EDGEWISE SUBDIVISIONS, LOCAL h-POLYNOMIALS, AND EXCEDANCES IN THE WREATH PRODUCT $\mathbb{Z}_r \wr \mathfrak{S}_n$*

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Abstract. The coefficients of the local h-polynomial of the barycentric subdivision of the simplex with n vertices are known to count derangements in the symmetric group $\mathfrak{S}_n$ by the number of excedances. A generalization of this interpretation is given for the local h-polynomial of the rth edgewise subdivision of the barycentric subdivision of the simplex. This polynomial is shown to be γ-nonnegative and a combinatorial interpretation to the corresponding γ-coefficients is provided. The new combinatorial interpretations involve the notions of flag excedance and descent in the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$. A related result on the derangement polynomial for $\mathbb{Z}_r \wr \mathfrak{S}_n$, studied by Chow and Mansour, is also derived from results of Linusson, Shareshian, and Wachs on the homology of Rees products of posets.

Key words. barycentric subdivision, edgewise subdivision, local h-polynomial, γ-polynomial, colored permutation, derangement, flag excedance, Rees product

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1. Introduction and results. Local h-polynomials were introduced by Stanley [27] as a fundamental tool in his theory of face enumeration for subdivisions of simplicial complexes. Given a (finite, geometric) simplicial subdivision (triangulation) Γ of the abstract simplex 2^V on an n-element vertex set V, the local h-polynomial $\ell_V(\Gamma, x)$ is defined by the formula

$$(1) \qquad \ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x),$$

where Γ_F is the restriction of Γ to the face $F \in 2^V$ and $h(\Delta, x)$ stands for the h-polynomial of the simplicial complex Δ; missing definitions can be found in sections 2 and 4. (The general notion of topological subdivision introduced in [27] will not concern us here.)

The importance of local h-polynomials stems from their appearance in the locality formula [27, Theorem 3.2], which expresses the h-polynomial of a simplicial subdivision of a pure simplicial complex Δ as a sum of local contributions, one for each face of Δ.

The polynomial $\ell_V(\Gamma, x)$ enjoys a number of pleasant properties, interpretations, and outstanding open problems. For instance, it was shown in [27, sections 3–5] that $\ell_V(\Gamma, x)$ has nonnegative and symmetric coefficients (with center of symmetry $n/2$) for every simplicial subdivision Γ of 2^V and unimodal coefficients for every regular such subdivision. Moreover, it is conjectured [2, Question 3.5], [27, Conjecture 5.4] that $\ell_V(\Gamma, x)$ has unimodal coefficients for every simplicial subdivision Γ of 2^V (although this property fails for the more general class of quasi-geometric subdivisions [2, Example 3.4]) and that it is γ-nonnegative for every flag such subdivision [2, Con-

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jecture 5.4], meaning that the integers $\xi_i(\Gamma)$ uniquely determined by

$$(2) \qquad\qquad \ell_V(\Gamma, x) \; = \; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Gamma) \, x^i (1 + x)^{n-2i}$$

are nonnegative for all $0 \le i \le \lfloor n/2 \rfloor$. Combinatorial interpretations for the coefficients of $\ell_V(\Gamma, x)$ and the integers $\xi_i(\Gamma)$ have been given for several interesting classes of (flag) simplicial subdivisions of the simplex in [3], [22], [27, section 2].

     We now recall these interpretations for the (first, simplicial) barycentric subdivision of the simplex, which will be generalized by the first main result of this paper. We will denote by $\mathrm{sd}(2^V)$ the barycentric subdivision of the simplex 2^V, by $\mathfrak{S}_n$ the symmetric group of permutations of $\{1, 2, \ldots, n\}$, and by $\mathcal{D}_n$ the set of derangements (permutations without fixed points) in $\mathfrak{S}_n$. We recall that an excedance of $w \in \mathfrak{S}_n$ is an index $1 \le i \le n$ such that $w(i) > i$. A descending run of $w \in \mathfrak{S}_n$ is a maximal string $\{a, a + 1, \ldots, b\}$ of integers, such that $w(a) > w(a + 1) > \cdots > w(b)$. The following statement is a combination of results of [3, section 4] and [27, section 2]. The expression for the local h-polynomial of $\mathrm{sd}(2^V)$ given in (4) may also be derived from an unpublished result of Gessel [25, Theorem 7.3] (see [25, Remark 5.5]), from Corollary 9 and Theorem 11 in [26], and, as will be explained in what follows, from results of [19, 24] on the homology of Rees products of posets.

     THEOREM 1.1 (see [3, Theorem 1.4], [27, Proposition 2.4]). *Let V be an n-element set. The local h-polynomial of the barycentric subdivision of the $(n-1)$-dimensional simplex 2^V can be expressed as*

$$(3) \qquad\qquad \ell_V(\mathrm{sd}(2^V), x) = \sum_{w \in \mathcal{D}_n} x^{\mathrm{exc}(w)}$$

$$(4) \qquad\qquad = \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,i} \, x^i (1 + x)^{n-2i},$$

where $\mathcal{D}_n$ is the set of derangements in $\mathfrak{S}_n$, $\mathrm{exc}(w)$ is the number of excedances of $w \in \mathfrak{S}_n$, and $\xi_{n,i}$ stands for the number of permutations $w \in \mathfrak{S}_n$ with i descending runs and no descending run of size one.

