GAMMA-POSITIVITY AND REES PRODUCT HOMOLOGY

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Abstract. The Rees product of certain partially ordered sets was studied using the theory of lexicographic shellability by Linusson, Shareshian and Wachs. These authors proved formulas, of relevance in the theory of gamma-positivity, for the dimension of its homology and provided several examples of combinatorial interest. A new proof of these formulas which does not require assumptions on shellability and leads to natural equivariant analogues is given in this paper. As applications, generalizations of results of Gessel, Shareshian and Wachs which establish the Schur gamma-positivity of certain symmetric functions arising in enumerative and geometric combinatorics, are obtained.

1. Introduction

The Rees product $P * Q$ of two partially ordered sets (posets, for short) was introduced and studied by Björner and Welker [4] as a combinatorial analogue of the Rees construction in commutative algebra (a precise definition of $P * Q$ can be found in Section 2). The connection of the Rees product of posets to enumerative combinatorics was hinted in [4, Section 5], where it was conjectured that the rank of the top homology group of the Rees product of the truncated Boolean algebra $B_n \smallsetminus \{\varnothing\}$ of rank n with an n-element chain equals the number of permutations of $\{1, 2, \dots, n\}$ without fixed points. This statement was generalized in several ways in [10], using enumerative and representation theoretic methods, and in [6], using the theory of lexicographic shellability, by replacing the Boolean algebra and the n-element chain by variants, or more general classes of posets, and by considering symmetric group actions on them. The results of [10] led to significant advances [11, 12, 13], summarised in the survey article [14], in permutation enumeration and other important themes in combinatorics.

The motivation behind this paper comes from results of [6], as well as unpublished results of Gessel, which we now describe in some detail. Let P be a finite bounded poset, with minimum element $\hat{0}$ and maximum element $\hat{1}$, which is graded of rank $n+1$, with rank function $\rho : P \to \{0, 1, \dots, n+1\}$. For $S \subseteq \{1, 2, \dots, n\}$ we denote by $a_P(S)$ the number of maximal chains of the rank-selected subposet

$$P_S \;=\; \{x \in P : \rho(x) \in S\} \,\cup\, \{\hat{0}, \hat{1}\} \tag{1}$$

of P and set

$$b_P(S) \;=\; \sum_{T \subseteq S} (-1)^{|S-T|} \, a_P(T). \tag{2}$$

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The numbers $a_P(S)$ and $b_P(S)$ are known as the entries of the flag f-vector and the flag h-vector of P, respectively. The numbers $b_P(S)$ can be equivalently defined by the equations

$$a_P(T) \;=\; \sum_{S \subseteq T} b_P(S) \tag{3}$$

for $T \subseteq \{1, 2, \dots, n\}$. They are nonnegative if P is Cohen–Macaulay over some field, and afford a simple combinatorial interpretation if P admits an R-labeling; see [18, Sections 3.13–3.14] [22, Section 3.4] for more information.

The following theorem is a restatement of [6, Corollary 3.8]. As in the references [6, 10], poset homology is taken with integer coefficients, the poset whose Hasse diagram is a complete t-ary tree of height n, rooted at the minimum element, is denoted by $T_{t,n}$, the set of all subsets (called stable) of $\Theta \subseteq \mathbb{Z}$ which do not contain two consecutive integers is denoted by $\mathrm{Stab}(\Theta)$ and the notation $[a, b] := \{a, a+1, \dots, b\}$ for integers $a \le b$ and $[m] := \{1, 2, \dots, m\}$ for positive integers m is used. We note that $T_{t,n-1}$ reduces to the n-element chain for $t = 1$. The poset obtained from P by removing its maximum or minimum element, or both, is denoted by P^-, P_- and $\bar{P}$, respectively. The reader is referred to [3] [22, Section 3.2] for the notion of EL-shellability.

Theorem 1.1. (Linusson–Shareshian–Wachs [6]) *Let P be a finite bounded graded poset of rank $n+1$. If P is EL-shellable, then*

$$\mathrm{rank}\ \widetilde{H}_{n-1}((P^- * T_{t,n})_-) \;=\; \sum_{S \in \mathrm{Stab}([n-2])} b_P([n-1] \smallsetminus S)\ t^{|S|+1}(1+t)^{n-1-2|S|} \;+\; \sum_{S \in \mathrm{Stab}([n-1])} b_P([n] \smallsetminus S)\ t^{|S|}(1+t)^{n-2|S|} \tag{4}$$

and

$$\mathrm{rank}\ \widetilde{H}_{n-1}(\bar{P} * T_{t,n-1}) \;=\; \sum_{S \in \mathrm{Stab}([2,n-2])} b_P([n-1] \smallsetminus S)\ t^{|S|+1}(1+t)^{n-2-2|S|} \;+\; \sum_{S \in \mathrm{Stab}([2,n-1])} b_P([n] \smallsetminus S)\ t^{|S|}(1+t)^{n-1-2|S|} \tag{5}$$

for every positive integer t.

This result, along with some unpublished symmetric function identities due to Gessel, were applied in [6] to provide various examples in which the rank of the top homology group of the Rees product of two posets admits interesting combinatorial interpretations. Two of Gessel's identities can be written in the form

$$\frac{(1-t)E(\mathbf{x}; tz)}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)} \;=\; 1 + \sum_{n \ge 1} z^n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(\mathbf{x})\, t^{k+1}(1+t)^{n-1-2k} \tag{6}$$

and

$$\frac{1-t}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)} \;=\; 1 + \sum_{n \ge 1} z^n \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \xi_{n,k}(\mathbf{x})\, t^{k+1}(1+t)^{n-2k-2}, \tag{7}$$

where $E(\mathbf{x}; z) = \sum_{n \ge 0} e_n(\mathbf{x}) z^n$ is the generating function for the elementary symmetric functions in $\mathbf{x} = (x_1, x_2, \dots)$ and the $\gamma_{n,k}(\mathbf{x})$ and $\xi_{n,k}(\mathbf{x})$ are Schur-positive symmetric functions.

Gessel essentially showed that these functions are sums of border-strip skew Schur functions (see, for instance, [15, Section 3]) and hence, when expanded in the Schur basis, the coefficients count certain standard Young tableaux by the number of descents; see Corollary 4.1. Various interesting combinatorial and algebraic-geometric interpretations of the left hand-sides of Equations (6) and (7) are discussed in [11, Section 7] [6, Section 4] [14].

