Face numbers of barycentric subdivisions of cubical polytopes

Polytopics: Recent Advances on Polytopes

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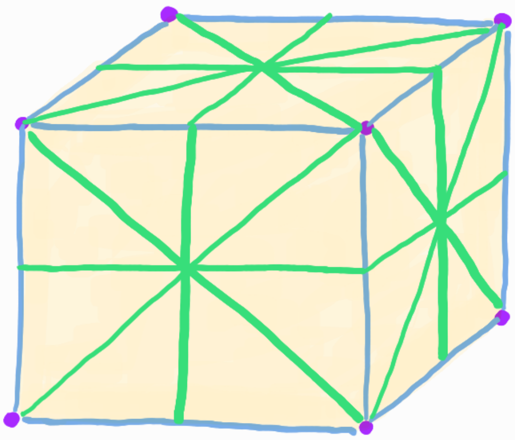
We are concerned with the following question.

Question (Brenti-Welker, 2008)

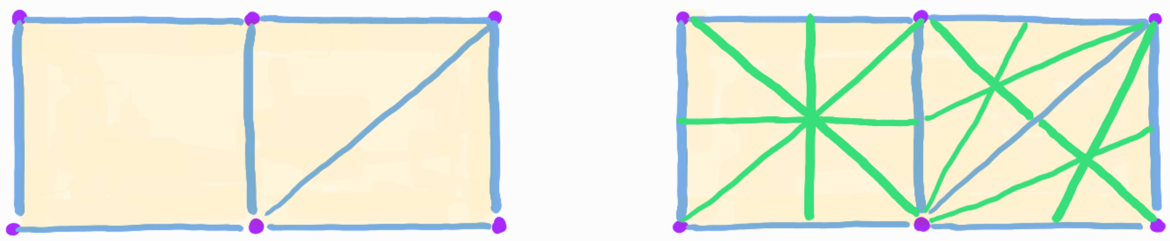
Does the face enumerating polynomial of the barycentric subdivision of a convex polytope have only real roots?



Recall that the barycentric subdivision of a polyhedral complex $\mathcal{L}$ is defined as

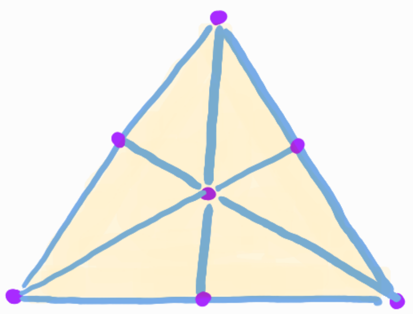
$\mathrm{sd}(\mathcal{L})$ = simplicial complex of all chains of nonempty faces of $\mathcal{L}$

= order complex of the poset of nonempty faces of $\mathcal{L}$.

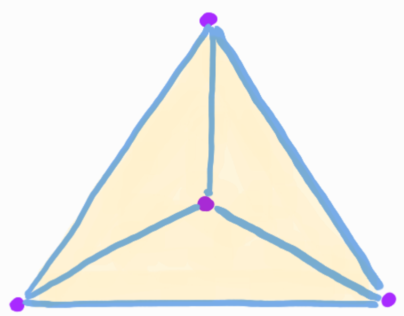


$\mathcal{L}$ $\qquad\qquad$ $\mathrm{sd}(\mathcal{L})$

Note. Order complexes form a special class of flag simplicial complexes (clique complexes of graphs).



flag



not flag

The f- and h-polynomials of a (finite, abstract) $(n-1)$-dimensional simplicial complex Δ are defined as

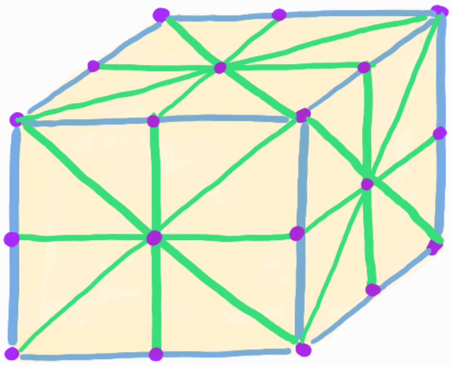
- $$f(\Delta, x) = \sum_{F \in \Delta} x^{|F|}$$

- $$h(\Delta, x) = (1-x)^n f\left(\Delta, \frac{x}{1-x}\right).$$

Thus

$f(sd(\mathcal{L}), x)$ = chain polynomial of the poset of nonempty faces of $\mathcal{L}$.

Example. For the boundary complex of the 3-dimensional cube



- $f(\Delta, x) = 1 + 26x + 72x^2 + 48x^3$
- $h(\Delta, x) = (1-x)^3 + 26x(1-x)^2 + 72x^2(1-x)$
$$+ 48x^3$$
$$= 1 + 23x + 23x^2 + x^3.$$

Note. $f(\Delta, x)$ is real-rooted iff so is $h(\Delta, x)$.

Brenti-Welker (2008)

- showed that $h(sd(P), x)$ is real-rooted for every simplicial polytope P
- asked whether the same holds for all polytopes.

