On the Möbius function of a lower Eulerian Cohen-Macaulay poset

Christos A. Athanasiadis
Department of Mathematics
University of Athens
Athens 15784, Hellas (Greece)
Email: caath@math.uoa.gr

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Abstract

A certain inequality is shown to hold for the values of the Möbius function of the poset obtained by attaching a maximum element to a lower Eulerian Cohen-Macaulay poset. In two important special cases, this inequality provides partial results supporting Stanley's nonnegativity conjecture for the toric h-vector of a lower Eulerian Cohen-Macaulay meet-semilattice and Adin's nonnegativity conjecture for the cubical h-vector of a Cohen-Macaulay cubical complex.

Keywords: Eulerian poset, Cohen-Macaulay poset, Möbius function, cubical h-vector, toric h-vector, Buchsbaum complex.

1 Introduction

Let P be a finite poset which has a minimum element $\hat{0}$ (for background and any undefined terminology on partially ordered sets we refer the reader to [12, Chapter 3] and Section 2). Such a poset is called *lower Eulerian* [13, Section 4] if every interval $[x, y]$ in P is graded and satisfies $\mu_P(x,y) = (-1)^{\rho(x,y)}$, where μ_P is the Möbius function of P and $\rho(x,y)$ is the common length of all maximal chains of $[x, y]$. Examples of lower Eulerian posets are the face posets of finite regular cell complexes; see [4, Section 12.4] [12, Section 3.8]. The main result of this paper concerns the Möbius function of a poset obtained by attaching a maximum element to a lower Eulerian Cohen-Macaulay poset (here, the Cohen-Macaulay property is defined with respect to an arbitrary field).

Theorem 1.1 *Let P be a lower Eulerian Cohen-Macaulay poset, with minimum element $\hat{0}$ and set of atoms $\mathcal{A}(P)$. Let $\hat{P} = P \cup \{\hat{1}\}$ be the poset obtained from P by attaching a maximum element $\hat{1}$ and let $\mu_{\hat{P}}$ denote the Möbius function of $\hat{P}$. Then we have*

$$\sum_{x \in \mathcal{A}(P)} |\mu_{\hat{P}}(x, \hat{1})| \; \geq \; \alpha_P \, |\mu_{\hat{P}}(\hat{0}, \hat{1})|, \tag{1}$$

where α_P is the minimum cardinality of the sets $\{x \in \mathcal{A}(P) : x \leq_P y\}$ of atoms of P in the interval $[\hat{0}, y]$, when y runs through all maximal elements of P.

To explain the motivation behind Theorem 1.1 and discuss some of its consequences, we consider two important classes of lower Eulerian posets. We recall that a finite poset P having a minimum element $\hat{0}$ is called *simplicial* [12, p. 135] (respectively, *cubical* [3]) if for every $x \in P$, the interval $[\hat{0}, x]$ in P is isomorphic to a Boolean lattice (respectively, to the face poset of a cube). The h-vector $(h_0(P), h_1(P), \ldots, h_d(P))$ is a fundamental enumerative invariant of a simplicial poset P (see, for instance, [13, p. 113]) defined by the formula

$$\sum_{i=0}^{d} h_i(P)q^i \;=\; \sum_{i=0}^{d} f_{i-1}(P)\, q^i (1-q)^{d-i}, \tag{2}$$

where d is the largest length of a chain in P and $f_{i-1}(P)$ is the number of elements $x \in P$ such that the interval $[\hat{0}, x]$ is isomorphic to a Boolean lattice of rank i. For a graded cubical poset P of rank d, Adin [1] introduced the cubical h-vector $(h_0^{(c)}(P), h_1^{(c)}(P), \ldots, h_d^{(c)}(P))$ of P as a cubical analogue of the h-vector of a simplicial poset. It can be defined by the formula

$$\sum_{i=0}^{d} h_i^{(c)}(P)q^i \;=\; \frac{1}{1+q}\left(2^{d-1} \,+\, q\sum_{i=0}^{d-1} f_i(P)\,(2q)^i (1-q)^{d-i-1} \,+\, (-2)^{d-1}\,\widetilde{\chi}(P)\, q^{d+1} \right),$$

where $f_i(P)$ denotes the number of elements of P of rank $i+1$, for $0 \le i \le d-1$, and $\widetilde{\chi}(P) = -1 + \sum_{i=0}^{d-1}(-1)^i f_i(P)$.

One of Stanley's early results in f-vector theory (see [9, Corollary 4.3]) states that the h-vector of P has nonnegative entries if P is the face poset of a Cohen-Macaulay simplicial complex. This statement was extended to all Cohen-Macaulay simplicial posets in [14] and, in fact, the h-vectors of Cohen-Macaulay simplicial complexes and posets can both be completely characterized [10, 14] (see also Sections II.3 and III.6 in [15]). On the contrary, far less is known about f-vectors of cubical complexes and posets. Adin [1, Question 1] (see also [16, Problem 8 (a)]) raised the question whether the cubical h-vector of P has nonnegative entries if P is the face poset of a Cohen-Macaulay cubical complex and gave an affirmative answer in the special case of shellable cubical complexes [1, Theorem 5 (iii)]. For the importance of Adin's question we refer the reader to [7], where an affirmative answer is given for cubical barycentric subdivisions (implicitly defined in Section 2.4) of simplicial complexes.