The main result of this paper provides a strengthening and generalization of Theorem 1.1 which can be shown to reduce to the two identities of Gessel when P^- is the Boolean algebra of rank n. To state it, we need to recall a few more definitions. Let P be bounded and graded of rank $n+1$, as before, fix a field $\mathbf{k}$ and let G be a finite group which acts on P by order preserving bijections (so that P becomes a G-poset). Then G defines a permutation representation $\alpha_P(S)$ over $\mathbf{k}$, induced by the action of G on the set of maximal chains of the rank-selected subposet P_S, for every $S \subseteq \{1, 2, \ldots, n\}$. Note that the dimension of $\alpha_P(S)$ is equal to $a_P(S)$. Following [17], one can consider the virtual G-representation

$$\beta_P(S) \;=\; \sum_{T \subseteq S} (-1)^{|S-T|} \, \alpha_P(T), \tag{8}$$

defined equivalently by the equations

$$\alpha_P(T) \;=\; \sum_{S \subseteq T} \beta_P(S) \tag{9}$$

for $T \subseteq \{1, 2, \ldots, n\}$. When P is Cohen–Macaulay over $\mathbf{k}$, $\beta_P(S)$ coincides with the non-virtual G-representation, of dimension equal to $b_P(S)$, induced on the top homology group of $\bar{P}_S$; see [17] [22, Section 3.4] for more information. Following [6], we write $\beta(\bar{P})$ in place of $\beta_P(\{1, 2, \ldots, n\})$ and recall from [10] that the action of G on P induces actions on the posets obtained from $(P^- * T_{t,n})_-$ and $\bar{P} * T_{t,n-1}$ by adding maximum and minimum elements.

Theorem 1.2. *Let G be a finite group acting on a finite bounded graded poset P of rank $n+1$ by order preserving bijections. Then*

$$\beta((P^- * T_{t,n})_-) \;\cong_G\; \sum_{S \in \mathrm{Stab}([n-2])} \beta_P([n-1] \smallsetminus S) \; t^{|S|+1} (1+t)^{n-1-2|S|} \;+\; \sum_{S \in \mathrm{Stab}([n-1])} \beta_P([n] \smallsetminus S) \; t^{|S|} (1+t)^{n-2|S|} \tag{10}$$

and

$$\beta(\bar{P} * T_{t,n-1}) \;\cong_G\; \sum_{S \in \mathrm{Stab}([2,n-2])} \beta_P([n-1] \smallsetminus S) \; t^{|S|+1} (1+t)^{n-2-2|S|} \;+\; \sum_{S \in \mathrm{Stab}([2,n-1])} \beta_P([n] \smallsetminus S) \; t^{|S|} (1+t)^{n-1-2|S|} \tag{11}$$

for every positive integer t.

In particular, if P is Cohen–Macaulay over $\mathbf{k}$, then the left hand-sides of (10) and (11) may be replaced by the G-representations $\widetilde{H}_{n-1}((P^- * T_{t,n})_-; \mathbf{k})$ and $\widetilde{H}_{n-1}(\bar{P} * T_{t,n-1}; \mathbf{k})$, respectively, and all representations which appear in these formulas are non-virtual.

A polynomial in t with real coefficients is said to be gamma-positive if it can be written, for some $m \in \mathbb{N}$, as a nonnegative linear combination of the binomials $t^i (1+t)^{m-2i}$ for $0 \le i \le m/2$.

All such polynomials have symmetric and unimodal coefficients and, in fact, gamma-positivity has developed into a powerful method to prove these properties; see, for instance, [5, Section 3] and [7, Chapter 4]. Theorem 1.1 can be viewed as one of few general results on gamma-positivity which exist in the literature (an application of this theorem on the gamma-positivity of local h-polynomials of triangulations of simplices appears in [2]) and Theorem 1.2 provides a natural equivariant analogue. Possible equivariant gamma-positivity phenomena arising in geometric combinatorics, instances of which are Gessel's identities, are discussed in [15, Sections 5–6].

The proof of Theorem 1.2 is given in Section 3, after relevant background and definitions are explained in Section 2. This short proof is different from the one of Theorem 1.1 given in [6], uses only elementary computations and does not require any shellability properties for P. Section 4 explains why Theorem 1.2 reduces to the two aforementioned identites of Gessel when P^- is the Boolean algebra B_n and the action is the usual symmetric group action. Another application, in which P^- is the natural signed analogue of B_n on which the hyperoctahedral group acts, results in two new identities of the from

$$\frac{(1-t)E(\mathbf{x};z)E(\mathbf{x};tz)E(\mathbf{y};tz)}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} = 1 + \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor (n+1)/2\rfloor} \gamma^{+}_{n,k}(\mathbf{x},\mathbf{y})\, t^k (1+t)^{n+1-2k} + \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \gamma^{-}_{n,k}(\mathbf{x},\mathbf{y})\, t^k (1+t)^{n-2k} \tag{12}$$

and

$$\frac{(1-t)E(\mathbf{x};z)}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} = 1 + \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \xi^{+}_{n,k}(\mathbf{x},\mathbf{y})\, t^k (1+t)^{n-2k} + \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \xi^{-}_{n,k}(\mathbf{x},\mathbf{y})\, t^k (1+t)^{n-1-2k}, \tag{13}$$

where the $\gamma^{\pm}_{n,k}(\mathbf{x},\mathbf{y})$ and $\xi^{\pm}_{n,k}(\mathbf{x},\mathbf{y})$ are Schur-positive symmetric functions in the sets of variables $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ separately. When these functions are expanded in the Schur basis, the coefficients count certain standard Young bitableaux by the number of descents, for a suitable notion of descent; see Corollaries 4.4 and 4.7.

The previous identities specialize to Equations (6) and (7) for $\mathbf{x} = 0$. Setting $\mathbf{y} = 0$ to (12) yields another interesting identity, recently shown by Shareshian and Wachs as one of the main results of [15] (see Proposition 3.3 and Theorem 3.4 there) in order to prove the gamma-positivity of the q-binomial Eulerian polynomials and to establish the equivariant Gal phenomenon for the n-dimensional stellohedron. These examples indicate the potential of Theorem 1.2 for further applications.

2. Preliminaries

This section briefly records definitions and background on partially ordered sets, group representations and (quasi-)symmetric functions which are needed to understand the results and their proofs. For basic notions and more information on these topics the reader is referred to the sources [9] [17] [18, Chapter 3] [19, Chapter 7] [22]. The symmetric group of permutations of the set $[n] := \{1, 2, \dots, n\}$ is denoted by $\mathfrak{S}_n$ and the cardinality of a finite set S by $|S|$.

Rees products and group actions. All groups and posets considered here are assumed to be finite. Homological notions for posets always refer to those of their order complex; see [22, Lecture 1]. A poset P has the structure of a *G-poset* if the group G acts on P by order preserving bijections. Then G induces a representation on every reduced homology group $\widetilde{H}_i(P; \mathbf{k})$, for every field $\mathbf{k}$.

Suppose that P is a G-poset with minimum element $\hat{0}$ and maximum element $\hat{1}$. Sundaram [21] (see also [22, Theorem 4.4.1]) established the isomorphism of G-representations

$$\bigoplus_{k \ge 0} (-1)^k \bigoplus_{x \in P/G} \widetilde{H}_{k-2}((\hat{0}, x); \mathbf{k}) \uparrow_{G_x}^{G} \ \cong_G \ 0. \tag{14}$$

Here P/G stands for a complete set of G-orbit representatives, $(\hat{0}, x)$ denotes the open interval of elements of P lying strictly between $\hat{0}$ and x, G_x is the stabilizer of x and $\uparrow$ denotes induction. Moreover, $\widetilde{H}_{k-2}((\hat{0}, x); \mathbf{k})$ is understood to be the trivial representation 1_{G_x} if $x = \hat{0}$ and $k = 0$, or x covers $\hat{0}$ and $k = 1$.

