

ISSN 0041-9109

**ANALELE ȘTIINȚIFICE**  
**ALE**  
**UNIVERSITĂȚII «AL. I. CUZA»**  
**DIN IASI**  
**(SERIE NOUĂ)**

**INFORMATICĂ**



**TOMUL III**

**1994**

**U** Editura Universității «Al. I. Cuza»



# Hypercompositional Structures in the Theory of the Languages and Automata

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## Abstract

The theory of Languages and Automata is being viewed under the light of theory of the Hipercompositional Structures. For this purpose, not only the already existing structures have been properly used, but new ones are introduced as well. This article is a brief presentation of such Hypercompositional Structures, the way there are being used and some results that they can lead to.

ASM-Clasification number: 68Q45, 08A70, 20N20, 68Q70

## 1 Automata and Hipercompositional Structures

An automaton is a 5-tuple  $(A, S, s_0, \delta, F)$  where  $A$  is the input alphabet,  $S$  is a finite nonempty set of states,  $s_0 \in S$  the start state,  $\delta$  the state transition function:  $\delta : S \times A \rightarrow S$  and  $F \subseteq S$  the set of final states. Sometimes it is convenient to use the extended transition function  $\delta^* : S \times A^* \rightarrow S$ , instead of  $\delta$ , when working in the set of the words  $A^*$  over the alphabet  $A$ , which is defined recursively as follows:

- i)  $(\forall s \in S)(\forall a \in A) \delta^*(s, a) = \delta(s, a)$
- ii)  $(\forall s \in S) \delta^*(s, \lambda) = s$  where  $\lambda$  is the empty string.
- iii)  $(\forall s \in S)(\forall x \in A^*)(\forall a \in A) \delta^*(s, ax) = \delta^*(\delta(s, a), x)$

From the quintuple defining the automaton, the set of the states  $S$  can receive the structure of a hypergroup through the proper definition of certain hypercompositions. In this way hypergroups are been *attached* to the automaton and describe its structure and operation. According to their definition, these *attached hypergroups* are:

- a) The attached order hypergroup,
- b) The attached grade hypergroup,

- c) The attached hypergroup of the **paths**, and
- d) The attached hypergroup of the **operation**.

Especially the order hypergroup and the grade hypergroup can lead to the creation of the minimum automaton that accepts the same language with the initial one [2].

For the following let  $M = (A, S, s_0, \delta, F)$  be an automaton, deterministic or not. Then:

#### a. The attached ORDER hypergroup

The operation of the automaton can be seen as the movement from state to state, according to the input letter and function  $\delta$ . So the automaton can reach a state different than  $s_0$  only after it has read a string of letters from  $A$ . Also it is possible for some state  $s_i$  to be reached from  $s_0$ , through different paths, that is after the input of different words, or words with different number of letters, i.e. different lengths.

**Definition 1** The "order" of a state  $s \in S$ , denoted by  $\text{ord } s$ , is the minimum of the length of words that lead from the start state  $s_0$  to  $s$ .

Obviously  $\text{ord } s_0 = 0$ .

Thus it is possible to exist one or more "unreacheable from  $s_0$ " states. Apparently these states have no influence in the operation of the automaton and therefore their order is not necessary to be defined.

This order defines an equivalence relation and therefore it leads to the creation of a partition of the set of states. So,  $s_1 \circ s_2$  if  $\text{ord } s_1 = \text{ord } s_2$ , where  $s_1$  and  $s_2$  are two states from  $S$  and  $\circ$  is the *order equivalence*.

This equivalence relation on the set of the states have the following properties generalized as:

- i The set of the classes is isomorphic to a subset of the set of the natural numbers  $\mathbf{N}$ , and therefore
- ii The set of the classes mod  $\circ$  is totally ordered.
- iii There exists a minimal class.
- iv Every class "covers" another, except that of  $s_0$  which does not cover other class.

In a similar way it is possible to define the order of an element of an arbitrary set  $T$ , provided that there exists an equivalence relation  $R$  in  $T$  which satisfies these properties. In this case the order of the element is the order of its class, i.e. the corresponding natural number according to the considered isomorphism.