One can easily check directly that $h_d(P) \ge 0$ (respectively, $h_d^{(c)}(P) \ge 0$) holds for every simplicial (respectively, cubical) Cohen-Macaulay poset P of rank d. Using the notation of Theorem 1.1, we claim (see Section 5) that

$$h_{d-1}(P) \;=\; (-1)^d \sum_{x \in \mathcal{A}(P)} \mu_{\hat{P}}(x, \hat{1}) \,-\, d\,(-1)^{d+1}\mu_{\hat{P}}(\hat{0}, \hat{1}) \tag{3}$$

and

$$h_{d-1}^{(c)}(P) \;=\; (-1)^d \sum_{x \in \mathcal{A}(P)} \mu_{\hat{P}}(x, \hat{1}) \,-\, 2^{d-1}\,(-1)^{d+1}\mu_{\hat{P}}(\hat{0}, \hat{1}) \tag{4}$$

holds for every simplicial (respectively, cubical) graded poset P of rank d. Thus, Theorem 1.1 applies to both of these situations and gives the following partial information on the nonnegativity of simplicial and cubical h-vectors. Since the proof of Theorem 1.1 uses tools from topological combinatorics, it specializes to a new proof that $h_{d-1}(P) \ge 0$ holds for Cohen-Macaulay simplicial posets P of rank d. The case of cubical posets yields a new result as follows.

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Corollary 1.2 *For every Cohen-Macaulay cubical poset P of rank d we have $h_{d-1}^{(c)}(P) \geq 0$.*

Motivated by results on the intersection cohomology of toric varieties, Stanley [13, Section 4] defined the (generalized, or toric) h-vector $(h_0(P), h_1(P), \ldots, h_d(P))$ for an arbitrary lower Eulerian poset P (where again, d is the largest length of a chain in P). For simplicial posets, this h-vector reduces to the one defined by (2). Stanley [13, Conjecture 4.2 (b)] conjectured that the generalized h-vector has nonnegative entries for every lower Eulerian Cohen-Macaulay meet-semilattice. The following partial result is also a consequence of Theorem 1.1 (see Section 5.1).

Corollary 1.3 *For every lower Eulerian Cohen-Macaulay meet-semilattice P of rank d we have $h_{d-1}(P) \geq 0$.*

This paper is organized as follows. Section 2 reviews basic definitions and background on the enumerative and topological combinatorics of partially ordered sets and establishes some preliminary results on (lower Eulerian) Cohen-Macaulay posets. Section 3 proves a certain statement (Corollary 3.4) on upper truncations of lower Eulerian Cohen-Macaulay posets which will be essential in the proof of Theorem 1.1. This statement follows from more general statements on rank-selections of Cohen-Macaulay (or Buchsbaum) posets and balanced simplicial complexes (Theorems 3.3 and 3.5), essentially established by Browder and Klee in [5]. Theorem 1.1 is proved in Section 4. The applications to cubical and toric h-vectors are discussed in Section 5.

2 Preliminaries

This section reviews basic background on the enumerative and topological combinatorics of simplicial complexes and partially ordered sets (posets), fixes notation and establishes some preliminary results which will be useful in the sequel. For more information on these topics we refer the reader to [12, Chapter 3] and [4]. Basic background on algebraic topology can be found in [8].

2.1 Simplicial complexes

Given a finite set E, an (abstract) *simplicial complex* on the ground set E is a collection Δ of subsets of E such that $\sigma \subseteq \tau \in \Delta$ implies $\sigma \in \Delta$. The elements of Δ are called *faces*. The dimension of a face σ is defined as one less than the cardinality of σ. A *facet* of Δ is a face which is maximal with respect to inclusion. The *reduced Euler characteristic* of Δ is defined as

$$\widetilde{\chi}(\Delta) \; = \; \sum_{i=0}^{d} (-1)^{i-1} f_{i-1}(\Delta), \tag{5}$$

where $d - 1 = \dim \Delta$ is the dimension (maximum dimension of a face) of Δ and $f_{i-1}(\Delta)$ denotes the number of faces of Δ of dimension $i - 1$, for $0 \leq i \leq d$. By the Euler-Poincaré formula we have

$$\widetilde{\chi}(\Delta) \; = \; \sum_{i=0}^{d} (-1)^{i-1} \dim_{\mathbf{k}} \widetilde{H}_{i-1}(\Delta; \mathbf{k}), \tag{6}$$

where $\widetilde{H}_*(\Delta; \mathbf{k})$ stands for the reduced simplicial homology of Δ over the field $\mathbf{k}$. The *link* of a face $\sigma \in \Delta$ is defined as the subcomplex $\mathrm{lk}_\Delta(\sigma) = \{\tau \smallsetminus \sigma : \tau \in \Delta, \, \sigma \subseteq \tau\}$ of Δ. The simplicial