Given finite graded posets P and Q with rank functions ρ_P and ρ_Q, respectively, their *Rees product* is defined in [4] as $P * Q = \{(p, q) \in P \times Q : \rho_P(p) \ge \rho_Q(q)\}$, with partial order defined by setting $(p_1, q_1) \preceq (p_2, q_2)$ if all of the following conditions are satisfied:

- $p_1 \preceq p_2$ holds in P,
- $q_1 \preceq q_2$ holds in Q and
- $\rho_P(p_2) - \rho_P(p_1) \ge \rho_Q(q_2) - \rho_Q(q_1)$.

Equivalently, (p_1, q_1) is covered by (p_2, q_2) in $P * Q$ if and only if (a) p_1 is covered by p_2 in P; and (b) either $q_1 = q_2$, or q_1 is covered by q_2 in Q. We note that the Rees product $P * Q$ is graded with rank function given by $\rho(p, q) = \rho_P(p)$ for $(p, q) \in P * Q$, and that if P is a G-poset, then so is $P * Q$ with the G-action defined by setting $g \cdot (p, q) = (g \cdot p, q)$ for $g \in G$ and $(p, q) \in P * Q$.

Permutations, Young tableaux and symmetric functions. Our notation concerning these topics follows mostly that of [9] [18, Chapter 1] [19, Chapter 7]. In particular, the set of standard Young tableaux of shape λ is denoted by $\mathrm{SYT}(\lambda)$, the descent set $\{i \in [n-1] : w(i) > w(i+1)\}$ of a permutation $w \in \mathfrak{S}_n$ is denoted by $\mathrm{Des}(w)$ and that of a tableau $Q \in \mathrm{SYT}(\lambda)$, consisting of those entries i for which $i+1$ appears in Q in a lower row than i, is denoted by $\mathrm{Des}(Q)$. We recall that the Robinson–Schensted correspondence is a bijection from the symmetric group $\mathfrak{S}_n$ to the set of pairs $(\mathcal{P}, Q)$ of standard Young tableaux of the same shape and size n. This correspondence has the property [19, Lemma 7.23.1] that $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$ and $\mathrm{Des}(w^{-1}) = \mathrm{Des}(\mathcal{P}(w))$, where $(\mathcal{P}(w), Q(w))$ is the pair of tableaux associated to $w \in \mathfrak{S}_n$.

We will consider symmetric functions in the indeterminates $\mathbf{x} = (x_1, x_2, \dots)$ over the complex field $\mathbb{C}$. We denote by $E(\mathbf{x}; z) := \sum_{n \ge 0} e_n(\mathbf{x}) z^n$ and $H(\mathbf{x}; z) := \sum_{n \ge 0} h_n(\mathbf{x}) z^n$ the generating functions for the elementary and complete homogeneous symmetric functions, respectively, and recall the identity $E(\mathbf{x}; z) H(\mathbf{x}; -z) = 1$. The characteristic map, a $\mathbb{C}$-linear isomorphism of fundamental importance from the space of virtual $\mathfrak{S}_n$-representations to that of homogeneous symmetric functions of degree n, will be denoted by ch.

The *fundamental quasisymmetric function* associated to $S \subseteq [n-1]$ is defined as

$$F_{n,S}(\mathbf{x}) \ = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in S \,\Rightarrow\, i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{15}$$

The following well-known expansion [19, Theorem 7.19.7]

$$s_\lambda(\mathbf{x}) \ = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(Q)}(\mathbf{x}) \tag{16}$$

of the Schur function $s_\lambda(\mathbf{x})$ associated to $\lambda \vdash n$ will be used in Section 4.

For the applications given there we need the analogues of these concepts in the representation theory of the hyperoctahedral group of signed permutations of the set $[n]$, denoted here by $\mathcal{B}_n$. We will keep this discussion rather brief and refer to [1, Section 2] for more information.

A *bipartition* of a positive integer n, written $(\lambda, \mu) \vdash n$, is any pair (λ, μ) of integer partitions of total sum n. A *standard Young bitableaux* of shape $(\lambda, \mu) \vdash n$ is any pair $Q = (Q^+, Q^-)$ of Young tableaux such that Q^+ has shape λ, Q^- has shape μ and every element of $[n]$ appears exactly once as an enrty of Q^+ or Q^-. The tableaux Q^+ and Q^- are called the *parts* of Q and the number n is its *size*. The Robinson–Schensted correspondence of type B, as described in [17, Section 6] (see also [1, Section 5]) is a bijection from the group $\mathcal{B}_n$ to the set of pairs $(\mathcal{P}, Q)$ of standard Young bitableaux of the same shape and size n.

The characteristic map for the hyperoctahedral group, denoted by $\mathrm{ch}_\mathcal{B}$, is a $\mathbb{C}$-linear isomorphism from the space of virtual $\mathcal{B}_n$-representations to that of homogeneous symmetric functions of degree n in the sets of indeterminates $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ separately; see, for instance, [20, Section 5]. The following basic properties of $\mathrm{ch}_\mathcal{B}$ will be useful in Section 4:

- $\mathrm{ch}_\mathcal{B}(1_{\mathcal{B}_n}) = h_n(\mathbf{x})$, where $1_{\mathcal{B}_n}$ is the trivial $\mathcal{B}_n$-representation,
- $\mathrm{ch}_\mathcal{B}(\sigma \otimes \tau \uparrow^{\mathcal{B}_n}_{\mathcal{B}_k \times \mathcal{B}_{n-k}}) = \mathrm{ch}_\mathcal{B}(\sigma) \cdot \mathrm{ch}_\mathcal{B}(\tau)$ for all representations σ and τ of $\mathcal{B}_k$ and $\mathcal{B}_{n-k}$, respectively, where $\mathcal{B}_k \times \mathcal{B}_{n-k}$ is a Young subgroup of $\mathcal{B}_n$,
- $\mathrm{ch}_\mathcal{B}(\uparrow^{\mathcal{B}_n}_{\mathfrak{S}_n} \rho) = \mathrm{ch}(\rho)(x, y)$ for every $\mathfrak{S}_n$-representation ρ, where $\mathfrak{S}_n \subset \mathcal{B}_n$ is the standard embedding.

