y \\ \{0, x\} & \text{if } x = y \end{cases}.$$

Now, if in the definition of the hyperringoid the additive part is a fortified join hypergroup, then the hyperringoidis called *fortified join hyperringoid* or *join hyperring*. Yet, if this additive part is a dilated B–hypergroup we have the *dilated B–hyperringoid* [5].

2. Fortified join hypergroup. For the following let $(H, +)$ be a FJH. Then:

Theorem 2.1. *For every* $x \in H$ *holds:* $0 + x \subseteq \{0, x\}$.

Proof. Obviously this holds for $x = 0$. Let $x \neq 0$ and $y \in 0 + x$, then $x \in y : 0$ (1). Also $0 \in x - x$, so $x \in 0 : (-x)$ (2). From (1) and (2) it derives that $(y : 0) \cap [0 : (-x)] \neq \emptyset$, from which, according to the join axiom, $(y - x) \cap (0 + 0) \neq \emptyset$, and so $0 \in y - x$. If $y \neq 0$, then $y = x$ and if this were valid for every $x \in H$, i.e. $0 + x = x$, then 0 would have been a scalar element of H and therefore H would have been a canonical hypergroup [7], [2]. So there exist $x \in H$, such that $0 \in 0 + x$ and therefore $x + 0 = \{x, 0\}$. Consequently, in a FJH, it holds $0 + x \subseteq \{0, x\}$.

According to the above theorem, there are two kinds of elements in every Fortified Join Hypergroup:

Definition 2.1. An element $x \in H$ is called c–element if $0 + x$ is the singleton $\{x\}$, while it is called attractive or a–element if $0 + x$ is the biset $\{0, x\}$.

A FJH has at least one c–element, the 0 element, since $0 + 0 = 0$. All the c–elements of a FJH consist its subset C, while all the a–elements consists its subset A. Obviously, these two sets form a partition in H.

Example 2.1. Let H be a set totally ordered and symmetrical around a center denoted by $0 \in H$, as for which a partition $H = H_1 \cup \{0\} \cup H_2 =$ $= H^- \cup \{0\} \cup H^+$ can be defined, such that $x < 0 < y$ for every $x \in H^-$ and $y \in H^+$, $x \le y \Rightarrow -y \le -x$ for every $x, y \in H$ (where $-x$ is the symmetrical of x as for 0).

Enriching $H^+ \cup \{0\}$ with the hypercomposition:

$$x \dot{+} y = \begin{cases} \max\{x,y\} & \text{if } x \neq y \\ [0,x] & \text{if } x = y \end{cases}$$

we get a canonical hypergroup [6], [8].

Also $H^- \cup \{0\}$ with the hypercomposition:

$$x \dot{+} y = \begin{cases} \{x,y\} & \text{if } x \neq y \\ [x,0] & \text{if } x = y \end{cases}$$

becomes a fortified join hypergroup.

The hypercomposition "+" defined in the following way:

$$x + y = \begin{cases} \max\{x,y\} & \text{if } x \neq y \text{ and } x \in H^+ \text{ or } y \in H^+ \\ \{x,y\} & \text{if } x \neq y \textit{ and } x, y \in H^- \\ \{x,0\} & \text{if } y = 0 \textit{ and } x \in H^- \\ x & \text{if } y = 0 \textit{ and } x \in H^+ \\ [x,0] & \text{if } x = y \in H^- \cup \{0\} \\ [0,x] U H^- & \text{if } x = y \in H^+ \end{cases}$$

makes $(H, +)$ a fortified join hypergroup with selfopposite elements. The elements of H that belong to $H^+ \cup \{0\}$ are c–elements, while all the elements that belong to H^- are a–elements.

Proposition 2.1. *Every canonical hypergroup can be embedded in a* FJH.

For the canonical and the attractive elements we have the properties:

Proposition 2.2.

i. $x \in A \Rightarrow -x \in A$

ii. *If* $x, y \in A$, *then* $\{x, y\} \subseteq x + y$

iii. *If* $x, y \in A$, *then* $x + y \subseteq A \cup \{0\}$

iv. $0 : 0 = A \cup \{0\}$.

Proposition 2.3.

i. $x \in C \Rightarrow -x \in C$

ii. *If* $x \in C \setminus \{0\}$, *then* $A \subseteq x - x$

iii. *If* $x, y \in C$, *with* $y \neq -x$, *then* $x + y \subseteq C$.

Proposition 2.4. *If* $y \in C \setminus \{0\}$ *and* $x \in A$, *then* $y = x + y$.