According to the definition of the hypercomposition, different types of hypergroups can be introduced in  $S$ , with the use of the notion of order. The following are different cases of commutative (by definition) attached order hypergroups:

If for every  $\chi, \Psi \in S$ :

1<sup>st</sup>

$$\chi + \Psi = \begin{cases} \Psi, & \text{if } \text{ord}\chi < \text{ord}\Psi \\ \bigcup_{\text{ord } z \leq \text{ord } x} C_z, & \text{if } \text{ord}\chi = \text{ord}\Psi \end{cases}$$

the deriving structure  $(S, +)$  is a canonical polysymmetrical hypergroup [8] with neutral element the minimal element of  $S$  (the start state) in which for every  $\chi \in S$  it is true that  $S(\chi) = C_x$ , and so  $\chi \in S(\chi)$  (selfopposite, where  $S(\chi)$  is the set of the symmetrical elements of  $\chi$ ).

2<sup>nd</sup>. If in the above definition the strict order  $<$  is used instead of  $\leq$ , i.e. if:

$$\chi + \chi' = \bigcup_{\text{ord } z < \text{ord } x} C_z \quad \text{for } \chi' \in C_x$$

the structure  $(S, +)$  is not a hypergroup, except for the case when  $S$  is totally ordered. Indeed, for every  $\chi \in S$  and  $\chi' \in C_x$ , the associativity:

$$(\chi + \chi) + \chi' = \chi + (\chi + \chi') \implies \bigcup_{\text{ord } w \leq \text{ord } x} C_w + \chi' = \chi + \bigcup_{\text{ord } z \leq \text{ord } x} C_z$$

holds only if  $\chi = \chi'$  and so  $C_x = \{\chi\}$ . Then we have the superiorly canonical hypergroup [6], [7] with

$$\chi + \Psi = \begin{cases} \max\{x, \Psi\}, & \text{if } \chi \neq \Psi \\ [s_0, \chi], & \text{if } \chi = \Psi \end{cases}$$

This is the case, when the states of the automaton is totally ordered.

3<sup>rd</sup>

$$\chi + \Psi = \begin{cases} \Psi, & \text{if } \text{ord}\chi < \text{ord}\Psi \\ \bigcup_{0 \neq \text{ord } z < \text{ord } x} C_z, & \text{when } \text{ord}\chi = \text{ord}\Psi \text{ and } s_0 \neq \chi \neq \Psi \neq s_0 \\ \bigcup_{\text{ord } z \leq \text{ord } x} C_z, & \text{when } \chi = \Psi \end{cases}$$

Now the hypergroup is canonical with selfopposite elements.

4<sup>th</sup>

$$\chi + \Psi = \begin{cases} C_\Psi, & \text{if } \text{ord}\chi < \text{ord}\Psi \\ \bigcup_{\text{ord } z \leq \text{ord } x} C_z, & \text{if } \text{ord}\chi = \text{ord}\Psi \end{cases}$$

The structure  $(S, +)$  is a generalised canonical polysymmetrical hypergroup [8] having  $s_0$  as a non scalar neutral element, in which  $S(\chi) = C_x, \forall \chi \in S$ .

5<sup>th</sup>. If the case  $\text{ord } \chi \neq \text{ord } \Psi$  of the above definition is altered as bellow:

$$\begin{aligned} \chi + \Psi &= C_\Psi, & \text{for } \text{ord } s_0 \neq \text{ord } \chi < \text{ord } \Psi \\ \text{and } s_0 + \chi &= \chi + s_0 = \chi, & \text{for every } \chi \in S \end{aligned}$$

then the structure is again a canonical polysymmetrical hypergroup with  $S(\chi) = C_x$

6<sup>th</sup>

$$\chi + \Psi = \begin{cases} C_\Psi, & \text{if } \text{ord } \chi = \text{ord } \Psi \\ \bigcup_{\text{ord } z \leq \text{ord } x} C_z, & \text{if } \text{ord } \chi < \text{ord } \Psi \end{cases}$$

Now the structure is a generalised canonical polysymmetrical hypergroup having a non scalar neutral element ( $s_0$ ) and  $S(\chi) = S \setminus C_x$  for every  $\chi \in S, \chi \neq s_0$ .

If in the case  $\text{ord } \chi \neq \text{ord } \Psi$  of this definition, it is assumed that  $\text{ord } z \leq \min(\text{ord } \chi, \text{ord } \Psi)$  instead of  $\text{ord } z \leq \max(\text{ord } \chi, \text{ord } \Psi)$ , then the hypermonoid does not satisfy the "reproductivity" and therefore the hypercomposition does not "give" a hypergroup.