We denote by $E(\mathbf{x}, \mathbf{y}; z) := \sum_{n \ge 0} e_n(\mathbf{x}, \mathbf{y}) z^n$ and $H(\mathbf{x}, \mathbf{y}; z) := \sum_{n \ge 0} h_n(\mathbf{x}, \mathbf{y}) z^n$ the generating function for the elementary and complete homogeneous, respectively, symmetric functions in the variables $(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \dots, y_1, y_2, \dots)$ and note that $E(\mathbf{x}, \mathbf{y}; z) = E(\mathbf{x}; z) E(\mathbf{y}; z)$, since $e_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^n e_k(\mathbf{x}) e_{n-k}(\mathbf{y})$, and similarly that $H(\mathbf{x}, \mathbf{y}; z) = H(\mathbf{x}; z) H(\mathbf{y}; z)$.

The *signed descent set* [1, Section 2] [8] of $w \in \mathcal{B}_n$, denoted $\mathrm{sDes}(w)$, records the positions of the increasing (in absolute value) runs of constant sign in the sequence $(w(1), w(2), \dots, w(n))$. Formally, we may define $\mathrm{sDes}(w)$ as the pair $(\mathrm{Des}(w), \varepsilon)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{-, +\}^n$ is the sign vector with ith coordinate equal to the sign of $w(i)$ and $\mathrm{Des}(w)$ consists of the indices $i \in [n-1]$ for which either $\varepsilon_i = +$ and $\varepsilon_{i+1} = -$, or else $\varepsilon_i = \varepsilon_{i+1}$ and $|w(i)| > |w(i+1)|$ (this definition is slightly different from, but equivalent to, the ones given in [1, 8]). The fundamental quasisymmetric function associated to w, introduced by Poirier [8] in a more general setting, is defined as

$$F_{\mathrm{sDes}(w)}(\mathbf{x}, \mathbf{y}) \ = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ j \in \mathrm{Des}(w) \,\Rightarrow\, i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}, \tag{17}$$

where $z_{i_j} = x_{i_j}$ if $\varepsilon_j = +$, and $z_{i_j} = y_{i_j}$ if $\varepsilon_j = -$. Given a standard Young bitableau Q of size n, one defines the signed descent set $\mathrm{sDes}(Q)$ as the pair $(\mathrm{Des}(Q), \varepsilon)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{-,+\}^n$ is the sign vector with ith coordinate equal to the sign of the part of Q in which i appears and $\mathrm{Des}(Q)$ is the set of of indices $i \in [n-1]$ for which either $\varepsilon_i = +$ and $\varepsilon_{i+1} = -$, or $\varepsilon_i = \varepsilon_{i+1}$ and $i+1$ appears in Q in a lower row than i. The function $F_{\mathrm{sDes}(Q)}(\mathbf{x}, \mathbf{y})$ is then defined by Equation (17), with w replaced by Q; see [1, Section 2]. The analogue

$$s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) \ = \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} F_{s\mathrm{Des}(Q)}(\mathbf{x}, \mathbf{y}) \tag{18}$$

of the expansion (16) holds ([1, Proposition 4.2]) and the Robinson–Schensted correspondence of type B has the properties that $\mathrm{sDes}(w) = \mathrm{sDes}(Q^B(w))$ and $\mathrm{sDes}(w^{-1}) = \mathrm{sDes}(\mathcal{P}^B(w))$, where $(\mathcal{P}^B(w), Q^B(w))$ is the pair of bitableaux associated to $w \in \mathcal{B}_n$; see [1, Proposition 5.1].

3. Proof of Theorem 1.2

This section proves Theorem 1.2 using only the definition of Rees product and the defining equation (2) of the representations $\beta_P(S)$. For $S = \{s_1, s_2, \dots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \cdots < s_k$ we set

$$\begin{aligned} \varphi_t(S) &:= [s_1+1]_t \, [s_2 - s_1 + 1]_t \cdots [s_k - s_{k-1} + 1]_t \\ \psi_t(S) &:= [s_1]_t \, [s_2 - s_1 + 1]_t \cdots [s_k - s_{k-1} + 1]_t, \end{aligned}$$

where $[m]_t := 1 + t + \cdots + t^{m-1}$ for positive integers t.

Lemma 3.1. *Let G be a finite group, P be a finite bounded graded G-poset of rank $n+1$, as in Theorem 1.2, and Q, R be the posets defined by $\bar{Q} = (P^- * T_{t,n})_-$ and $\bar{R} = \bar{P} * T_{t,n-1}$. Then*

$$\begin{aligned} \alpha_Q(S) &\cong_G \varphi_t(S) \, \alpha_P(S) \\ \alpha_R(S) &\cong_G \psi_t(S) \, \alpha_P(S) \end{aligned}$$

for all positive integers t and $S \subseteq [n]$.

Proof. Let $S = \{s_1, s_2, \dots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \cdots < s_k$ and let $\rho : T_{t,n} \to \mathbb{N}$ be the rank function of $T_{t,n}$. By the definition of Rees product, the maximal chains in Q_S have the form

$$\hat{0} \prec (p_1, \tau_1) \prec (p_2, \tau_2) \prec \cdots \prec (p_k, \tau_k) \prec \hat{1} \tag{19}$$

where $\hat{0} \prec p_1 \prec p_2 \prec \cdots \prec p_k \prec \hat{1}$ is a maximal chain in P_S and $\tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_k$ is a multichain in $T_{t,n}$ such that

- $0 \le \rho(\tau_1) \le s_1$ and
- $\rho(\tau_j) - \rho(\tau_{j-1}) \le s_j - s_{j-1}$ for $2 \le j \le k$.

Let $m_t(S)$ be the number of these multichains. Since the elements of G act on (19) by fixing the τ_j and acting on the corresponding maximal chain of P_S, we have $\alpha_Q(S) \cong_G m_t(S) \, \alpha_P(S)$. To choose such a multichain $\tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_k$, we need to specify the sequence $i_1 \le i_2 \le \cdots \le i_k$ of ranks of its elements so that $i_j - i_{j-1} \le s_j - s_{j-1}$ for $1 \le j \le k$, where $i_0 := s_0 := 0$, and choose its maximum element τ_k in t^{i_k} possible ways. Summing over all such sequences, we get

$$m_t(S) \ = \sum_{(i_1, i_2, \dots, i_k)} t^{i_k} \ = \sum_{0 \le a_j \le s_j - s_{j-1}} t^{a_1 + a_2 + \cdots + a_k} \ = \ \varphi_t(S)$$

and the result for $\alpha_Q(S)$ follows. The same argument applies to $\alpha_R(S)$; one simply has to switch the condition for the rank of τ_1 to $0 \le \rho(\tau_1) \le s_1 - 1$. □

The proof of the following technical lemma will be given after that of Theorem 1.2.