Corollary 2.1.

i. *If* $x, y \in A$, *then* $x : y \subseteq A \cup \{0\}$

ii. *If* $z \in C$, *then* $A \subseteq z : z$.

In this point let's see again Example 2.1, where $x, y \in x + y$ for every x, y from H^-, with $x \neq y$. Here $x \in x + y \Rightarrow x \in x - y = x + y$, while $y \notin x - x = x + x = [x, 0]$ except the case that $x < y$ in which $y \in x - x$. Therefore the reversibility, i.e. the property $z \in x + y \Rightarrow y \in z - x$, for every three elements $x, y, z \in H$, is not generally valid. In a FJH it holds:

Proposition 2.5. *For every* $x, y, z \in H$ *with* $z \in x + y$, *the reversibility is generally valid, except for the cases:*

i. $z = y = 0$, $x \in A$ *and*

ii. $z = x \neq y$ *with* $x, y \in A$,

for which respectively it holds:

$0 \in x + 0 \Rightarrow 0 \in 0 - x$, *while* $x \notin 0 - 0$

$x \in x + y \Rightarrow x \in x - y$, *while* $y \notin x - x$ *in general.*

Thus in the FJHs the reversibility can be viewed as a property that is "partially" valid and so:

Proposition 2.6. *Every* FJH $(H, +)$ *is partially reversible, that is, for every* $x, y, z \in H$ *it holds:* $z \in x + y \Rightarrow$ *either* $y \in z - x$ *or* $x \in z - y$ (*without the one always rulling out the other*).

Corollary 2.2. $z \in x + y \Rightarrow y \in z - x$, *for every* $x, y \in C$.

Another aspect worth mentioning for the "partial" reversibility is its relation to the property: $-(x+y) = -x-y$. In a canonical hypergroup $(K,+)$, the above property is true for every $x, y \in K$ and it is equivalent to the reversibility [7]. In a FJH though, it holds:

Proposition 2.7. *The partial reversibility in the* FJHs *is equivalent to the property:* $-(x+y) = -x-y$, *for every* $x, y \in H$ *with* $y \neq -x$.

When $y = -x$ this property is not generally valid, i.e. in the general case $-(x-x) \neq x-x$, as it can be seen in the Example:

Example 2.2. Let again H be a set totally ordered and dense as for the order relation and also symmetrical around a center denoted by $0 \in H$. Using the same notation as in Example 2.1, let's introduce the hypercompositions:

$$x+y = \{x,y\}, \text{ if } y \neq -x \quad \text{and} \quad x+(-x) = [0,|x|] \cup \{-|x|\}.$$

The deriving structure is a FJH in which for every $x \neq 0$ it holds $x - x \neq -(x-x)$, since $-(x-x) = [-|x|, 0] \cup \{|x|\}$.

Relatively we also have the Proposition:

Proposition 2.8. *If* x *is a c-element of a* FJH, *then* $-(x-x) = x-x$.

Definition 2.2. An element $x \in H$ such that $-(x-x) = x-x$ is called *normal*. Otherwise it is called *abnormal*. A FJH having only normal elements is called *normal* FJH, while if it contains at least one abnormal element it is called *abnormal* FJH.

In the following , a few elements on the subhypergroups of a FJH are being presented. Since a FJH is a join hypergroup, in accordance to the theory of the join hypergroups [9], [1], [2], [5], there will exists subhypergroups that are *join* and others that are not join, that is subhypergroups that satisfy the join axiom inside them and others that don't.

Proposition 2.9. *A subhypergroup of a join hypergroup is join if and only if it is closed.*

Yet, since every closed subhypergroup of a hypergroup contains all the neutral elements [7], every join subhypergroup h of a FJH contains the zero element. Moreover $-x \in 0 : x$ and since in a closed subhypergroup h $0 : x \subseteq h$ for every $x \in h$ [3], it derives that $-x \in h$. So:

Proposition 2.10. *Every join subhypergroup of* FJH *is a* FJH *itself with the same zero.*

Proposition 2.11. *A non void subset* h *of* H *is a join subhypergroup of* H *if and only if* $x : y \subseteq h$ *and* $x - y \subseteq h$, *for every* $x, y \in h$.