complex Δ is called *Cohen-Macaulay* over $\mathbf{k}$ if $\widetilde{H}_i\left(\mathrm{lk}_\Delta(\sigma); \mathbf{k}\right) = 0$ for every $\sigma \in \Delta$ (including $\sigma = \varnothing$) and all $i < \dim \mathrm{lk}_\Delta(\sigma)$. Such a complex is *pure*, meaning that every facet of Δ has dimension equal to $\dim \Delta$. All topological properties of Δ we mention in the sequel will refer to those of the geometric realization [4, Section 9] of Δ, uniquely defined up to homeomorphism.

2.2 The Möbius function and the order complex

A poset P is called *locally finite* if every closed interval $[x, y]$ in P is finite. The *Möbius function* μ_P of such a poset is defined on pairs (x, y) of elements of P satisfying $x \leq_P y$ by the recursive formula

$$\mu_P(x, y) \;=\; \begin{cases} 1, & \text{if } x = y, \\ -\displaystyle\sum_{x \leq_P z <_P y} \mu_P(x, z), & \text{if } x <_P y. \end{cases} \tag{7}$$

Given a finite poset Q, we denote by $\Delta(Q)$ the simplicial complex on the ground set Q whose faces are chains (totally ordered subsets) of Q, known as the *order complex* of Q; see [12, p. 120] [4, Section 9]. The following proposition gives a fundamental interpretation of the Möbius function of a locally finite poset.

Proposition 2.1 ([12, Proposition 3.8.6]) *For every locally finite poset P and all $x, y \in P$ with $x <_P y$ we have $\mu_P(x, y) = \widetilde{\chi}(\Delta(x, y))$, where $\Delta(x, y)$ denotes the order complex of the open interval (x, y) in P.*

2.3 Lower Eulerian posets

A locally finite poset P is called *locally graded* if for every closed interval $[x, y]$ in P there exists a nonnegative integer $\rho(x, y)$, called the rank of $[x, y]$, such that every maximal chain in $[x, y]$ has length equal to $\rho(x, y)$. A locally graded poset which has a minimum element will be called *lower graded*. For such a poset P with minimum element $\hat{0}$, we will refer to the rank of $[\hat{0}, x]$ simply as the rank of x and will denote it by $\rho(x)$. A locally graded poset P is called *locally Eulerian* if $\mu_P(x, y) = (-1)^{\rho(x,y)}$ holds for all $x, y \in P$ with $x \leq_P y$. A *lower Eulerian* (respectively, *Eulerian*) poset is a locally Eulerian poset which has a minimum (respectively, a minimum and a maximum) element.

Given a finite poset P with a minimum element $\hat{0}$, we will denote by $\bar{P}$ (respectively, by $\hat{P}$) the poset which is obtained from P by removing $\hat{0}$ (respectively, by attaching a maximum element $\hat{1}$). We set

$$\widetilde{\chi}(P) \;=\; \mu_{\hat{P}}(\hat{0}, \hat{1}) \;=\; \widetilde{\chi}(\Delta(\bar{P})), \tag{8}$$

where the second equality is due to Proposition 2.1. For a finite, lower graded poset P we also set

$$\psi(P) \;=\; \sum_{i=0}^{d} (-1)^{i-1} f_{i-1}(P), \tag{9}$$

where d is the maximum rank of an element of P and $f_{i-1}(P)$ denotes the number of elements of P of rank i, for $0 \leq i \leq d$.

Lemma 2.2 *We have $\widetilde{\chi}(P) = \psi(P)$ for every lower Eulerian poset P.*

Proof. Using the defining recurrence (7) for the Möbius function and the fact that each closed interval in P is Eulerian, we find that

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) \; = \; - \sum_{x \in P} \mu_P(\hat{0}, x) \; = \; \sum_{x \in P} (-1)^{\rho(x) - 1}$$

and the proof follows. $\qquad\square$

The *rank* of an Eulerian poset is defined as the rank of its maximum element. The following elementary lemma will be used in Section 4.

Lemma 2.3 *Let P be an Eulerian poset of rank $d \geq 2$ with minimum element $\hat{0}$ and maximum element $\hat{1}$ and let Q be the poset which is obtained from P by removing all its atoms. Then we have*

$$\mu_Q(\hat{0}, \hat{1}) \; = \; (-1)^{d-1}(f_0(P) - 1),$$

where $f_0(P)$ is the number of atoms of P.

Proof. This statement is a special case of [12, Proposition 3.14.5]. $\qquad\square$

2.4 The poset of intervals

We denote by $\mathrm{int}(P)$ the set of (nonempty) closed intervals $[x, y]$ of a locally finite poset P, partially ordered by inclusion. For the enumerative, order-theoretic and topological properties of $\mathrm{int}(P)$ we refer the reader to Exercises 7 and 58 of [12, Chapter 3] and to [3, Section 2] [17, Section 4]. The following proposition (in the form of [17, Theorem 4.1]) also appears as Theorem 2.3 in [3].