     Our generalization of Theorem 1.1 concerns the rth edgewise subdivision $\mathrm{sd}(2^V)^{\langle r \rangle}$ of the barycentric subdivision $\mathrm{sd}(2^V)$ and the flag excedance statistic on the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$, where $\mathbb{Z}_r$ is the cyclic group of order r. The rth edgewise subdivision $\Delta^{\langle r \rangle}$ of a simplicial complex Δ is a standard way to subdivide a simplicial complex Δ so that each face $F \in \Delta$ is subdivided into $r^{\dim(F)}$ faces of the same dimension. This construction has appeared is several mathematical contexts; see, for instance, [10], [11], [15], [18]. Figure 1 shows the barycentric subdivision of the two-dimensional simplex and its third edgewise subdivision. The elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$ will be thought of as r-colored permutations, meaning permutations in $\mathfrak{S}_n$ with each coordinate in their one–line notation colored with an element of the set $\{0, 1, \ldots, r - 1\}$. The flag excedance of $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is defined as

$$(5) \qquad\qquad \mathrm{fexc}(w) = r \cdot \mathrm{exc}_A(w) + \mathrm{csum}(w),$$

where $\mathrm{exc}_A(w)$ is the number of excedances of w at coordinates of zero color and $\mathrm{csum}(w)$ is the sum of the colors of all coordinates of w; more explanation appears in section 2. The flag excedance statistic was introduced (although not with this name)
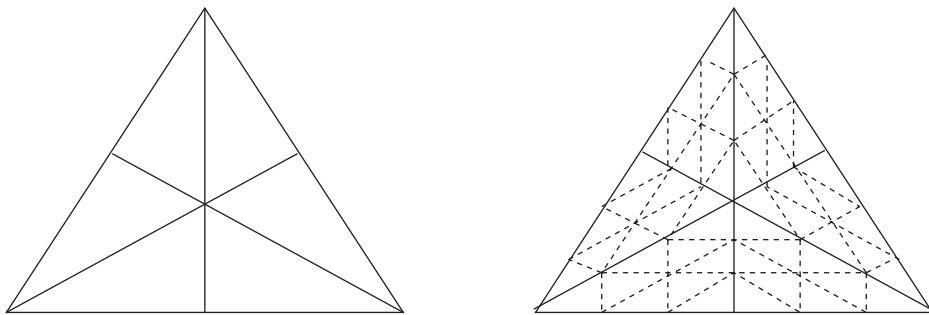
FIG. 1. *The barycentric subdivision of the 2-simplex and its third edgewise subdivision.*

by Bagno and Garber [5] and was further studied in [17] and, in the special case $r = 2$, in [16, 21]. We will denote by $\mathcal{D}_n^r$ the set of derangements (elements without fixed points of zero color) in $\mathbb{Z}_r \wr \mathfrak{S}_n$. We will call an element $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ balanced if $\mathrm{csum}(w)$ (equivalently, $\mathrm{fexc}(w)$) is divisible by r. Decreasing runs for elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$ are defined as for those of $\mathfrak{S}_n$, after the r-colored integers have been totally ordered in a standard lexicographic way; see section 2. The following statement provides a new context in which the flag excedance statistic naturally arises; it reduces to Theorem 1.1 for $r = 1$.

THEOREM 1.2. *Let V be an n-element set and let r be a positive integer. The local h-polynomial of the rth edgewise subdivision of the barycentric subdivision of the $(n-1)$-dimensional simplex 2^V can be expressed as*

$$(6) \qquad \ell_V(\mathrm{sd}(2^V)^{\langle r \rangle}, x) = \sum_{w \in (\mathcal{D}_n^r)^b} x^{\mathrm{fexc}(w)/r}$$

$$(7) \qquad = \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+ \, x^i (1+x)^{n-2i},$$

where $(\mathcal{D}_n^r)^b$ is the set of balanced derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$, $\mathrm{fexc}(w)$ is the flag excedance of $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ and $\xi_{n,r,i}^+$ stands for the number of elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$ with i descending runs, no descending run of size one, and last coordinate of zero color.

Theorem 1.2 will be derived from a related result on the derangement polynomial for $\mathbb{Z}_r \wr \mathfrak{S}_n$, studied by Chow and Mansour [14]. This polynomial, denoted here by $d_n^r(x)$, is defined as the generating polynomial for the excedance statistic, in the sense of Steingrímsson [30], [31], on the set $\mathcal{D}_n^r$ of derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$. It specializes to the right-hand side of (3), the derangement polynomial for $\mathfrak{S}_n$, introduced by Brenti [9], for $r = 1$ and to its type B analogue introduced by Chen, Tang, and Zhao [12] and, independently (in a variant form), by Chow [13] for $r = 2$. The following statement is the second main result of this paper.

THEOREM 1.3. *For all positive integers n, r*

$$(8) \qquad d_n^r(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+ \, x^i (1+x)^{n-2i} + \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \xi_{n,r,i}^- \, x^i (1+x)^{n+1-2i},$$

where $\xi_{n,r,i}^+$ is as in Theorem 1.2 and $\xi_{n,r,i}^-$ stands for the number of elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$ with i descending runs, no descending run of size one other than $\{n\}$, and last coordinate of nonzero color.

The previous theorem implies that $d_n^r(x)$ can be written as a sum of two symmetric and unimodal (in fact, γ-nonnegative) polynomials whose centers of symmetry differ by a half. Moreover, one of these polynomials is equal to the local h-polynomial of a suitable simplicial subdivision of the $(n-1)$-dimensional simplex. These facts were first observed in the special case $r = 2$ in joint work of the author with Savvidou [22, Chapter 3] using methods different from those in this paper. (The two main results of this paper, however, are new even for $r = 2$.) Although a direct combinatorial proof should be possible, Theorem 1.3 will be derived from [19, Corollary 3.8] by exploiting a connection of $d_n^r(x)$ to the homology of Rees products of posets, previously noticed only in the special case $r = 1$.

This paper is structured as follows. Section 2 includes background material on the combinatorics of colored permutations. Some preliminary results on the various excedance statistics on derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$ are also included. Section 3 reviews the necessary background on Rees product homology and proves Theorem 1.3. Theorem 1.2 is proven in section 5 after the relevant definitions on simplicial subdivisions and local h-polynomials have been explained in section 4. Section 6 concludes with some corollaries, remarks, and open problems.

2. Colored permutation statistics. This section recalls basic definitions and useful facts about colored permutations and the various notions of descent and excedance for them. It also includes some new results and formulas, in preparation for the proofs of Theorems 1.2 and 1.3 in the following sections.