7<sup>th</sup>

$$\chi + \Psi = \begin{cases} C_\Psi, & \text{if } \text{ord } \chi < \text{ord } \Psi \\ \bigcup_{0 \neq \text{ord } z \leq \text{ord } x} C_z, & \text{when } \text{ord } \chi = \text{ord } \Psi \text{ and } s_0 \neq \chi \neq \Psi \neq s_0 \\ \bigcup_{\text{ord } z \leq \text{ord } x} C_z, & \text{when } s_0 \neq \chi = \Psi \end{cases}$$

and

$$s_0 + \chi = \chi + s_0 = \chi \text{ for every } \chi \in S$$

The derived hypergroup is a canonical one with selfopposite elements.

#### b. The attached GRADE hypergroup

It is possible in an automaton to have states that lead to final states with exactly the same word. If such states exist, it is really of no importance to keep track from which one the automaton passes in order to reach a final state. They are considered equivalent. So in the set of the states of an automaton another equivalence relation  $R$  can be introduced through the notion of the grade.

**Definition 2** Let  $s$  be a state of an automaton  $(A, S, s_0, \delta, F)$ . The grade of this state  $s$  is the set:

$$\text{grad } s = \{x \in A^* \mid sx \in F\}$$

where  $A^*$  is the set of the words over the alphabet  $A$ . From the definition it follows that  $\text{grad } s_0$  is the language accepted by the automaton.

Two states  $s_1$  and  $s_2$  are equivalent, if they have the same grade, i.e.

$$s_1 \text{ R } s_2 \text{ if } \text{grad } s_1 = \text{grad } s_2$$

Let's denote by  $C(s)^R = C_s^R$  the class which contains the state  $s$ . The hypercomposition:

$$s_1 + s_2 = C_{s_1}^R \cup C_{s_2}^R$$

makes the set of the states  $S$  a join hypergroup. If the hypercomposition is defined in the following way:

$$s_1 + s_2 = \begin{cases} C_{s_1}^R \cup C_{s_2}^R & \text{if } C_{s_1}^R \neq C_{s_2}^R \text{ and } s_1, s_2 \neq s_t \\ C_{s_1}^R \cup \{s_t\} & \text{if } C_{s_1}^R = C_{s_2}^R \end{cases}$$

where  $s_t$  is the only final state of the automaton or the conventional final state, then the hypergroup  $(S, +)$  is a join polysymmetrical one [2], [3].

### c. The attached hypergroup of the PATHS

In the set  $S$  there may exist states for which no letter, or word can lead from the one to the other. Of course there exist states that can be reached, one from the another. The latter are called connected, according to the definition:

**Definition 3** *The state  $s_2$  of  $S$  will be named **connected** to the state  $s_1$  of  $S$  if there exists  $\omega \in A^*$  such that  $s_2 = s_1\omega$ . If  $\omega$  is one letter only, then the state  $s_2$  is called **successive** to  $s_1$ .*

The fact that  $s_2$  is connected (successive) to  $s_1$  does not imply that  $s_1$  is connected (successive) to  $s_2$ .

The set of the states with the hypercomposition:

$$s_1 + s_2 = \begin{cases} \{s \in S | s = s_1\omega \text{ and } s_2 = s_y \text{ with } \omega, y \in A^*\} & \text{if } s_2 \text{ is connected to } s_1 \\ \{s_1, s_2\}, & \text{if } s_2 \text{ is not connected to } s_1 \end{cases}$$

becomes a non commutative hypergroup [4]. This hypergroup is needed for the proof of Kleene's theorem with the use of tools and methods from the Theory of Hypercompositional Structures [4].

### d. The attached hypergroup of the OPERATION

In the previous three kinds of the attached hypergroups, the automaton has been approached as a static mechanism and references were made to the possible movements from one states to another. The operation of the automaton though

takes place in real time and so "time" is one of the factors that are involved. This basic ideas gave birth to the attached hypergroup of the operation.