Problem. Find a counterexample, or a proof, at least for a broad class of nonsimplicial, nonsimple polytopes.

Why is it plausible that the question has an affirmative answer?

- $h(sd(P),x) = h(sd(\partial P),x)$ has palindromic and nonnegative coefficients and (as observed by Gal) can be expressed as

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i(P)\, x^i (1+x)^{n-2i}$$

for some $\gamma_i(P) \in \mathbb{N}$ $(n = \dim(P))$.

Why is the question interesting?

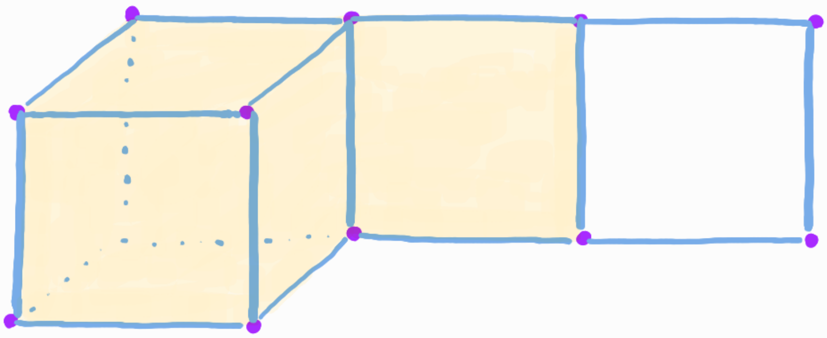
- It concerns flag f-vectors of polytopes.
- It is part of a broader class of questions about
  - flag triangulations of balls and spheres
  - triangulations of polyhedral and cell complexes
  - order complexes of posets.

Theorem (A, 2020+) $h(sd(\mathcal{L}),x)$ is real-rooted for every pure shellable cubical complex $\mathcal{L}$.

In particular, $h(sd(P),x)$ is real-rooted for every cubical polytope P.

Recall that

- a polyhedral complex $\mathcal{L}$ is said to be cubical if every face of $\mathcal{L}$ is combinatorially isomorphic to a cube



- a polytope P is cubical if so is its boundary complex ∂P.

Note. Cubical complexes and polytopes are important and mysterious objects in mathematics with highly intricate combinatorial properties.

Review of the simplicial case.

Given an $(n-1)$-dimensional simplicial complex Δ we write

- $$h(\Delta, x) = \sum_{k=0}^{n} h_k(\Delta)\, x^k.$$

Then, there exist $p_{n,k}(x) \in \mathbb{N}[x]$ such that

- $$h(sd(\Delta), x) = \sum_{k=0}^{n} h_k(\Delta)\, p_{n,k}(x).$$

$$p_{n,k}(x)$$

$n \backslash k$	0	1	2	3
0	1			
1	1	x		
2	$1+x$	$2x$	$x+x^2$	
3	$1+4x+x^2$	$4x+2x^2$	$2x+4x^2$	$x+4x^2+x^3$

$$p_{4,k}(x) = \begin{cases} 1+11x+11x^2+x^3, & k=0 \\ 8x+14x^2+2x^3, & k=1 \\ 4x+16x^2+4x^3, & k=2 \\ 2x+14x^2+8x^3, & k=3 \\ x+11x^2+11x^3+x^4, & k=4. \end{cases}$$

Recurrence:

$$p_{n+1,k}(x) = x \sum_{i=0}^{k-1} p_{n,i}(x) + \sum_{i=k}^{n} p_{n,i}(x)$$

$$\Longrightarrow$$

$\sum_{k=0}^{n} c_k p_{n,k}(x)$ is real-rooted whenever $c_k \geq 0$ for all k.

One possible approach: Could it be that for every n-dimensional polytope P

$$h(sd(P), x) = \sum_{k=0}^{n} c_k(P)\, p_{n,k}(x)$$

for some $c_k(P) \geq 0$?

Note. The $c_k(P) \geq 0$ are linear inequalities on the flag f-vector of P.

Unfortunately, no.

- There exist 4-dimensional cubical neighborly polytopes for which this fails.
- There exists a 6-dimensional zonotope for which this fails

Question (a) For which polytopes $c_k(P) \geq 0$ for all k? (b) For which polytopes $c_k(P) \in \mathbb{N}$ for all k?

Another approach: Let $\mathcal{L}$ be an n-dimensional cubical complex and

- $f_k(\mathcal{L})$ = # of k-dimensional faces
- $\tilde{\chi}(\mathcal{L})$ = reduced Euler characteristic

of $\mathcal{L}$. The cubical h-vector and h-polynomial

$$h(\mathcal{L},x) = \sum_{k=0}^{n+1} h_k(\mathcal{L})\, x^k$$

were defined by Adin (1996) by the formula

$$(1+x)h(\mathcal{L},x) = 1 + \sum_{k=0}^{n} f_k(\mathcal{L})\, x^{k+1} \left(\frac{1-x}{2}\right)^{n-k} + (-1)^n \tilde{\chi}(\mathcal{L})\, x^{n+2}.$$

Note. $h_k(\mathcal{L}) \geq 0$ for every pure shellable cubical complex $\mathcal{L}$ and all k.