2.1. Colored permutations. The symmetric group of permutations of $\{1, 2, \ldots, n\}$ will be denoted by $\mathfrak{S}_n$. The elements of the cyclic group $\mathbb{Z}_r$ of order r will be represented by those of $\{0, 1, \ldots, r-1\}$ and will be thought of as colors.

The wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$ consists of all r-colored permutations of the form $\sigma \times \mathbf{z}$, where $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathfrak{S}_n$ and $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \{0, 1, \ldots, r-1\}^n$. The number z_i will be thought of as the color assigned to $\sigma(i)$. The product in the group $\mathbb{Z}_r \wr \mathfrak{S}_n$ is given by the rule $(\sigma \times \mathbf{z})(\tau \times \mathbf{w}) = \sigma\tau \times (\mathbf{w} + \tau(\mathbf{z}))$, where the composition $\sigma\tau = \sigma \circ \tau$ is evaluated from right to left, $\tau(\mathbf{z}) = (z_{\tau(1)}, z_{\tau(2)}, \ldots, z_{\tau(n)})$, and the addition is coordinatewise modulo r. The inverse of $\sigma \times \mathbf{z} \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is the element $\sigma^{-1} \times \mathbf{w}$, where σ^{-1} is the inverse of σ in $\mathfrak{S}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is such that

$$w_i = \begin{cases} 0 & \text{if } z_{\sigma^{-1}(i)} = 0, \\ r - z_{\sigma^{-1}(i)} & \text{otherwise.} \end{cases}$$

A *fixed point* of $\sigma \times \mathbf{z} \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is an index $i \in \{1, 2, \ldots, n\}$ such that $\sigma(i) = i$ and $z_i = 0$. Elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$ without fixed points are called *derangements*. The set of derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$ will be denoted by $\mathcal{D}_n^r$.

2.2. Statistics. Throughout this section $w = \sigma \times \mathbf{z} \in \mathbb{Z}_r \wr \mathfrak{S}_n$ will be a colored permutation as in section 2.1, where $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathfrak{S}_n$ and $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \{0, 1, \ldots, r-1\}^n$.

We first recall the notions of descent and excedance for elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$, introduced in [30, section 2], [31, section 3.1]. A descent of w is an index $i \in \{1, 2, \ldots, n\}$ such that either $z_i > z_{i+1}$, or $z_i = z_{i+1}$ and $\sigma(i) > \sigma(i+1)$, where $\sigma(n+1) = n+1$ and $z_{n+1} = 0$. Thus, in particular, n is a descent of w if and only if $\sigma(n)$ has nonzero color. A *descending run* of w is a maximal string $\{a, a+1, \ldots, b\}$ of integers such that i is a descent of w for every $a \leq i \leq b-1$. An excedance of w is an index $i \in \{1, 2, \ldots, n\}$ such that either $\sigma(i) > i$, or $\sigma(i) = i$ and $z_i > 0$. The number of

descents and excedances, respectively, of w will be denoted by $\mathrm{des}(w)$ and $\mathrm{exc}(w)$. The equidistribution of the descent and excedance statistics [30, Theorem 3.15], [31, Theorem 15] on $\mathbb{Z}_r \wr \mathfrak{S}_n$ allows one to define the Eulerian polynomial for $\mathbb{Z}_r \wr \mathfrak{S}_n$ as

$$(9) \qquad A_n^r(x) := \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\mathrm{des}(w)} = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\mathrm{exc}(w)}.$$

For $r = 1$, this polynomial specializes to the classicial Eulerian polynomial

$$A_n(x) := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{exc}(w)},$$

although this definition differs from the standard one [29, p. 32] by a power of x.

The *flag descent* statistic on $\mathbb{Z}_r \wr \mathfrak{S}_n$ was defined first in the special case $r = 2$ by Adin, Brenti, and Roichman [1, section 4] and later by Bagno and Biagoli [4, section 3] for general $r \geq 1$. We find it convenient to define it here as

$$(10) \qquad \mathrm{fdes}^*(w) = \begin{cases} r \cdot \mathrm{des}(w) & \text{if } z_n = 0, \\ r \cdot \mathrm{des}(w) - r + z_n & \text{otherwise} \end{cases}$$

for $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$, where, as explained earlier, $z_n \in \{0, 1, \ldots, r-1\}$ is the color of the last coordinate $\sigma(n)$ of w. (This definition differs by a minor twist from the one given in [1], [4]; see the proof of Proposition 2.2.) The *flag excedance* number $\mathrm{fexc}(w)$ of $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is defined by (5), where $\mathrm{exc}_A(w)$ is the number of indices $i \in \{1, 2, \ldots, n\}$ such that $\sigma(i) > i$ and $z_i = 0$ and $\mathrm{csum}(w) = z_1 + z_2 + \cdots + z_n$, the sum being computed in $\mathbb{Z}$. We will call w *balanced* if $\mathrm{csum}(w)$ is an integer multiple of r.

Example 2.1. Suppose that $w = \sigma \times \mathbf{z}$ is the colored permutation in $\mathbb{Z}_3 \wr \mathfrak{S}_6$ with $\sigma = (2, 5, 1, 4, 6, 3)$ and $\mathbf{z} = (1, 0, 0, 2, 0, 1)$. Then w is a derangement with descents 1, 2, 4, and 6, excedances 1, 2, 4, and 5, and type A excedances 2 and 5, so that $\mathrm{des}(w) = \mathrm{exc}(w) = 4$, $\mathrm{exc}_A(w) = 2$, and $\mathrm{fdes}^*(w) = 10$. Moreover, we have $\mathrm{csum}(w) = 4$ and hence $\mathrm{fexc}(w) = 10$.

The following statement combines results from [1], [4], [17] (where [1] contributes to the special case $r = 2$).