Lemma 3.2. *We have*

$$\sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \ = \ \begin{cases} 0, & \textit{if } [n] \smallsetminus S \textit{ is not stable}, \\ t^k (1+t)^{n-2k}, & \textit{if } [n] \smallsetminus S \textit{ is stable and } n \in S, \\ t^k (1+t)^{n+1-2k}, & \textit{if } [n] \smallsetminus S \textit{ is stable and } n \notin S \end{cases} \tag{20}$$

and

$$\sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|} \, \psi_t(T) \ = \ \begin{cases} 0, & \textit{if } 1 \notin S, \\ 0, & \textit{if } [n] \smallsetminus S \textit{ is not stable}, \\ t^k (1+t)^{n-1-2k}, & \textit{if } [n] \smallsetminus S \textit{ is stable and } 1, n \in S, \\ t^k (1+t)^{n-2k}, & \textit{if } [n] \smallsetminus S \textit{ is stable, } 1 \in S \textit{ and } n \notin S \end{cases} \tag{21}$$

for every $S \subseteq [n]$, where $k := n - |S|$.

Proof of Theorem 1.2. Using Equations (8) and (9), as well as Lemma 3.1, we compute that

$$\begin{aligned} \beta_Q([n]) \ &= \ \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \alpha_Q(T) \ \cong_G \ \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \, \alpha_P(T) \\ &= \ \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \sum_{S \subseteq T} \beta_P(S) \\ &= \ \sum_{S \subseteq [n]} \beta_P(S) \sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \end{aligned}$$

and find similarly that

$$\beta_R([n]) \ \cong_G \ \sum_{S \subseteq [n]} \beta_P(S) \sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|} \, \psi_t(T).$$

The proof follows from these formulas and Lemma 3.2. For the last statement of the theorem one has to note that, as a consequence of [4, Corollary 2], if P is Cohen–Macaulay over $\mathbf{k}$, then so are the Rees products $P^- * T_{t,n}$ and $\bar{P} * T_{t,n-1}$. □

Proof of Lemma 3.2. We denote by $\chi_t(S)$ the left hand-side of (20) and write $S = \{s_1, s_2, \dots, s_k\}$ with $1 \le s_1 < s_2 < \cdots < s_k \le n$. By definition, we have

$$\chi_t(S) \ = \ \chi_t(s_1) \, \chi_t(s_2 - s_1) \cdots \chi_t(s_k - s_{k-1}) \, \omega_t(n - s_k), \tag{22}$$

where

$$\begin{aligned} \chi_t(n) \ &:= \ \sum_{n \in T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \\ \omega_t(n) \ &:= \ \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \end{aligned}$$

for $n \geq 1$ and $\omega_t(0) := 1$. We claim that

$$\chi_t(n) \ = \ \begin{cases} 1+t, & \text{if } n=1, \\ t, & \text{if } n=2, \\ 0, & \text{if } n \geq 3 \end{cases} \qquad \text{and} \qquad \omega_t(n) \ = \ \begin{cases} 1, & \text{if } n=0, \\ t, & \text{if } n=1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

Equation (20) is a direct consequence of (22) and this claim. To verify the claim, note that the defining equation for $\chi_t(n)$ can be rewritten as

$$\chi_t(n) \ = \ \sum_{(a_1, a_2, \dots, a_k) \models n} (-1)^{n-k} \, [a_1+1]_t \, [a_2+1]_t \cdots [a_k+1]_t,$$

where the sum ranges over all sequences (compositions) $(a_1, a_2, \dots, a_k)$ of positive integers summing to n. By a standard computation of the generating function of the right-hand side (or by interpreting it combinatorially and then double-counting) one shows that $\chi_t(n) = 0$ for every $n \geq 3$. The claim follows from this fact and the obvious recurrence $\omega_t(n) = \chi_t(n) - \omega_t(n-1)$, valid for $n \geq 1$.

Finally, note that Equation (21) is equivalent to (20) in the case $1 \in S$. Otherwise, the terms in the left hand-side can be partitioned into pairs of terms, corresponding to pairs $\{T, T \cup \{1\}\}$ of subsets with $1 \notin T$, cancelling each other. This shows that the left hand-side vanishes. $\square$

4. Applications

This section discusses the promised applications (Corollaries 4.1, 4.4 and 4.7) of Theorem 1.2. The proofs partially follow the methods of [10], with which we assume some familiarity. We first explain why Gessel's identities are special cases of this theorem. The set of ascents of a permutation $w \in \mathfrak{S}_n$ is defined as $\mathrm{Asc}(w) := [n-1] \smallsetminus \mathrm{Des}(w)$ and, similarly, we have $\mathrm{Asc}(\mathcal{P}) := [n-1] \smallsetminus \mathrm{Des}(\mathcal{P})$ for every standard Young tableau $\mathcal{P}$ of size n.

Corollary 4.1. *Equations (6) and (7) are valid for the functions*

$$\gamma_{n,k}(\mathbf{x}) \ = \ \sum_{\lambda \vdash n} c_{\lambda,k} \cdot s_\lambda(\mathbf{x}) \ = \ \sum_{w} F_{n,\mathrm{Des}(w)}(\mathbf{x}) \tag{23}$$

and

$$\xi_{n,k}(\mathbf{x}) \ = \ \sum_{\lambda \vdash n} d_{\lambda,k} \cdot s_\lambda(\mathbf{x}) \ = \ \sum_{w} F_{n,\mathrm{Des}(w)}(\mathbf{x}), \tag{24}$$

where $c_{\lambda,k}$ *(respectively,* $d_{\lambda,k}$*) stands for the number of tableaux* $\mathcal{P} \in \mathrm{SYT}(\lambda)$ *for which* $\mathrm{Asc}(\mathcal{P}) \in \mathrm{Stab}([n-2])$ *(respectively,* $\mathrm{Asc}(\mathcal{P}) \in \mathrm{Stab}([2, n-2])$*) has* k *elements and, similarly,* $w \in \mathfrak{S}_n$ *runs through all permutations for which* $\mathrm{Asc}(w^{-1}) \in \mathrm{Stab}([n-2])$ *(respectively,* $\mathrm{Asc}(w^{-1}) \in \mathrm{Stab}([2, n-2])$*) has* k *elements.*

Proof. We will apply Theorem 1.2 when P^- is the Boolean lattice B_n of subsets of $[n]$, partially ordered by inclusion, considered as an $\mathfrak{S}_n$-poset. On the one hand, the proof of Equation (3.3) in [10, pp. 15–16] shows that

$$1 \ + \sum_{n \geq 1} \mathrm{ch}\left(\widetilde{H}_{n-1}((B_n * T_{t,n})_-; \mathbb{C})\right) z^n \ = \ \frac{(1-t)E(\mathbf{x}; tz)}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)},$$

where the left hand-side is equal to $-F_t(-z)$ in the notation used in that proof. On the other hand, since B_n has a maximum element, the second summand in the right hand-side of Equation (10) vanishes and hence this equation gives

$$\mathrm{ch}\left(\widetilde{H}_{n-1}((B_n * T_{t,n})_-;\mathbb{C})\right) z^n \;=\; \sum_{S\in\mathrm{Stab}([n-2])} \mathrm{ch}\left(\beta_{B_n}([n-1]\smallsetminus S)\right) t^{|S|+1}\,(1+t)^{n-1-2|S|}$$