Proposition 2.4 ([17, Theorem 6.1 (a)]) *For every finite poset P, the order complex $\Delta(\mathrm{int}(P))$ is homeomorphic to $\Delta(P)$.*

Proposition 2.5 *For every locally Eulerian poset P, the poset which is obtained from $\mathrm{int}(P)$ by attaching a minimum element is lower Eulerian.*

Proof. Let Q be the poset in question. We denote by $\hat{0}$ the minimum element of Q and choose any elements $x, y \in Q$ with $x <_Q y$. Assume first that $x \neq \hat{0}$. Then we may write $x = [b, c] \in \mathrm{int}(P)$ and $y = [a, d] \in \mathrm{int}(P)$ with $a \leq_P b \leq_P c \leq_P d$. Clearly, the interval $[x, y]$ in Q is isomorphic to the direct product of the intervals $[a, b]$ and $[c, d]$ in P and hence we have $\mu_Q(x, y) = \mu_P(a, b)\mu_P(c, d)$ by [12, Proposition 3.8.2]. Suppose now that $x = \hat{0}$ and let $y = [a, d] \in \mathrm{int}(P)$, as before. Then y (as a subposet of P) is an Eulerian poset and the interval $[x, y]$ in Q is the poset obtained from $\mathrm{int}(y)$ by attaching a minimum element $\hat{0}$. By the result of [12, Excercise 3.58 (b)] we have $\mu_Q(x, y) = \mu_Q(\hat{0}, y) = -\mu_P(a, d)$. The previous observations and the fact that each closed interval in P is Eulerian imply that Q is locally Eulerian as well. $\square$

2.5 Cohen-Macaulay posets

A finite poset P is *graded* of rank d (respectively, Cohen-Macaulay over the field $\mathbf{k}$) if the order complex $\Delta(P)$ is pure of dimension d (respectively, Cohen-Macaulay over $\mathbf{k}$). For the remainder of this section we assume that P has a minimum element $\hat{0}$. Then P is Cohen-Macaulay over $\mathbf{k}$ if and only if $\bar{P}$ is as well. Moreover, in that case $\bar{P}$ and P are graded of rank $d-1$ and d, respectively, and by (6) and (8) we have

$$\widetilde{\chi}(P) \;=\; (-1)^{d-1} \dim_{\mathbf{k}} \widetilde{H}_{d-1}(\Delta(\bar{P}); \mathbf{k}). \tag{10}$$

Suppose now that P is graded of rank $d \geq 2$ and let Q be the poset which is obtained from P by removing the set $\mathcal{M}(P)$ of maximal elements of P (to follow the proofs in the present and the following two sections, it may be helpful for the reader to keep in mind the special case in which P is the face poset of a regular cell complex). Given $y \in \mathcal{M}(P)$, the order complex $\Delta(\hat{0}, y)$ of the open interval $(\hat{0}, y)$ of P is a subcomplex of $\Delta(\bar{Q})$ and hence there is a map $\widetilde{H}_{d-2}(\Delta(\hat{0}, y); \mathbf{k}) \to \widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k})$, induced by inclusion. We denote by $\Omega(y)$ the image of this map. Equations (8) and (10) imply that if P is lower Eulerian and Cohen-Macaulay over $\mathbf{k}$, then $\Omega(y)$ is a one-dimensional $\mathbf{k}$-vector space for every $y \in \mathcal{M}(P)$. We will then denote by $\omega(y)$ any basis (nonzero) element of $\Omega(y)$.

Lemma 2.6 *Suppose that $\bar{P}$ is Cohen-Macaulay over $\mathbf{k}$ of rank $d-1 \geq 1$ and let $\bar{Q}$ be the poset which is obtained from $\bar{P}$ by removing all maximal elements. Then the $\mathbf{k}$-vector space $\widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k})$ is equal to the sum of its subspaces $\Omega(y)$ for $y \in \mathcal{M}(\bar{P})$.*

Proof. Let c be a $(d-2)$-cycle in the chain complex of $\Delta(\bar{Q})$ over $\mathbf{k}$. We will show that c can be written as a sum of $(d-2)$-cycles in the chain complexes of $\Delta(\hat{0}, y)$ over $\mathbf{k}$, for $y \in \mathcal{M}(\bar{P})$. We observe that c is also a $(d-2)$-cycle in the chain complex of $\Delta(\bar{P})$ over $\mathbf{k}$. Since $\Delta(\bar{P})$ is Cohen-Macaulay over $\mathbf{k}$ of dimension $d-1$, we have $\widetilde{H}_{d-2}(\Delta(\bar{P}); \mathbf{k}) = 0$ and hence $c = \partial_{d-1}(\tilde{c})$ for some $(d-1)$-chain $\tilde{c}$ in the chain complex of $\Delta(\bar{P})$ over $\mathbf{k}$, where ∂_* denotes the boundary map of this complex. Clearly $\tilde{c}$ can be written (uniquely) as the sum of $(d-1)$-chains $\tilde{c}_y$ for $y \in \mathcal{M}(\bar{P})$, where $\tilde{c}_y$ is a $(d-1)$-chain in the chain complex of $\Delta(\hat{0}, y]$ over $\mathbf{k}$. Then c is equal to the sum of the boundaries $\partial_{d-1}(\tilde{c}_y)$. Since c is supported on $\Delta(\bar{Q})$, each chain $\partial_{d-1}(\tilde{c}_y)$ is supported on $\Delta(\hat{0}, y)$. Since $\partial_{d-2}\partial_{d-1} = 0$, this implies that $\partial_{d-1}(\tilde{c}_y)$ is a $(d-2)$-cycle in the chain complex of $\Delta(\hat{0}, y)$ over $\mathbf{k}$. This proves the desired statement for c. $\qquad\square$