PROPOSITION 2.2 (see [1], [4], [17]). *For all positive integers n, r*

$$(11) \qquad \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\mathrm{fdes}^*(w)} = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\mathrm{fexc}(w)} = (1 + x + x^2 + \cdots + x^{r-1})^n A_n(x).$$

Proof. Given $w = \sigma \times \mathbf{z} \in \mathbb{Z}_r \wr \mathfrak{S}_n$, as in the beginning of this section, we set $\tilde{w} = \tilde{\sigma} \times \tilde{\mathbf{z}} \in \mathbb{Z}_r \wr \mathfrak{S}_n$, where $\tilde{\sigma} = (n - \sigma(n) + 1, n - \sigma(n-1) + 1, \ldots, n - \sigma(1) + 1) \in \mathfrak{S}_n$ and $\tilde{\mathbf{z}} = (z_n, z_{n-1}, \ldots, z_1)$. The flag descent of w is defined in [4, section 3] as $\mathrm{fdes}(w) = r \cdot \mathrm{des}^*(w) + z_1$, where $\mathrm{des}^*(w)$ is the number of indices $i \in \{1, 2, \ldots, n-1\}$ such that either $z_i < z_{i+1}$, or $z_i = z_{i+1}$ and $\sigma(i) > \sigma(i+1)$. We note that this happens if and only if $n - i$ is a descent of $\tilde{w}$ and conclude that

$$\mathrm{fdes}(w) = r \cdot \mathrm{des}_A(\tilde{w}) + z_n(\tilde{w}) = \mathrm{fdes}^*(\tilde{w}),$$

where $\mathrm{des}_A(\tilde{w})$ is the number of descents of $\tilde{w}$ other than n. Since the map which sends w to $\tilde{w}$ is a bijection from $\mathbb{Z}_r \wr \mathfrak{S}_n$ to itself, this implies that

$$\sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\mathrm{fdes}^*(w)} = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\mathrm{fdes}(w)}.$$

Given that, the equality between the leftmost and rightmost expressions in (11) follows by setting $q = 1$ in [4, Theorem A.1]. The first equality in (11) follows from [17, Theorem 1.4]. ☐

We now consider the generating polynomials for the excedance and flag excedance statistics on the set $\mathcal{D}_n^r$ of derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$. The generating polynomial

$$(12) \qquad d_n^r(x) := \sum_{w \in \mathcal{D}_n^r} x^{\text{exc}(w)}$$

for the excedance statistic was studied by Chow and Mansour [14, section 3]. As already mentioned in the introduction, $d_n^r(x)$ generalizes both the right-hand side of (3) and (as can be inferred, for instance, from the discussion in [4, p. 428]) the type B derangement polynomial, introduced and studied in [12], [13]. For $x \in \mathbb{R}$, we will denote by $\lceil x \rceil$ the smallest integer which is not strictly less than x.

PROPOSITION 2.3. *For all positive integers n, r*

$$(13) \qquad d_n^r(x) = \sum_{w \in \mathcal{D}_n^r} x^{\lceil \frac{\text{fexc}(w)}{r} \rceil}.$$

Proof. We define the linear operator $\widetilde{\text{E}}_r : \mathbb{R}[x] \to \mathbb{R}[x]$ by setting $\widetilde{\text{E}}_r(x^m) = x^{\lceil m/r \rceil}$ for $m \in \mathbb{N}$. The defining equation (10) shows that $\widetilde{\text{E}}_r(x^{\text{fdes}^*(w)}) = x^{\text{des}(w)}$ for every $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$. Applying operator $\widetilde{\text{E}}_r$ to the first equality in (11) we get

$$A_n^r(x) = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\text{des}(w)} = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\lceil \frac{\text{fexc}(w)}{r} \rceil}.$$

Because of (9), the previous equalities can be rewritten as

$$A_n^r(x) = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\text{exc}(w)} = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} x^{\lceil \frac{\text{fexc}(w)}{r} \rceil}.$$

Since adding or removing fixed points from $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ does not affect $\text{exc}(w)$ or $\text{fexc}(w)$, the previous equalities, the defining equation (12), and two easy applications of the principle of inclusion-exclusion show that

$$d_n^r(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_k} x^{\text{exc}(w)}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_k} x^{\lceil \frac{\text{fexc}(w)}{r} \rceil}$$

$$= \sum_{w \in \mathcal{D}_n^r} x^{\lceil \frac{\text{fexc}(w)}{r} \rceil},$$

and the proof follows. ☐

PROPOSITION 2.4. *For all positive integers n, r*

$$(14) \qquad x^n d_n^r(1/x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} r^k A_k(x),$$

where $A_0(x) = 1$.

Proof. It was shown in [14, Theorem 5 (iv)] that

$$\sum_{n\geq 0} d_n^r(x) \frac{t^n}{n!} = \frac{(1-x)e^{(r-1)xt}}{e^{rxt} - xe^{rt}}.$$

Replacing x by $1/x$ and t by xt we get

(15) $$\sum_{n\geq 0} x^n d_n^r(1/x) \frac{t^n}{n!} = \frac{(1-x)e^{(r-1)t}}{e^{rxt} - xe^{rt}}.$$

Similarly, denoting by $S_n(x)$ the right-hand side of (14) and using the well-known expression for the exponential generating function for the Eulerian polynomials $A_n(x)$ [29, Proposition 1.4.5], we get

$$\sum_{n\geq 0} S_n(x) \frac{t^n}{n!} = e^{-t} \cdot \sum_{n\geq 0} A_n(x) \frac{(rt)^n}{n!} = \frac{(1-x)e^{(r-1)t}}{e^{rxt} - xe^{rt}}.$$

Comparing with (15) shows that $x^n d_n^r(1/x) = S_n(x)$ for every $n \in \mathbb{N}$ and the proof follows. □

The following observation on the generating polynomial for the flag excedance statistic on $\mathcal{D}_n^r$ specializes to [21, Proposition 3.5] for $r = 2$.

PROPOSITION 2.5. *For all positive integers n, r, the polynomial*

(16) $$f_n^r(x) := \sum_{w \in \mathcal{D}_n^r} x^{\mathrm{fexc}(w)}$$

is symmetric with center of symmetry $rn/2$, i.e., we have $x^{rn} f_n^r(1/x) = f_n^r(x)$. Moreover, $f_n^r(x)$ is monic of degree $rn - 1$.