In the FJHs there exists another kind of subhypergroups, the symmetrical ones. Symmetrical is a subhypergroup of a FJH for which $-x \in h$ for every $x \in h$. From Proposition 2.11 it derives that every join hypergroup is a symmetrical one. The opposite is not valid though. Therefore there exist three kinds of subhypergroups in a FJH:

i. *the join subhypergroups,*
ii. *the symmetrical subhypergroups that are not join and*
iii. *subhypergroups that are not symmetrical (and so not join).*

The study of these subhypergroups is extensive and it is worth mentioning that the minimum, in the sense of inclusion, join subhypergroup h of H is the set $A^{\hat{}}$ of the attractive elements and the zero element. Also it has been proved [5] that $A^{\hat{}} = 0 : 0$. Moreover if a symmetrical subhypergroup contains a c–element different than 0, then it is a join one.

3. Join hyperrings. Let $(Y, +, \cdot)$ be a Join Hyperring (JH). The distinction of the elements of the additive hypergroup of Y into c and a–elements endows its multiplication with certain properties:

Proposition 3.1. *The product of two c–element, while the result of the product of a c–element with an a–element is the element 0.*

Corollary 3.1. *If a proper* JH *has no zero divisors then it does not contain any c–elements other than 0.*

Proposition 3.2. *In a* JH *which contains a c–element different from zero, the product of two a–elements equals to 0.*

As it is known, in a hyperring hold the "classical" algebra properties:

i. $x(-y) = (-x)y = -xy$

ii. $(-x)(-y) = xy$

iii. $\omega(x - y) = \omega x - \omega y, \quad (x - y)\omega = x\omega - y\omega.$

(that are being proved with the help of the addition). These properties are not generally valid though in a JH as it can be seen in the following Example:

Example 3.1. Let S be a multiplicative semigroup having a bilaterally asborbing element 0. Consider the set: $P = (\{0\} \times S) \cup (S \times \{0\})$. In this set introduce a hypercomposition "+" as follows:

$(x, 0) + (y, 0) = \{(x, 0), (y, 0)\}$
$(0, x) + (0, y) = \{(0, x), (0, y)\}$

$(x,0)+(0,y)=(0,y)+(x,0)=\{(x,0),(0,y)\}$ for $x \neq y$ and
$(x,0)+(0,x)=(0,x)+(x,0)=\{(x,0),(0,x),(0,0)\}$.
The structure $(P,+)$ which derives in this way is a FJH having neutral element the $(0,0)$. Also for every element $(x.0)$ the opposite is the element $(0,x)$. For this multiplication we observe that

$$(x,0)(y,0)=(xy,0)$$

while $(x,0)(0,y)=(0,)$ and $(0,x)(y,0)=(0,0)$. That is, if $x'=(x,0)$ and $y'=(y,0)$, it is $x'y'\neq 0'$, and so $-x'y'\neq 0'$ while $x'(-y')=(-x')y'=0'$. Moreover $(0,x)(0,y)=(0,xy)$, which is the opposite of $(xy,0)$, i.e.,

$$(-x')(-y')=-x'y'.$$

It derives therefore that from the above properties neither the first nor the the second (and consequently nor the third one) is valid in this example. The following proposition gives conditions under which these properties are valid in a JH $(Y,+,\cdot)$.

Proposition 3.3. *The properties under consideration are valid:*

i. *if* $-x,-y,x,\omega$ *are not divisors of 0.*

ii. *if at least one of the participating elements is a c–element.*

The consideration of the corresponding properties of the FJH, which is the additive part of the JH and the interaction of the hypercomposition on the operation leads to a great ammount of interesting results and properties of the JH. Among them it is worth mentioning thet every JH having no divisors of zero, contains only normal elements.

In the FJHs certain types of subhypergroups appeared . In an analogous way they appear here the join subhyperrings, and the symmetrical subhyperrings as well as the semi-subhyperrings. Among the results that have been proved are the ones concerning the c– and a–elements. Indeed, the set $A\hat{}$ containing all the a–elements and the zero element is a bilaterally join hyperideal of Y and furthermore it is the minimum one, in the sense of inclusion. Moreover every symmetrical subhyperring is a subset of the minimum join subhyperring. Also a symmetrical subhyperring containing a non zero c–element is a join one.

The study that has been carried out up to this point is deep and extensive. For example, concerning the characteristic of the hyperringoids it has been proved that every proper normal join hyperringoid without divisors of 0 is of characteristic 1 and therefore every dilated B–hyperringoid is of characteristic 1. Also systems and inequalities are being solved and their solutions create sets with special interest in the Theory of Languages and Automata.

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54, Klious st.,
155 61 Cholargos
Athens
GREECE