Then

$$h(sd(\mathcal{L}),x) = \sum_{k=0}^{n+1} h_k(\mathcal{L})\, p^B_{n,k}(x)$$

for some $p^B_{n,k}(x) \in \mathbb{Z}[x]$.

$n \backslash k$	0	1	2	3
0	1	x		
1	$1+x$	$4x$	$x+x^2$	
2	$1+6x+x^2$	$12x+4x^2$	$4x+12x^2$	$x+6x^2+x^3$

$$p^B_{n,k}(x)$$

$$P^B_{3,k}(x) = \begin{cases} 1 + 23x + 23x^2 + x^3 \\ 36x + 54x^2 + 4x^3 \\ 12x + 72x^2 + 12x^3 \\ 4x + 54x^2 + 36x^3 \\ x + 23x^2 + 23x^3 + x^4. \end{cases}$$

Recurrence:

$$x \sum_{i=0}^{k-1} P^B_{n,i}(x) + \sum_{i=k}^{n+1} P^B_{n,i}(x) =$$

$$= \begin{cases} P^B_{n+1,k}(x), & k \in \{0, n+2\} \\ P^B_{n+1,k}(x)/2, & 1 \leq k \leq n+1 \end{cases}$$

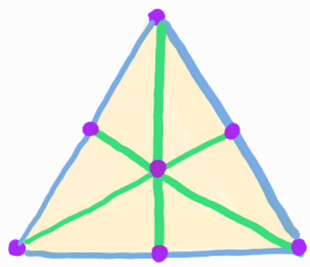
$$\Rightarrow \sum_{k=0}^{n+1} c_k \, p^{B}_{n,k}(x) \quad \text{is real-rooted whenever } c_k \geq 0.$$

Geometric interpretation: Let

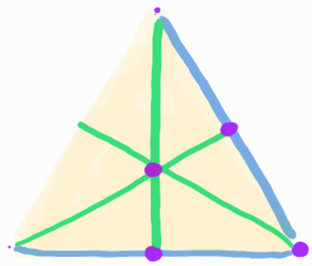
- σ_n = $(n-1)$-dimensional simplex
- τ_n = n-dimensional cube.

Then

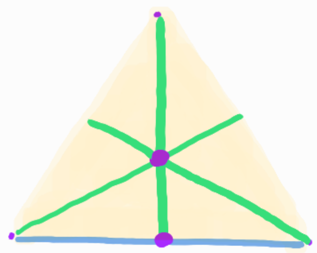
$p_{n,k}(x)$ = h-polynomial of relative simplicial complex obtained from $sd(\sigma_n)$ by removing all faces on k facets of $\partial\sigma_n$.



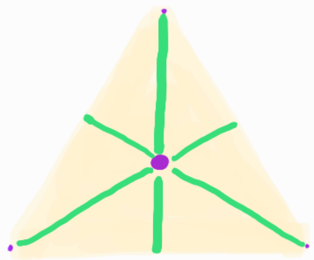
$k=0$



$k=1$



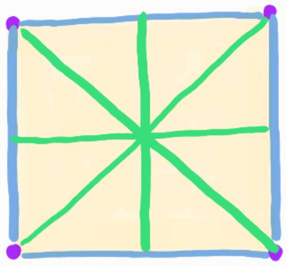
$k=2$



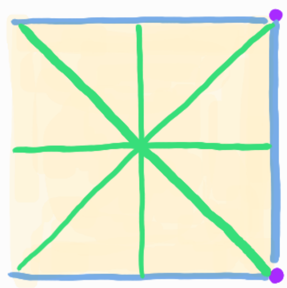
$k=3$

$n=3$

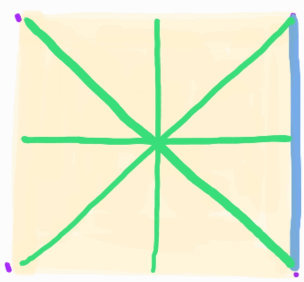
There is a similar interpretation for the $p^B_{n,k}(x)$:



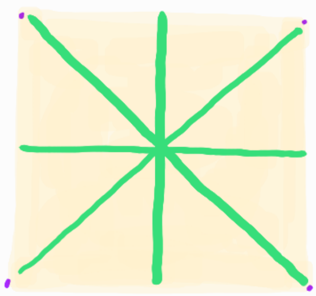
$k=0$



$k=1$



$k=2$



$k=3$

$n=2$

One removes all faces of $sd(\tau_n)$ from

- no facet of $\partial\tau_n$ for $k=0$
- one facet of $\partial\tau_n$ for $k=1$
- $k-1$ more pairs of antipodal facets of $\partial\tau_n$ for $2\leq k\leq n$
- all facets of $\partial\tau_n$ for $k=n+1$.

Open: Find a combinatorial interpretation of $p^{B}_{n,k}(x)$.

Open: Is $h(sd(Z),x)$ real-rooted for every zonotope Z?

Thank you for your attention!