Proof. Suppose that $w \in \mathcal{D}_n^r$ is a colored permutation with k coordinates of zero color. From the description of w^{-1} given in section 2.1 we get $\mathrm{exc}_A(w^{-1}) = k - \mathrm{exc}_A(w)$ and $\mathrm{csum}(w^{-1}) = r(n - k) - \mathrm{csum}(w)$. These equalities imply that $\mathrm{fexc}(w^{-1}) = rn - \mathrm{fexc}(w)$. Since the map $\varphi : \mathcal{D}_n^r \to \mathcal{D}_n^r$ defined by $\varphi(w) = w^{-1}$ for $w \in \mathcal{D}_n^r$ is a well-defined bijection, the first claim in the statement of the proposition follows. The second can be left to the reader. □

3. Rees product homology and proof of Theorem 1.3. This section derives Theorem 1.3 from results of Shareshian and Wachs [24] and Linusson, Shareshian, and Wachs [19] on the homology of Rees products of posets. Some of the background needed to understand and apply these results will first be explained. Throughout this section we will assume familiarity with the basics of finite partially ordered sets [29, Chapter 3]. We will also write $[a, b] = \{a, a + 1, \ldots, b\}$ for integers a, b with $a \leq b$ and set $[n] := [1, n]$.

3.1. Shellability and Rees products of posets. Given a finite poset P with a minimum element $\hat{0}$ and maximum element $\hat{1}$, we will denote by $\bar{P}$ the poset which is obtained from P by removing $\hat{0}$ and $\hat{1}$. Similarly, given any finite poset Q, we will denote by $\hat{Q}$ the poset which is obtained from Q by adding elements $\hat{0}$ and $\hat{1}$ so that $\hat{0} < x < \hat{1}$ holds for every $x \in Q$. We will then write $\mu(Q)$ for the value $\mu_{\hat{Q}}(\hat{0}, \hat{1})$ of the Möbius function [29, section 3.7] of $\hat{Q}$ between $\hat{0}$ and $\hat{1}$. We will denote by $Q_{\leq x}$ the principal order ideal $\{y \in Q : y \leq x\}$ of Q generated by $x \in Q$.

We now briefly recall the definition of EL-shellability and refer the reader to [7], [32, Lecture 3] for more information. Suppose that P is a finite graded poset of rank $n + 1$ with minimum element $\hat{0}$ and maximum element $\hat{1}$. Consider a map $\lambda : \mathcal{E}(P) \to \Lambda$, where $\mathcal{E}(P)$ is the set of cover relations of P and Λ is a set equipped with a total order $\preceq$. To any unrefinable chain $c : x_0 < x_1 < \cdots < x_r$ of elements of P one can associate the sequence $\lambda(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{r-1}, x_r))$, called here the *label* of c. The chain c is said to be *rising* or *falling* if the entries of $\lambda(c)$ weakly increase or strictly decrease, respectively, in the total order $\preceq$. The map λ is called an *EL-labeling* if the following hold for every nonsingleton interval $[u, v]$ in P: (a) there is a unique rising maximal chain in $[u, v]$, and (b) the label of this chain is lexicographically smallest among all labels of maximal chains in $[u, v]$. The poset P is called *EL-shellable* if it has an EL-labeling for some totally ordered set Λ. Assuming P is EL-shellable and given $S \subseteq [n]$, we will denote by $\beta_P(S)$ the number of maximal chains $c : \hat{0} = x_0 < x_1 < \cdots < x_{n+1} = \hat{1}$ in P such that for $1 \le i \le n$ we have $\lambda(x_{i-1}, x_i) \succ \lambda(x_i, x_{i+1}) \Leftrightarrow i \in S$. The numbers $\beta_P(S)$ are independent of the EL-labeling λ.

Given two finite graded posets P and Q with rank functions ρ_P and ρ_Q, respectively, the Rees product of P and Q is defined in [8] as $P * Q = \{(p, q) \in P \times Q : \rho_P(p) \ge \rho_Q(q)\}$, with partial order defined by setting $(p_1, q_1) \le (p_2, q_2)$ if all of the following conditions are satisfied:

- $p_1 \le p_2$ holds in P,
- $q_1 \le q_2$ holds in Q, and
- $\rho_P(p_2) - \rho_P(p_1) \ge \rho_Q(q_2) - \rho_Q(q_1)$.

Equivalently, (p_1, q_1) is covered by (p_2, q_2) in $P * Q$ if and only if (a) p_1 is covered by p_2 in P, and (b) either $q_1 = q_2$, or q_1 is covered by q_2 in Q. As a consequence of the definition, the principal order ideal of $P * Q$ generated by $x = (p, q) \in P * Q$ satisfies

$$(17) \qquad (P * Q)_{\le x} = (P_{\le p} * Q_{\le q})_{\le x}.$$

For positive integers x, n, we will denote by $T_{x,n}$ the poset whose Hasse diagram is a complete x-ary tree of height $n - 1$, with root at the bottom (so that $T_{x,n}$ is an n-element chain for $x = 1$). The following theorem is a restatement of the first part of [19, Corollary 3.8].

THEOREM 3.1 (see[19]). *For every EL-shellable poset P of rank $n + 1$ and every positive integer x, we have*

$$(18) \quad |\mu(\bar{P} * T_{x,n})| = \sum_{S \in \mathcal{P}_{\mathrm{stab}}([2, n-1])} \beta_P([n] \smallsetminus S)\, x^{|S|}\, (1 + x)^{n-1-2|S|}$$

$$+ \sum_{S \in \mathcal{P}_{\mathrm{stab}}([2, n-2])} \beta_P([n-1] \smallsetminus S)\, x^{|S|+1}\, (1 + x)^{n-2-2|S|},$$

where $\mathcal{P}_{\mathrm{stab}}(\Theta)$ denotes the set of all subsets of $\Theta \subseteq \mathbb{Z}$ which do not contain two consecutive integers.