Corollary 2.7 *Let P be a lower Eulerian Cohen-Macaulay poset of rank $d \geq 2$ and let Q denote the poset which is obtained from P by removing all maximal elements. Then the $\mathbf{k}$-vector space $\widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k})$ is spanned by the classes $\omega(y)$ for $y \in \mathcal{M}(P)$.*

Proof. This statement is a special case of Lemma 2.6. $\qquad\square$

3 Doubly Cohen-Macaulay and Buchsbaum* posets

This section proves a certain property (Corollary 3.4) of the first upper truncation Q of a lower Eulerian Cohen-Macaulay poset which will be essential in the proof of Theorem 1.1 in Section 4. This property follows from the Buchsbaum* condition, recently introduced in [2], for the order complex $\Delta(\bar{Q})$, so we begin by recalling the relevant definitions. Throughout this section, we

write $H_i(\Delta, \Gamma; \mathbf{k})$ for the relative simplicial homology of the pair of simplicial complexes (Δ, Γ), where Γ is a subcomplex of Δ.

A simplicial complex Δ is *Buchsbaum* over the field $\mathbf{k}$ if Δ is pure and $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma); \mathbf{k}) = 0$ for every nonempty face $\sigma \in \Delta$ and all $i < \dim \mathrm{lk}_\Delta(\sigma)$. Recall that the *contrastar* of a face $\sigma \in \Delta$ is defined as the subcomplex $\mathrm{cost}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \not\subseteq \tau\}$ of Δ.

Proposition 3.1 (cf. [15, Theorem 8.1]) *The following conditions on a simplicial complex Δ are equivalent:*

(i) *Δ is Buchsbaum over $\mathbf{k}$.*

(ii) *Δ is pure and $\mathrm{lk}_\Delta(\sigma)$ is Cohen-Macaulay over $\mathbf{k}$ for every $\sigma \in \Delta \smallsetminus \{\varnothing\}$.*

(iii) *$H_i(\Delta, \mathrm{cost}_\Delta(\sigma); \mathbf{k}) = 0$ holds for every $\sigma \in \Delta \smallsetminus \{\varnothing\}$ and all $i < \dim \Delta$.*

The following definition of a Buchsbaum* simplicial complex is equivalent to [2, Definition 1.2] (see Propositions 2.3 and 2.8 in [2]) and will be convenient for the purposes of this section.

Definition 3.2 *Let Δ be a $(d-1)$-dimensional simplicial complex which is Buchsbaum over $\mathbf{k}$. The complex Δ is called* Buchsbaum* *over $\mathbf{k}$ if the canonical map*

$$\rho_* : \widetilde{H}_{d-1}(\Delta; \mathbf{k}) \to H_{d-1}(\Delta, \mathrm{cost}_\Delta(\sigma); \mathbf{k}),$$

induced by inclusion, is surjective for every $\sigma \in \Delta \smallsetminus \{\varnothing\}$.

Assume now that Δ is Cohen-Macaulay over $\mathbf{k}$. Then Δ is called *doubly Cohen-Macaulay* over $\mathbf{k}$ [15, p. 71] if for every vertex v of Δ, the complex $\Delta \smallsetminus v = \{\tau \in \Delta : v \notin \tau\}$ (obtained from Δ by removing all faces which contain v) is Cohen-Macaulay over $\mathbf{k}$ of the same dimension as Δ. Given that Δ is Cohen-Macaulay over $\mathbf{k}$, it was shown in [2, Proposition 2.5] that Δ is doubly Cohen-Macaulay over $\mathbf{k}$ if and only if Δ is Buchsbaum* over $\mathbf{k}$. A Cohen-Macaulay complex Δ over $\mathbf{k}$ is called *Gorenstein* * over $\mathbf{k}$ [15, Section II.5] if for every $\sigma \in \Delta$ and for $i = \dim \mathrm{lk}_\Delta(\sigma)$ we have $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma); \mathbf{k}) \cong \mathbf{k}$. Every such complex is doubly Cohen-Macaulay over $\mathbf{k}$ (see, for instance, Theorem II.5.1 and Proposition III.3.7 in [15]). A finite poset Q is *Buchsbaum* (respectively, *Buchsbaum** or *doubly Cohen-Macaulay* or *Gorenstein**) over $\mathbf{k}$ if $\Delta(Q)$ is a Buchsbaum (respectively, Buchsbaum* or doubly Cohen-Macaulay or Gorenstein*) simplicial complex over $\mathbf{k}$.