for $n \ge 1$. The representations $\beta_{B_n}(S)$ for $S \subseteq [n-1]$ are known to satisfy [16, Section 6] (see also [17, Theorem 4.3])

$$\mathrm{ch}\left(\beta_{B_n}(S)\right) \;=\; \sum_{\lambda\vdash n} c_{\lambda,S}\cdot s_\lambda(\mathbf{x}),$$

where $c_{\lambda,S}$ is the number of standard Young tableaux of shape λ and descent set equal to S. Combining the previous three equalities yields the first equality in Equation (23). The second equality follows from the first by expanding $s_\lambda(\mathbf{x})$ according to Equation (16) to get

$$\gamma_{n,k}(\mathbf{x}) \;=\; \sum_{\lambda\vdash n}\sum_{\mathcal{P}}\sum_{Q\in\mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(Q)}(\mathbf{x})$$

where, in the inner sum, $\mathcal{P}$ runs over all tableaux in $\mathrm{SYT}(\lambda)$ for which $\mathrm{Asc}(\mathcal{P}) \in \mathrm{Stab}([n-2])$ has k elements, and then using the Robinson–Schensted correspondence and its standard properties $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$ and $\mathrm{Des}(w^{-1}) = \mathrm{Des}(\mathcal{P}(w))$ to replace the summations with one running over elements of $\mathfrak{S}_n$, as in the statement of the corollary.

The proof of (24) is entirely similar; one has to use Equation (11) instead of (10), as well as the equality

$$1 + \sum_{n\ge 1} \mathrm{ch}\left(\widetilde{H}_{n-1}((B_n\smallsetminus\{\varnothing\}) * T_{t,n-1};\mathbb{C})\right) z^n \;=\; \frac{1-t}{E(\mathbf{x};tz)-tE(\mathbf{x};z)}.$$

The latter, although not explicitly stated in [10], follows as in the proof of its special case $t = 1$ [10, Corollary 5.2]. ☐

Example 4.2. The coefficient of z^4 in the left hand-sides of Equations (6) and (7) equals

- $e_4(\mathbf{x})\,(t+t^2+t^3+t^4) + e_1(\mathbf{x})e_3(\mathbf{x})\,(t^2+t^3) + e_2(\mathbf{x})^2\,(t^2+t^3)$, and
- $e_4(\mathbf{x})\,(t+t^2+t^3) + e_2(\mathbf{x})^2\,t^2$,

respectively. These expressions may be rewritten as

- $s_{(1,1,1,1)}(\mathbf{x})\,t(1+t)^3 + 2s_{(2,1,1)}(\mathbf{x})\,t^2(1+t) + s_{(2,2)}(\mathbf{x})\,t^2(1+t)$, and
- $s_{(1,1,1,1)}(\mathbf{x})\,t(1+t)^3 + s_{(2,1,1)}(\mathbf{x})\,t^2 + s_{(2,2)}(\mathbf{x})\,t^2$,

respectively, and hence $\gamma_{4,0}(\mathbf{x}) = s_{(1,1,1,1)}(\mathbf{x})$, $\gamma_{4,1}(\mathbf{x}) = 2s_{(2,1,1)}(\mathbf{x})+s_{(2,2)}(\mathbf{x})$, $\xi_{4,0}(\mathbf{x}) = s_{(1,1,1,1)}(\mathbf{x})$ and $\xi_{4,1}(\mathbf{x}) = s_{(2,1,1)}(\mathbf{x}) + s_{(2,2)}(\mathbf{x})$. We leave it to the reader to verify that these formulas agree with Corollary 4.1. ☐

We now focus on the identities (12) and (13). We consider the collection sB_n of all subsets of $\{1, 2, \ldots, n\}\cup\{-1, -2, \ldots, -n\}$ which do not contain $\{i, -i\}$ for any index i, partially ordered by inclusion. This signed analogue of the Boolean lattice B_n is a graded poset of rank n, having the empty set as its minimum element, on which the hyperoctahedral group $\mathcal{B}_n$ acts in the obvious way, turning it into a $\mathcal{B}_n$-poset. It was considered (under different notation) in [10, Section 6] and, in a more general form, in [2, Section 3.2]. The poset sB_n is shellable (see, for instance, [2, p. 1487]) and hence Cohen–Macaulay over $\mathbb{Z}$ and any field. We recall that the Lefschetz character of a finite G-poset P is defined as the virtual G-representation

$$L(P;G) \ := \ \bigoplus_{i\geq 0} (-1)^i \, \widetilde{H}_i(P;\mathbb{C})$$

and note that $L(P;G) = (-1)^r \widetilde{H}_r(P;\mathbb{C})$, if P is Cohen–Macaulay over $\mathbb{C}$ of rank r.

Proposition 4.3. *For the $\mathcal{B}_n$-poset sB_n we have*

$$1 + \sum_{n\geq 1} \mathrm{ch}_{\mathcal{B}} \left(\widetilde{H}_{n-1}((sB_n * T_{t,n})_-;\mathbb{C}) \right) z^n \ = \ \frac{(1-t)E(\mathbf{y};z)E(\mathbf{x};tz)E(\mathbf{y};tz)}{E(\mathbf{x};tz)E(\mathbf{y};tz) - tE(\mathbf{x};z)E(\mathbf{y};z)}. \tag{25}$$

Proof. Following the reasoning in the proof of [10, Equation (3.3)], we set

$$L_n(\mathbf{x},\mathbf{y};t) \ := \ \mathrm{ch}_{\mathcal{B}} \left(L((sB_n * T_{t,n})_-;\mathcal{B}_n) \right).$$

Since $(sB_n * T_{t,n})_-$ is Cohen–Macaulay over $\mathbb{C}$ of rank $n-1$, we have

$$\mathrm{ch}_{\mathcal{B}} \left(\widetilde{H}_{n-1}((sB_n * T_{t,n})_-;\mathbb{C}) \right) \ = \ (-1)^{n-1} \, L_n(\mathbf{x},\mathbf{y};t).$$

Thus, the right hand-side of Equation (25) is equal to $-\sum_{n\geq 0} L_n(\mathbf{x},\mathbf{y};t)(-z)^n$. The sequence of posets $(sB_0, sB_1, \dots, sB_n)$ is easily verified to be $(\mathcal{B}_0 \times \mathfrak{S}_n, \mathcal{B}_1 \times \mathfrak{S}_{n-1}, \dots, \mathcal{B}_n \times \mathfrak{S}_0)$-uniform, in the sense of [10, Section 3]. Moreover, there is a single $\mathcal{B}_n$-orbit of elements of sB_n of rank k for each $k \in \{0,1,\dots,n\}$. Thus, applying [10, Proposition 3.7] to this sequence gives

$$1_{\mathcal{B}_n} \oplus \bigoplus_{k=0}^{n} [k+1]_t \, L((sB_{n-k} * T_{t,n-k})_-; \mathcal{B}_{n-k} \times \mathfrak{S}_k) \uparrow_{\mathcal{B}_{n-k}\times\mathfrak{S}_k}^{\mathcal{B}_n} \ \cong_{\mathcal{B}_n} \ 0.$$