3.2. Proof of Theorem 1.3. We consider the subsets Ω of $[n] \times [0, r - 1]$ for which for every $i \in [n]$ there is at most one $j \in [0, r - 1]$ such that $(i, j) \in \Omega$. We will denote by B_n^r the set of all nonempty such subsets, partially ordered by inclusion. Thus, for $r = 1$ the poset B_n^r is isomorphic to the poset obtained from the Boolean lattice of subsets of $[n]$ by removing its minimum element (empty set). For $r = 2$ this poset was considered in [24, section 6].

The following statement provides the key connection of the derangement polynomial $d_n^r(x)$ to the Rees product construction. The special case $r = 1$ is equivalent to the special case $q = 1$ of [19, equation (1.3)].

PROPOSITION 3.2. *For all positive integers n, r, x, we have*

$$(19) \qquad |\mu(B_n^r * T_{x,n})| = x^n d_n^r(1/x).$$

Proof. Setting $Q = B_n^r * T_{x,n}$ and using the definition [29, section 3.7] of the Möbius function, we get

$$(20) \qquad \mu(B_n^r * T_{x,n}) = \mu(Q) = \mu_{\hat{Q}}(\hat{0}, \hat{1}) = - \sum_{y \in Q \cup \{\hat{0}\}} \mu_{\hat{Q}}(\hat{0}, y).$$

Suppose that $y = (a, b) \in Q$, where $a \in B_n^r$ has rank $k - 1$, $b \in T_{x,n}$ has rank i, and $k - 1, i \in [0, n-1]$ with $i \le k - 1$. Since the principal order ideal of B_n^r generated by a is isomorphic to B_k^1 and that of $T_{x,n}$ generated by b is an $(i+1)$-element chain, (17) and [24, Theorem 1.2] imply that $(-1)^k \mu_{\hat{Q}}(\hat{0}, y)$ is equal to the (Eulerian) number of elements of $\mathfrak{S}_k$ with i descents. Denoting this number by a_{ki} and noting that there are exactly $r^k \binom{n}{k}$ elements of B_n^r of rank $k-1$ and exactly x^i elements of $T_{x,n}$ of rank i, we conclude from (20) that

$$
\begin{aligned}
\mu(Q) &= -1 - \sum_{i=0}^{n-1} \sum_{k=i+1}^{n} r^k \binom{n}{k} x^i (-1)^k a_{ki} \\
&= -1 - \sum_{k=1}^{n} (-1)^k \binom{n}{k} r^k \sum_{i=0}^{k-1} a_{ki} x^i \\
&= \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} r^k A_k(x),
\end{aligned}
$$

and the result follows from Proposition 2.4. □

Proof of Theorem 1.3. Since both sides of (8) are polynomials in x, we may assume that x is a positive integer. We wish to apply Theorem 3.1 to the poset $P := \hat{B}_n^r$. Clearly, P is graded of rank $n + 1$. We set $\Lambda = ([n] \times [0, r-1]) \cup \{\delta\}$, where δ is a symbol not in $[n] \times [0, r-1]$, and totally order Λ as

$$(1, 0) \prec (2, 0) \prec \cdots \prec (n, 0) \prec \delta \prec (1, 1) \prec (2, 1) \prec \cdots \prec (n, 1) \prec \cdots \prec (1, r-1) \prec$$
$$(2, r-1) \prec \cdots \prec (n, r-1).$$

For a cover relation $(x, y) \in \mathcal{E}(P)$, we define $\lambda(x, y)$ as the unique element of the set $y \smallsetminus x$, if $y \in P \smallsetminus \{\hat{1}\}$, and set $\lambda(x, y) = \delta$ if $y = \hat{1}$. We leave to the reader to verify (by standard arguments; see, for instance, [32, section 3.2.1]) that $\lambda : \mathcal{E}(P) \to \Lambda$ is an EL-labeling and that the map which assigns to a maximal chain c of P the label $\lambda(c)$, with its last entry δ removed, induces a bijection from the set of maximal chains of P to $\mathbb{Z}_r \wr \mathfrak{S}_n$. Moreover, given such a chain $c : \hat{0} = x_0 < x_1 < \cdots < x_{n+1} = \hat{1}$ with corresponding colored permutation w, an index $i \in [n]$ satisfies $\lambda(x_{i-1}, x_i) \succ \lambda(x_i, x_{i+1})$ if and only if i is a descent of w, in the sense of section 2.2. As a result, for $S \subseteq [n]$ the number $\beta_P(S)$ is equal to the number of elements of $\mathbb{Z}_r \wr \mathfrak{S}_n$ whose set of descents is equal to S.

Combining Proposition 3.2 with Theorem 3.1 and the previous discussion (and remembering to replace x by $1/x$ in the right-hand side of (18)), we conclude that (8)

holds for some nonnegative integers $\xi_{n,r,i}^+$ and $\xi_{n,r,i}^-$. Moreover, denoting by $\mathrm{Asc}(w)$ the set of elements of $[n]$ which are not descents of $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$, we have

$$\xi_{n,r,i}^+ = \sum \beta_P([n-1] \smallsetminus S)$$
$$= \sum \# \{w \in \mathbb{Z}_r \wr \mathfrak{S}_n : \mathrm{Asc}(w) = S \cup \{n\}\},$$

where S runs through all $(i-1)$-element sets in $\mathcal{P}_{\mathrm{stab}}([2, n-2])$ and

$$\xi_{n,r,i}^- = \sum \beta_P([n] \smallsetminus S)$$
$$= \sum \# \{w \in \mathbb{Z}_r \wr \mathfrak{S}_n : \mathrm{Asc}(w) = S\},$$

where S runs through all $(i-1)$-element sets in $\mathcal{P}_{\mathrm{stab}}([2, n-1])$. These are exactly the combinatorial interpretations claimed for $\xi_{n,r,i}^+$ and $\xi_{n,r,i}^-$ in Theorem 1.3. □

4. Subdivisions. This section briefly reviews the background on simplicial subdivisions needed to understand Theorem 1.2 and its proof. Familiarity with basic notions on simplicial complexes, such as the correspondence between abstract and geometric simplicial complexes, will be assumed. For more information on this topic and any undefined terminology, the reader is referred to [7], [20], [28]. All simplicial complexes considered here will be finite. We will denote by $|S|$ the cardinality, and by 2^S the set of all subsets, of a finite set S.