Part (i) of the next theorem follows from [6, Corollary 2.7] in the special case in which P is the face poset of a regular cell complex with the intersection property and from [18, Theorem 4.5] in the special case in which P is a simplicial poset.

Theorem 3.3 *Let P be a graded poset of rank $d \geq 2$ with a minimum element $\hat{0}$ and let Q be the poset which is obtained from P by removing all maximal elements.*

(i) *If P is Cohen-Macaulay over $\mathbf{k}$ and the interval $(\hat{0}, y)$ in P is doubly Cohen-Macaulay over $\mathbf{k}$ for every maximal element y of P, then $\bar{Q}$ is doubly Cohen-Macaulay over $\mathbf{k}$.*

(ii) *If $\bar{P}$ is Buchsbaum over $\mathbf{k}$ and the interval $(\hat{0}, y)$ in P is doubly Cohen-Macaulay over $\mathbf{k}$ for every maximal element y of P, then $\bar{Q}$ is Buchsbaum* over $\mathbf{k}$.*

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Before we comment on the proof of Theorem 3.3, let us deduce the statement which will be needed in Section 4. Let Γ be a simplicial complex of dimension $d-2$. Recall that the *closed star* of a vertex v in Γ is the subcomplex of Γ defined as $\overline{\mathrm{st}}_\Gamma(v) = \{\sigma \in \Gamma : \sigma \cup \{v\} \in \Gamma\}$. Via the isomorphisms $H_i(\Gamma, \mathrm{cost}_\Gamma(v); \mathbf{k}) \cong H_i(\overline{\mathrm{st}}_\Gamma(v), \mathrm{lk}_\Gamma(v); \mathbf{k}) \cong \widetilde{H}_{i-1}(\mathrm{lk}_\Gamma(v); \mathbf{k})$, the canonical map $\rho_* : \widetilde{H}_{d-2}(\Gamma; k) \to H_{d-2}(\Gamma, \mathrm{cost}_\Gamma(v); \mathbf{k})$, considered in Definition 3.2, induces a map

$$\rho_v : \widetilde{H}_{d-2}(\Gamma; \mathbf{k}) \to \widetilde{H}_{d-3}(\mathrm{lk}_\Gamma(v); \mathbf{k}).$$

One can check that ρ_v is induced by a chain map from the (augmented) chain complex of Γ over $\mathbf{k}$ to that of $\mathrm{lk}_\Gamma(v)$ which sends a face $\sigma \in \Gamma$ to $\sigma \smallsetminus \{v\}$ (with appropriate sign), if $v \in \sigma$, and to zero otherwise. Suppose now that $\Gamma = \Delta(\bar{Q})$, where Q is a graded poset having a minimum element and rank $d-1$, and let $x \in Q$ be an atom. Since $\mathrm{lk}_\Gamma(x) = \Delta(Q_{>x})$, where $Q_{>x} = \{y \in Q : x <_Q y\}$ is considered as a subposet of Q, we get a map

$$\rho_x : \widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k}) \to \widetilde{H}_{d-3}(\Delta(Q_{>x}); \mathbf{k}).$$

Corollary 3.4 *Let P be a lower Eulerian Cohen-Macaulay poset (over $\mathbf{k}$) of rank $d \geq 2$ and let $\bar{Q}$ denote the poset which is obtained from P by removing the minimum and all maximal elements. Then the map $\rho_x : \widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k}) \to \widetilde{H}_{d-3}(\Delta(Q_{>x}); \mathbf{k})$ is surjective for every minimal element x of $\bar{Q}$.*

Proof. Our assumptions imply that the interval $(\hat{0}, y)$ in P is Gorenstein*, and hence doubly Cohen-Macaulay, over $\mathbf{k}$ for every maximal element y of P. Thus, Theorem 3.3 implies that $\Delta(\bar{Q})$ is Buchsbaum* over $\mathbf{k}$. In view of Definition 3.2, the proof follows from the discussion preceding the statement of the corollary. □

Theorem 3.3 will be deduced from the following more general statement on simplicial complexes. The proof of the latter will follow from that of [5, Theorem 3.1]. For a subset U of the set of vertices of Δ we write $\Delta \smallsetminus U$ for the subcomplex $\{\sigma \in \Delta : \sigma \cap U = \varnothing\}$ of Δ, obtained from Δ by removing all faces which intersect U.