Applying the characteristic map $\mathrm{ch}_{\mathcal{B}}$ and using the transitivity $\uparrow_{\mathcal{B}_{n-k}\times\mathfrak{S}_k}^{\mathcal{B}_n} \cong_{\mathcal{B}_n} \uparrow_{\mathcal{B}_{n-k}\times\mathfrak{S}_k}^{\mathcal{B}_{n-k}\times\mathcal{B}_k} \uparrow_{\mathcal{B}_{n-k}\times\mathcal{B}_k}^{\mathcal{B}_n}$ of induction and properties of $\mathrm{ch}_{\mathcal{B}}$ discussed in Section 2 gives

$$\sum_{k=0}^{n} [k+1]_t \, h_k(\mathbf{x},\mathbf{y}) \, L_{n-k}(\mathbf{x},\mathbf{y};t) \ = \ -h_n(\mathbf{x}).$$

Standard manipulation with generating functions, just as in the proof of [10, Equation (3.3)], results in the formula

$$\sum_{n\geq 0} L_n(\mathbf{x},\mathbf{y};t) z^n \ = \ - \, \frac{H(\mathbf{x};z)}{\sum_{n\geq 0} [n+1]_t \, h_n(\mathbf{x},\mathbf{y}) z^n} \ = \ - \, \frac{(1-t)H(\mathbf{x};z)}{H(\mathbf{x},\mathbf{y};z) - tH(\mathbf{x},\mathbf{y};tz)}.$$

The proof now follows by switching z to $-z$ and using the identities $E(\mathbf{x};z)H(\mathbf{x};-z) = 1$ and $E(\mathbf{x},\mathbf{y};z) = E(\mathbf{x};z)E(\mathbf{y};z)$. □

Recall the definition of the sets $\mathrm{Des}(w)$ and $\mathrm{Des}(\mathcal{P})$ for signed permutations $w \in \mathcal{B}_n$ and standard Young bitableaux $\mathcal{P}$ of size n, respectively, from Section 2. Following [17, Section 6], we define the *type B descent set* of $\mathcal{P} = (\mathcal{P}^+, \mathcal{P}^-)$ as $\mathrm{Des}_B(\mathcal{P}) = \mathrm{Des}(\mathcal{P}) \cup \{n\}$, if n appears in $\mathcal{P}^+$, and $\mathrm{Des}_B(\mathcal{P}) = \mathrm{Des}(\mathcal{P})$ otherwise. The complement of $\mathrm{Des}_B(\mathcal{P})$ in the set $[n]$ is the *type B ascent set* of $\mathcal{P}$ and denoted by $\mathrm{Asc}_B(\mathcal{P})$. Similarly, we define the *type B descent set* of $w \in \mathcal{B}_n$ as $\mathrm{Des}_B(w) = \mathrm{Des}(w) \cup \{n\}$, if $w(n)$ is positive, and $\mathrm{Des}_B(w) = \mathrm{Des}(w)$ otherwise. The complement of $\mathrm{Des}_B(w)$ in the set $[n]$ is the *type B ascent set* of w and denoted by $\mathrm{Asc}_B(w)$. The sets $\mathrm{Des}_B(w)$ and $\mathrm{Des}_B(\mathcal{P})$ depend only in the signed descent sets $\mathrm{sDes}(w)$ and $\mathrm{sDes}(\mathcal{P})$, respectively, and [1, Proposition 5.1], mentioned at the end of Section 2, implies that $\mathrm{Des}_B(w) = \mathrm{Des}_B(Q^B(w))$ and $\mathrm{Des}_B(w^{-1}) = \mathrm{Des}_B(\mathcal{P}^B(w))$ for every $w \in \mathcal{B}_n$.

Corollary 4.4. *Equation (12) is valid for the functions*

$$\gamma^+_{n,k}(\mathbf{x},\mathbf{y}) \;=\; \sum_{(\lambda,\mu)\vdash n} c^+_{(\lambda,\mu),k} \cdot s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) \;=\; \sum_w F_{\mathrm{sDes}(w)}(\mathbf{x},\mathbf{y}) \tag{26}$$

and

$$\gamma^-_{n,k}(\mathbf{x},\mathbf{y}) \;=\; \sum_{(\lambda,\mu)\vdash n} c^-_{(\lambda,\mu),k} \cdot s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) \;=\; \sum_w F_{\mathrm{sDes}(w)}(\mathbf{x},\mathbf{y}), \tag{27}$$

where $c^+_{(\lambda,\mu),k}$ (respectively, $c^-_{(\lambda,\mu),k}$) is the number of bitableaux $\mathcal{P} \in \mathrm{SYT}(\lambda,\mu)$ for which $\mathrm{Asc}_B(\mathcal{P}) \in \mathrm{Stab}([n])$ has k elements and contains (respectively, does not contain) n and, similarly, $w \in \mathcal{B}_n$ runs through all signed permutations for which $\mathrm{Asc}_B(w^{-1}) \in \mathrm{Stab}([n])$ has k elements and contains (respectively, does not contain) n.

Proof. We apply the first part of Theorem 1.2 when $P^- = sB_n$, thought of as a $\mathcal{B}_n$-poset. The representations $\beta_P(S)$ for $S \subseteq [n]$ were computed in this case in [17, Theorem 6.4], which implies that

$$\mathrm{ch}_{\mathcal{B}}\left(\beta_{B_n}(S)\right) \;=\; \sum_{(\lambda,\mu)\vdash n} c_{(\lambda,\mu),S} \cdot s_\lambda(\mathbf{y}) s_\mu(\mathbf{x})$$

for $S \subseteq [n]$, where $c_{(\lambda,\mu),S}$ is the number of standard Young bitableaux $\mathcal{P}$ of shape (λ,μ) such that $\mathrm{Des}_B(\mathcal{P}) = S$. Switching the roles of $\mathbf{x}$ and $\mathbf{y}$ and combining this result with the first part of Theorem 1.2 and Proposition 4.3 yields the first equalities in (26) and (27). The second equalities follow from those by expanding $s_\lambda(\mathbf{x})s_\mu(\mathbf{y})$ according to Equation (18) and then using the Robinson–Schensted correspondence of type B and its properties $\mathrm{sDes}(w) = \mathrm{sDes}(Q^B(w))$ and $\mathrm{Des}_B(w^{-1}) = \mathrm{Des}_B(\mathcal{P}^B(w))$, exactly as in the proof of Corollary 4.1. □