Consider two geometric simplicial complexes Σ' and Σ in some Euclidean space $\mathbb{R}^m$ (so the elements of Σ' and Σ are geometric simplices in $\mathbb{R}^m$), with corresponding abstract simplicial complexes Δ' and Δ. We will say that Σ' is a *simplicial subdivision* of Σ, and that Δ' is a (finite, geometric) *simplicial subdivision* of Δ, if (a) every simplex of Σ' is contained in some simplex of Σ, and (b) the union of the simplices of Σ' is equal to the union of the simplices of Σ. Then, given any simplex $L \in \Sigma$ with corresponding face $F \in \Delta$, the subcomplex Σ'_L of Σ' consisting of all simplices of Σ' contained in L is called the *restriction* of Σ' to L. The subcomplex Δ'_F of Δ' corresponding to Σ'_L is the restriction of Δ' to F. Clearly, Δ'_F is a simplicial subdivision of the abstract simplex 2^F.

The more general notion of topological subdivision introduced in [27, section 2] will not be needed here.

Barycentric and edgewise subdivisions. The (first, simplicial) *barycentric subdivision* of a simplicial complex Δ, denoted $\mathrm{sd}(\Delta)$, is defined as the simplicial complex consisting of all (possibly empty) chains $F_0 \subset F_1 \cdots \subset F_k$ of nonempty faces of Δ. It is well known that $\mathrm{sd}(\Delta)$ is a simplicial subdivision of Δ. The restriction of $\mathrm{sd}(\Delta)$ to a face $F \in \Delta$ coincides with the barycentric subdivision of 2^F.

To define the rth edgewise subdivision of a simplicial complex, we follow the discussions in [10, section 4], [11, section 6]. Suppose that Δ is a simplicial complex on the vertex set $\Omega_1 = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ of coordinate vectors in some Euclidean space $\mathbb{R}^m$. For $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$, we will denote by $\mathrm{supp}(\mathbf{a})$ the set of indices $i \in \{1, 2, \ldots, m\}$ for which $a_i \neq 0$ and write $\iota(\mathbf{a}) = (a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_m)$. The rth edgewise subdivision of Δ is the simplicial complex $\Delta^{\langle r \rangle}$ on the ground set $\Omega_r = \{(i_1, i_2, \ldots, i_m) \in \mathbb{N}^m : i_1 + i_2 + \cdots + i_m = r\}$ of which a set $G \subseteq \Omega_r$ is a face if the following two conditions are satisfied:

- $\bigcup_{\mathbf{u} \in G} \mathrm{supp}(\mathbf{u}) \in \Delta$ and
- $\iota(\mathbf{u}) - \iota(\mathbf{v}) \in \{0, 1\}^m$ or $\iota(\mathbf{v}) - \iota(\mathbf{u}) \in \{0, 1\}^m$ for all $\mathbf{u}, \mathbf{v} \in G$.

We leave it to the reader to verify that $\Delta^{\langle r \rangle} = \Delta$ for $r = 1$ and that if Δ is a flag simplicial complex (meaning that every minimal nonface has at most two elements), then so is $\Delta^{\langle r \rangle}$.

For any $r \geq 1$, the simplicial complex $\Delta^{\langle r \rangle}$ can be realized as a simplicial subdivision of Δ (see, for instance, [11, section 6]). The restriction of $\Delta^{\langle r \rangle}$ to a face $F \in \Delta$ coincides with the rth edgewise subdivision $(2^F)^{\langle r \rangle}$ of the simplex 2^F; it has exactly $r^{\dim(F)}$ faces of the same dimension as F.

Face enumeration. Consider an $(n-1)$-dimensional simplicial complex Δ and let $f_i(\Delta)$ denote the number of i-dimensional faces of Δ. The h-*polynomial* of Δ is defined as

$$h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta)\, x^i (1-x)^{n-i}.$$

For the importance of h-polynomials, the reader is referred to [28, Chapter II].

Example 4.1. For the barycentric subdivision $\Delta = \mathrm{sd}(2^V)$ of an $(n-1)$-dimensional simplex 2^V, it is well known that $h(\Delta, x)$ is equal to the Eulerian polynomial $A_n(x)$. For example, for $n = 3$ (see Figure 1) we have $f_{-1}(\Delta) = 1$, $f_0(\Delta) = 7$, $f_1(\Delta) = 12$, and $f_2(\Delta) = 6$ and hence $h(\Delta, x) = 1 + 4x + x^2$.

An explicit formula can be given for the h-polynomial of the rth edgewise subdivision of any $(n-1)$-dimensional simplicial complex Δ. Indeed, combining [11, Corollary 6.8] with [10, Corollary 1.2], one gets

$$(21) \qquad h(\Delta^{\langle r \rangle}, x) = \mathrm{E}_r\left((1 + x + x^2 + \cdots + x^{r-1})^n\, h(\Delta, x)\right),$$

where $\mathrm{E}_r : \mathbb{R}[x] \to \mathbb{R}[x]$ is the linear operator defined by setting $\mathrm{E}_r(x^k) = x^{k/r}$ if k is divisible by r, and $\mathrm{E}_r(x^k) = 0$ otherwise.

Given a simplicial subdivision Γ of an $(n-1)$-dimensional simplex 2^V, the *local h-polynomial*, denoted $\ell_V(\Gamma, x)$, of Γ is defined [27, Definition 2.1] by (1), where Γ_F is the restriction of Γ to the face $F \in 2^V$. Then $\ell_V(\Gamma, x)$ has degree at most $n - 1$ (unless $V = \varnothing$, in which case $\ell_V(\Gamma, x) = 1$) and nonnegative and symmetric coefficients, so that $x^n \ell_V(\Gamma, 1/x) = \ell_V(\Gamma, x)$. For examples and further properties of local h-polynomials, see [27, Part I], [28, section II.10], [2, 3], and [22].