Theorem 3.5 (cf. [5, Theorem 3.1]) *Let Δ be a pure simplicial complex of positive dimension and let U be a subset of the vertex set of Δ which has the following property: every facet of Δ contains exactly one element of U.*

(i) If Δ is Cohen-Macaulay over $\mathbf{k}$ and $\mathrm{lk}_\Delta(u)$ is doubly Cohen-Macaulay over $\mathbf{k}$ for every $u \in U$, then $\Delta \smallsetminus U$ is doubly Cohen-Macaulay over $\mathbf{k}$.

(ii) If Δ is Buchsbaum over $\mathbf{k}$ and $\mathrm{lk}_\Delta(u)$ is doubly Cohen-Macaulay over $\mathbf{k}$ for every $u \in U$, then $\Delta \smallsetminus U$ is Buchsbaum over $\mathbf{k}$.*

Proof. By [2, Proposition 2.5 (i)], it suffices to prove part (ii). Since Δ is pure, our assumption on U implies that $\Delta \smallsetminus U$ is pure as well. It also implies that Δ is balanced of type $(d-1, 1)$, in the sense of [11], where $d-1 = \dim\Delta$. Using the equivalence (i) $\Leftrightarrow$ (ii) in Proposition 3.1 and the rank-selection theorem [11, Theorem 4.3] for balanced Cohen-Macaulay simplicial complexes, it follows as in the beginning of [5, Section 3] that $\Delta \smallsetminus U$ is Buchsbaum over $\mathbf{k}$. The proof of [5, Theorem 3.1] shows (by induction on the cardinality of U) that $\Delta \smallsetminus U$ satisfies the condition of Definition 3.2, using exactly the hypotheses that Δ is Buchsbaum over $\mathbf{k}$ and that $\mathrm{lk}_\Delta(u)$ is doubly Cohen-Macaulay over $\mathbf{k}$ for every $u \in U$. Thus the proof follows from that of [5, Theorem 3.1]. □

Proof of Theorem 3.3. The statement follows by applying the appropriate part of Theorem 3.5 to the order complex $\Delta(\bar{P})$ and choosing U as the set of maximal elements of P. □

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4 Proof of Theorem 1.1

Throughout this section, P is a (finite) lower Eulerian Cohen-Macaulay (over $\mathbf{k}$) poset of rank d. When $d \geq 2$, we denote by Q the poset which is obtained from P by removing all maximal elements and by R the poset which is obtained from Q by removing all atoms. Thus Q is lower Eulerian and graded of rank $d-1$ and R is graded of rank $d-2$. Moreover, by the rank-selection theorem for balanced Cohen-Macaulay complexes [15, Theorem 4.5], both Q and R are Cohen-Macaulay over $\mathbf{k}$.

Lemma 4.1 *Under the assumptions and notation of Theorem 1.1, and assuming that P has rank $d \geq 2$, we have*

$$\sum_{x \in \mathcal{A}(P)} |\mu_{\hat{P}}(x, \hat{1})| - \alpha_P |\mu_{\hat{P}}(\hat{0}, \hat{1})| \;=\; (\alpha_P - 1)|\widetilde{\chi}(Q)| - |\widetilde{\chi}(R)| + \sum_{y \in \mathcal{M}(P)} (\alpha(y) - \alpha_P),$$

where $\mathcal{M}(P)$ is the set of maximal elements of P and $\alpha(y)$ is the number of atoms $x \in \mathcal{A}(P)$ satisfying $x \leq_P y$, for $y \in \mathcal{M}(P)$.

Proof. Since P is Cohen-Macaulay over $\mathbf{k}$, so is $\hat{P} = P \cup \{\hat{1}\}$ and hence [12, Proposition 3.8.11] we have $(-1)^d \mu_{\hat{P}}(x, \hat{1}) \geq 0$ for every $x \in \mathcal{A}(P)$. Using this fact and Lemma 2.2, we find that

$$\sum_{x \in \mathcal{A}(P)} |\mu_{\hat{P}}(x, \hat{1})| \;=\; (-1)^d \sum_{x \in \mathcal{A}(P)} \mu_{\hat{P}}(x, \hat{1}) \;=\; (-1)^d \sum_{x \in \mathcal{A}(P)} \sum_{x \leq_P y} (-1)^{\rho(y)}$$

$$=\; (-1)^d \sum_{x \in \mathcal{A}(Q)} \sum_{x \leq_Q y} (-1)^{\rho(y)} + \sum_{y \in \mathcal{M}(P)} \alpha(y),$$

where $\rho(y)$ is the rank of y in P. Similarly, we have

$$|\mu_{\hat{P}}(\hat{0}, \hat{1})| \;=\; (-1)^{d-1} \widetilde{\chi}(P) \;=\; (-1)^{d-1} \psi(P) \;=\; (-1)^{d-1} \psi(Q) + f_{d-1}(P)$$
$$=\; (-1)^{d-1} \widetilde{\chi}(Q) + f_{d-1}(P) \;=\; f_{d-1}(P) - |\widetilde{\chi}(Q)|$$