Let an automaton be in state  $s_i$  and in the next moment (clock pulse) it reads a letter from the alphabet  $A$  that causes either a movement to another (successive) state  $s_j$  or a loop back to  $s_i$ . Being in  $s_i$  in the next clock pulse is different than the previous situation, since this *new*  $s_i$  is a somehow different state. To point out exactly this difference, it is convenient to consider the cartesian product  $S \times \mathbf{N}$ , where  $S$  the set of the states and  $\mathbf{N}$  the non negative integers, presenting thus the states along with the corresponding clock pulse during which it has been reached. So the pair  $(s_i, t)$ , which means that the automaton was in state  $s_i$  during the clock pulse  $t$ , shows exactly the difference from  $(s_i, t + 1)$ , which means that the automaton  $s_i$  during the next clock pulse too. A simpler and more convenient notation that will be used from now on is  $s_i^t$  instead of  $(s_i, t)$ .

Now, in the set of states there can exist states in which the automaton can be "found" during the clock pulse  $t$  and others where it can not. For instance, the automaton cannot be found in a state with order 5 during the clock pulse 3. So a the notation for the "activated" elements may be introduced:

**Definition 4** *An element  $s_i^t$  of the cartesian product  $S \times \mathbf{N}$  is called activated if, after  $t$  clock pulses, the automaton can be found in the state  $s_i$ .*

Moreover the notions of the successive and the connected elements may be generalised for the set  $S \times \mathbf{N}$  and so the elements  $s_j^r$  and  $s_i^t$  are called successive if the state  $s_j$  is successive to the state  $s_i$  and  $r = t + 1$  and also the element  $s_j^r$  is called connected with  $s_i^t$  if the state  $s_j$  is connected with the state  $s_i$  and  $t < r$ . After this generalisation, the set  $A \subseteq S \times \mathbf{N}$  of the activated elements becomes a non commutative hypergroup with the hypercomposition [5]:

$$s_i^m + s_j^n = \begin{cases} \{\delta A^*(s_i^m, x) | x \in \text{Prefix}(r), \delta A^*(s_i^m, r) = s_j^n\} \\ \quad \text{if } m < n \text{ and } s_i^m, s_j^n \text{ are connected} \\ \{s_i^m, s_j^n\}, \text{ if } s_i^m, s_j^n \text{ are not connected} \end{cases}$$

where  $\delta A^*$  is the generalisation of the extended state transition function  $\delta^*$ , that is

$$\delta A^*(s_i^t, x) = \delta^*(s_i, x)^{t+|x|}$$

Using this hypergroup and through a certain procedure, among others, all the states at which the automaton can possibly be found, at a given time  $t$ , may be effectively determined [5].

## 2 Languages and Hypercompositional Structures

In the papers [2] and [4], presented at the 4th and 5th AHA, appears explicitly how the expression  $a + b$  from the Theory of Languages can be interpreted as a



hypercomposition  $(a + b) = \{a, b\}$ , leading thus to the consideration of a special kind of a join hypergroup [1], the B-hypergroup. Also the necessity of the existence of a unique, non scalar zero element led to the introduction of the Dilated B-hypergroup, which is a Fortified Join Hypergroup (FJH), i.e. a Join Hypergroup  $H$  having a unique neutral element  $0$ , such that  $x \in x + 0$  and  $0 = 0 + 0$  for every  $x \in H$  and also, for every  $x \in H \setminus \{0\}$  there exists one and only one element  $x' \in H \setminus \{0\}$  for which  $0 \in x + x'$  ( $x'$  is denoted by  $-x$ ) [3].

Yet in the Theory of Languages the words consist of letters from an alphabet, written down side by side. The same thing can happen with strings and this whole procedure is called concatenation. The concatenation of the word is an operation and the set of the words  $A^*$  over the alphabet  $A$  is a monoid with operation the concatenation [2]. Moreover, the operation of concatenation is bilaterally distributive to the above hypercomposition, giving birth thus to the B-hyperringoid [2]. Generally a hyperringoid  $(Y, +, \cdot)$  is a multiplicative-hyperadditive structure, where  $(Y, +)$  is a hypergroup,  $(Y, \cdot)$  is a semigroup and the multiplication is bilaterally distributive to the hypercomposition. If  $(Y, +)$  is a join Hypergroup, then  $(Y, +, \cdot)$  is a Join Hyperringoid, while if the additive hypergroup of the Hyperringoid is a FJH then the structure is a Fortified Join Hyperringoid or Join Hyperring. In the case of the languages the Hyperringoid  $A^*$  is called linguistic Hyperringoid and  $\underline{A}^* = A^* \cup \{0\}$  is called Dilated Linguistic Hyperringoid. It must be mentioned though that every B-Hyperringoid is not a linguistic one. Indeed every element of the Linguistic Hyperringoid (word) has a unique factorization into the elements (letters) of the alphabet. Therefore these elements (letters) from a finite prime subset of the Linguistic Hyperringoid, that is a finite set of prime and irreducible elements such that every one of this elements (word) has a unique factorisation with factors from the prime subset. Thus from every non commutative free monoid with finite basis a Linguistic Hyperringoid can be derived.