5. Proof of Theorem 1.2. This section deduces Theorem 1.2 from the results of section 2 and Theorem 1.3.

Proof of Theorem 1.2. We set $\Gamma = \mathrm{sd}(2^V)^{\langle r \rangle}$ and consider a k-element set $F \subseteq V$. As the relevant discussions in section 4 show, the restriction of Γ to F satisfies $\Gamma_F = \mathrm{sd}(2^F)^{\langle r \rangle}$ and $h(\mathrm{sd}(2^F), x) = A_k(x)$. Thus, combining (21) with Proposition 2.2 we get

$$h(\Gamma_F, x) = h(\mathrm{sd}(2^F)^{\langle r \rangle}, x) = \mathrm{E}_r\left((1 + x + x^2 + \cdots + x^{r-1})^k A_k(x)\right)$$

$$= \mathrm{E}_r\left(\sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_k} x^{\mathrm{fexc}(w)}\right) = \sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_k)^b} x^{\mathrm{fexc}(w)/r},$$

where $(\mathbb{Z}_r \wr \mathfrak{S}_k)^b$ is the set of balanced elements of $\mathbb{Z}_r \wr \mathfrak{S}_k$. Since adding or removing fixed points from $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ does not affect $\mathrm{fexc}(w)$, the previous equalities, the defining equation (1), and an easy application of the principle of inclusion-exclusion show that

$$\ell_V(\Gamma, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_k)^b} x^{\mathrm{fexc}(w)/r} = \sum_{w \in (\mathcal{D}_n^r)^b} x^{\mathrm{fexc}(w)/r},$$

and the proof of (6) follows.

We now assume that $r \geq 2$ and derive (7) from Theorem 1.3. Since $d_n^r(x)$ has degree n and zero constant term, it can be written uniquely in the form

$$(22) \qquad d_n^r(x) = f_{n,r}^+(x) + f_{n,r}^-(x),$$

where $f_{n,r}^+(x)$ and $f_{n,r}^-(x)$ are polynomials of degrees at most $n-1$ and n, respectively, satisfying

$$(23) \qquad f_{n,r}^+(x) = x^n f_{n,r}^+(1/x),$$

$$(24) \qquad f_{n,r}^-(x) = x^{n+1} f_{n,r}^-(1/x).$$

(See, for instance, [6, Lemma 2.4] for this elementary fact.) From Proposition 2.3 we get that

$$(25) \qquad d_n^r(x) = \sum_{w \in (\mathcal{D}_n^r)^b} x^{\mathrm{fexc}(w)/r} + \sum_{w \in \mathcal{D}_n^r \smallsetminus (\mathcal{D}_n^r)^b} x^{\left\lceil \frac{\mathrm{fexc}(w)}{r} \right\rceil}.$$

It is an easy consequence of Proposition 2.5 that the two summands in the right-hand side of (25) have degrees at most $n-1$ and n and satisfy (23) and (24), respectively. Thus, the uniqueness of the decomposition (22) implies that

$$f_{n,r}^+(x) = \sum_{w \in (\mathcal{D}_n^r)^b} x^{\mathrm{fexc}(w)/r}.$$

For similar reasons, the decomposition of $d_n^r(x)$ provided by Theorem 1.3 implies that

$$f_{n,r}^+(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+ \, x^i (1+x)^{n-2i},$$

and the proof of (7) follows.   □

6. Remarks and open problems.
1. Chow and Mansour showed [14, Theorem 5(ii)] that $d_n^r(x)$ is real-rooted and, as a result, that it has log-concave and unimodal coefficients. Theorem 1.3 provides a transparent proof of the unimodality of $d_n^r(x)$. It also shows that its peak occurs at $\lfloor (n+1)/2 \rfloor$.
2. Following Steingrímsson [31, Definition 36], we call a colored permutation $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$ *reverse alternating* if for $i \in \{1, 2, \ldots, n\}$ we have the following: i is a descent of w if and only if i is odd. For fixed r, the exponential generating function of these permutations was computed in [31, Theorem 39]. The following statement specializes to [34, Theorem 11] for $r = 2$.

COROLLARY 6.1. *For all positive integers n, r, the number $(-1)^{\lfloor \frac{n+1}{2} \rfloor} d_n^r(-1)$ is equal to the number of reverse alternating colored permutations in $\mathbb{Z}_r \wr \mathfrak{S}_n$.*

Proof. Setting $x = -1$ in (8), we get

$$d_n^r(-1) = \begin{cases} (-1)^{\frac{n}{2}} \, \xi_{n,r,\frac{n}{2}}^+ & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} \, \xi_{n,r,\frac{n+1}{2}}^- & \text{if } n \text{ is odd,} \end{cases}$$

and the proof follows from the combinatorial interpretations given to the $\xi_{n,r,i}^+$ and $\xi_{n,r,i}^-$ in Theorems 1.1 and 1.2.   □

3. The proof of Theorem 1.2 shows that the polynomials $d_n^2(x)$ and $f_n^2(x)$ can be computed one from the other in a simple way. Indeed, (25) shows that $d_n^2(x) = \alpha_n(x) + \beta_n(x)$, where $\alpha_n(x)$ and $\beta_n(x)$ satisfy $f_n^2(x) = \alpha_n(x^2) + \beta_n(x^2)/x$ and hence are uniquely determined by $f_n^2(x)$. Similarly, $\alpha_n(x)$ and $\beta_n(x)$ are uniquely determined by $d_n^2(x)$ by the uniqueness of the decomposition (22), which was already discussed.

4. Does the local h-polynomial of the subdivision $\mathrm{sd}(2^V)^{\langle r \rangle}$, computed in Theorem 1.2, have (when not identically zero) only real roots?

Since we are not aware of an explicit example of a flag (geometric) simplicial subdivision of the simplex whose local h-polynomial is not real-rooted, even a negative answer to this question would be of interest. The answer is affirmative for $r = 1$ by the result of [33] and, as was verified by Savvidou [23] by explicit computation, for $r = 2$ and $n \leq 50$.

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