Example 4.5. The coefficient of z^2 in the left hand-side of Equation (12) is equal to

$$e_2(\mathbf{x})\,(1+t+t^2) + e_1(\mathbf{x})^2\,t + e_1(\mathbf{x})e_1(\mathbf{y})\,(2t+t^2) + e_2(\mathbf{y})\,(t+t^2) \;=\; s_{(1,1)}(\mathbf{x})\,(1+t)^2 + s_{(2)}(\mathbf{x})\,t + s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})\,(2t+t^2) + s_{(2)}(\mathbf{y})\,t(1+t)$$

and hence $\gamma^+_{2,0}(\mathbf{x},\mathbf{y}) = 0$, $\gamma^+_{2,1}(\mathbf{x},\mathbf{y}) = s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y}) + s_{(2)}(\mathbf{y})$, $\gamma^-_{2,0}(\mathbf{x},\mathbf{y}) = s_{(1,1)}(\mathbf{x})$ and $\gamma^-_{2,1}(\mathbf{x},\mathbf{y}) = s_{(2)}(\mathbf{x}) + s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})$, in agreement with Corollary 4.4. □

Consider the n-element chain $C_n = \{0,1,\ldots,n-1\}$, with the usual total order. Following [10], we denote by $I_j(B_n)$ the order ideal of elements of $(B_n \smallsetminus \{\varnothing\}) * C_n$ which are strictly less than $([n], j)$. Then $I_j(B_n)$ is an $\mathfrak{S}_n$-poset for every $j \in C_n$ and one of the main results of [10] (see [10, p. 21] [14, Equation (2.5)]) states that

$$1 \,+\, \sum_{n\geq 1} z^n \sum_{j=0}^{n-1} t^j \,\mathrm{ch}\left(\widetilde{H}_{n-2}(I_j(B_n);\mathbb{C})\right) \;=\; \frac{(1-t)E(\mathbf{x};z)}{E(\mathbf{x};tz) - tE(\mathbf{x};z)}. \tag{28}$$

This result will be essential in the proof of the following statement.

Proposition 4.6. *For the $\mathcal{B}_n$-poset sB_n we have*

$$1 + \sum_{n\geq 1} \mathrm{ch}_{\mathcal{B}} \left(\widetilde{H}_{n-1}((sB_n \smallsetminus \{\varnothing\}) * T_{t,n-1}; \mathbb{C}) \right) z^n \;=\; \frac{(1-t)E(\mathbf{y};z)}{E(\mathbf{x};tz)E(\mathbf{y};tz) - tE(\mathbf{x};z)E(\mathbf{y};z)}. \tag{29}$$

Proof. Following the reasoning in the proof of [10, Corollary 5.2], we apply (14) to the Cohen–Macaulay $\mathcal{B}_n$-poset obtained from $(sB_n \smallsetminus \{\varnothing\}) * T_{t,n-1}$ by adding a minimum and a maximum element. For $0 \le j < k \le n$, there are exactly t^j elements x of rank k in this poset with second coordinate equal to j and for each one of these, the open interval $(\hat{0}, x)$ is isomorphic to $I_j(B_k)$ and the stabilizer of x is isomorphic to $\mathfrak{S}_k \times \mathcal{B}_{n-k}$. We conclude that

$$\widetilde{H}_{n-1}((sB_n \smallsetminus \{\varnothing\}) * T_{t,n-1}; \mathbb{C}) \;\cong_{\mathcal{B}_n}\; \bigoplus_{k=0}^{n} (-1)^{n-k} \bigoplus_{j=0}^{k-1} t^j \left(\widetilde{H}_{k-2}(I_j(B_k); \mathbb{C}) \otimes 1_{\mathcal{B}_{n-k}} \right) \uparrow_{\mathfrak{S}_k \times \mathcal{B}_{n-k}}^{\mathcal{B}_n} .$$

Applying the characteristic map $\mathrm{ch}_{\mathcal{B}}$, as in the proof of Proposition 4.3, the right hand-side becomes

$$\sum_{k=0}^{n} (-1)^{n-k} \sum_{j=0}^{k-1} t^j \,\mathrm{ch} \left(\widetilde{H}_{k-2}(I_j(B_k); \mathbb{C}) \right) (\mathbf{x}, \mathbf{y}) \cdot h_{n-k}(\mathbf{x}).$$

Thus, the left hand-side of Equation (29) is equal to

$$H(\mathbf{x}; -z) \cdot \left(1 + \sum_{n \ge 1} z^n \sum_{j=0}^{n-1} t^j \,\mathrm{ch} \left(\widetilde{H}_{n-2}(I_j(B_n); \mathbb{C}) \right) (\mathbf{x}, \mathbf{y}) \right)$$

and the result follows from Equation (28) and the identities $E(\mathbf{x}; z)H(\mathbf{x}; -z) = 1$ and $E(\mathbf{x}, \mathbf{y}; z) = E(\mathbf{x}; z)E(\mathbf{y}; z)$. □

Corollary 4.7. *Equation (13) is valid for the functions*

$$\xi^+_{n,k}(\mathbf{x}, \mathbf{y}) \;=\; \sum_{(\lambda,\mu) \vdash n} d^+_{(\lambda,\mu),k} \cdot s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) \;=\; \sum_{w} F_{\mathrm{sDes}(w)}(\mathbf{x}, \mathbf{y}) \tag{30}$$

and

$$\xi^-_{n,k}(\mathbf{x}, \mathbf{y}) \;=\; \sum_{(\lambda,\mu) \vdash n} d^-_{(\lambda,\mu),k} \cdot s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) \;=\; \sum_{w} F_{\mathrm{sDes}(w)}(\mathbf{x}, \mathbf{y}), \tag{31}$$

where $d^+_{(\lambda,\mu),k}$ (respectively, $d^-_{(\lambda,\mu),k}$) stands for the number of bitableaux $\mathcal{P} \in \mathrm{SYT}(\lambda, \mu)$ for which $\mathrm{Asc}_B(\mathcal{P}) \in \mathrm{Stab}([2, n])$ has k elements and contains (respectively, does not contain) n and where, similarly, $w \in \mathcal{B}_n$ runs through all signed permutations for which $\mathrm{Asc}_B(w^{-1}) \in \mathrm{Stab}([2, n])$ has k elements and contains (respectively, does not contain) n.

Proof. This statement follows by the same reasoning as in the proof of Corollary 4.4, provided one appeals to the second part of Theorem 1.2 and Proposition 4.6 instead. □

Example 4.8. The coefficient of z^2 in the left hand-side of Equation (13) is equal to

$$e_2(\mathbf{x})\,(1+t) + e_1(\mathbf{x})e_1(\mathbf{y})\,t + e_2(\mathbf{y})\,t \;=\; s_{(1,1)}(\mathbf{x})\,(1+t) + s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})\,t + s_{(1,1)}(\mathbf{y})\,t$$

and hence $\xi^+_{2,0}(\mathbf{x}, \mathbf{y}) = 0$, $\xi^+_{2,1}(\mathbf{x}, \mathbf{y}) = s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y}) + s_{(1,1)}(\mathbf{y})$ and $\xi^-_{2,0}(\mathbf{x}, \mathbf{y}) = s_{(1,1)}(\mathbf{x})$, in agreement with Corollary 4.7. □

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