### 3 Hypermoduloids – Supermoduloids

In an automaton the words of the language cause the system to move from state to state. There appears thus the action of a set of operators which are the elements of the set  $A^*$ , on the set of the states. In part two of this paper we show how can  $A^*$  be organised into the form of a certain hypercompositional structure, the Linguistic Hyperringoid. The definition of the hypermoduloid and the supermoduloid [4], can be naturally derived:

**Definition 5** *If  $M$  is a Hypergroup and  $Y$  a Hyperringoid of operators over  $M$  such that for every  $k, \lambda \in Y$  and  $s, t \in M$  the axioms:*

$$i (sk)\lambda = s(k\lambda),$$

$$\text{ii } (s + t)\lambda = s\lambda + t\lambda,$$

$$\text{iii } s(\lambda + k) \subseteq s\lambda + sk$$

hold, then  $M$  is called a **hypermuloid** over  $Y$ . If  $Y$  is a set of hyperoperators, that is if there exists an external hyperoperation from  $M \times Y$  to  $P(M)$  satisfying the axiom i, then  $M$  is called a **supermuloid** over  $Y$ .

If in a hyperringoid  $Y$ , a congruence relation  $R$  is defined, then the quotient set  $Y/R$  becomes a hypermuloid over  $Y$  [4]. Thus, depending on whether  $rk(R)$  is finite or not, the deriving hypermuloid will be finite or not. So if  $Y$  is a linguistic hypermuloid and  $rk(R) < +\infty$ , then  $Y/R$  is a finite hypermuloid. The elements of such a hypermuloid can represent the states of an automaton which can be completely defined if one of the classes of  $Y/R$  is viewed as the start state and a set of classes as the final states.

Let's see the above working on the binary counter. The set of the natural numbers becomes a linguistic hyperringoid having as a prime subset the singleton  $\{1\}$ , operation (concatenation) the  $xy = x + y$  and hypercomposition the  $x \uparrow y = \{x, y\}$  for every  $x, y \in \mathbf{N}$ . In  $\mathbf{N}$  we consider the equivalence relation *mod*  $n$  i.e.

$$x = y(\text{mod } n) \iff |x - y| = kn, \quad k \in \mathbf{N}$$

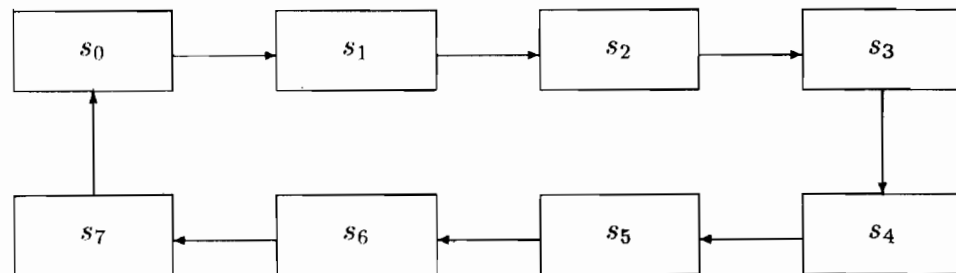
This relation is also a homomorphic one and thus a congruence relation. Then *mod*  $n$  creates  $n$  classes in  $\mathbf{N}$  and so  $\mathbf{N}/\text{mod } n$  is a hypermuloid with  $n$  elements. Consequently, according to the above, a set of  $n$  states can be considered.

Counters are special types of sequential circuits that are important building blocks in digital systems and they can appear on a variety of forms. One of these forms is the digital counter. A digital counter which can count in the binary system from 0 to  $2^n - 1$  consist of  $n$  states which are built from  $n$  flip-flops. The binary counter is the hypermuloid which derives from the partition of the linguistic hyperringoid of the natural numbers through the relation *mod*  $2^n$ .

For example the binary counter counting through the successive binary numbers 0 through 7 is the hypermuloid  $\mathbf{N}/\text{mod } 2^3$  having 8 states as shown in the diagram:

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