and hence

$$\sum_{x \in \mathcal{A}(P)} |\mu_{\hat{P}}(x, \hat{1})| - \alpha_P |\mu_{\hat{P}}(\hat{0}, \hat{1})| \;=\; (-1)^d \sum_{x \in \mathcal{A}(Q)} \sum_{x \leq_Q y} (-1)^{\rho(y)} + \alpha_P |\widetilde{\chi}(Q)| +$$

$$\sum_{y \in \mathcal{M}(P)} (\alpha(y) - \alpha_P).$$

Thus, it suffices to show that

$$(-1)^{d-1} \sum_{x \in \mathcal{A}(Q)} \sum_{x \leq_Q y} (-1)^{\rho(y)} \;=\; |\widetilde{\chi}(Q)| + |\widetilde{\chi}(R)|$$

or, equivalently, that

$$\sum_{x \in \mathcal{A}(Q)} \sum_{x \leq_Q y} (-1)^{\rho(y)-1} \;=\; \widetilde{\chi}(Q) - \widetilde{\chi}(R). \tag{11}$$

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Indeed, we have

$$\sum_{x \in \mathcal{A}(Q)} \sum_{x \leq_Q y} (-1)^{\rho(y)-1} \;=\; \sum_{\hat{0} \neq x \leq_Q y} (-1)^{\rho(y)-\rho(x)} \;-\; \sum_{\hat{0} \neq x \leq_R y} (-1)^{\rho(y)-\rho(x)}$$

$$=\; \psi(\mathrm{int}(\bar{Q})_{\circ}) \,-\, \psi(\mathrm{int}(\bar{R})_{\circ}),$$

where $\mathrm{int}(\bar{Q})_{\circ}$ (respectively, $\mathrm{int}(\bar{R})_{\circ}$) is the poset obtained from $\mathrm{int}(\bar{Q})$ (respectively, $\mathrm{int}(\bar{R})$) by adding a minimum element. Since $\bar{Q}$ and $\bar{R}$ are locally Eulerian, the posets $\mathrm{int}(\bar{Q})_{\circ}$ and $\mathrm{int}(\bar{R})_{\circ}$ are lower Eulerian by Proposition 2.5. Thus, Lemma 2.2 implies that $\psi(\mathrm{int}(\bar{Q})_{\circ}) = \widetilde{\chi}(\mathrm{int}(\bar{Q})_{\circ})$ and $\psi(\mathrm{int}(\bar{R})_{\circ}) = \widetilde{\chi}(\mathrm{int}(\bar{R})_{\circ})$. Finally, we note that $\widetilde{\chi}(\mathrm{int}(\bar{Q})_{\circ}) = \widetilde{\chi}(Q)$ and $\widetilde{\chi}(\mathrm{int}(\bar{R})_{\circ}) = \widetilde{\chi}(R)$ by Proposition 2.4 and hence (11) follows. $\qquad\square$

We now proceed with the proof of Theorem 1.1. We recall that for every atom x of Q we have the natural map $\rho_x : \widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k}) \to \widetilde{H}_{d-3}(\Delta(Q_{>x}); \mathbf{k})$, shown to be surjective in Corollary 3.4.

Proof of Theorem 1.1. Let d be the rank of P. We assume that $d \geq 2$, since the result is trivial otherwise. By Corollary 2.7, we may choose $\mathcal{B}(P) \subseteq \mathcal{M}(P)$ so that the classes $\omega(y)$ for $y \in \mathcal{B}(P)$ form a basis of the $\mathbf{k}$-vector space $\widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k})$. Since Q is Cohen-Macaulay over $\mathbf{k}$, it follows from (10) that the cardinality of $\mathcal{B}(P)$ is equal to $|\widetilde{\chi}(Q)|$. Hence Lemma 4.1 implies that

$$\sum_{x \in \mathcal{A}(P)} |\mu_{\hat{P}}(x, \hat{1})| - \alpha_P |\mu_{\hat{P}}(\hat{0}, \hat{1})| \;\geq\; (\alpha_P - 1)|\widetilde{\chi}(Q)| - |\widetilde{\chi}(R)| + \sum_{y \in \mathcal{B}(P)} (\alpha(y) - \alpha_P)$$

$$=\; \sum_{y \in \mathcal{B}(P)} (\alpha(y) - 1) - |\widetilde{\chi}(R)|.$$

Therefore, it suffices to show that

$$|\widetilde{\chi}(R)| \;\leq\; \sum_{y \in \mathcal{B}(P)} (\alpha(y) - 1). \tag{12}$$

For $x \in \mathcal{A}(Q)$ we consider the natural map $\rho_x : \widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k}) \to \widetilde{H}_{d-3}(\Delta(Q_{>x}); \mathbf{k})$ and the map $\widetilde{H}_{d-3}(\Delta(Q_{>x}); \mathbf{k}) \to \widetilde{H}_{d-3}(\Delta(\bar{R}); \mathbf{k})$, induced by inclusion. These maps yield the sequence of linear maps

$$\bigoplus_{x \in \mathcal{A}(Q)} \widetilde{H}_{d-2}(\Delta(\bar{Q}); \mathbf{k}) \;\to\; \bigoplus_{x \in \mathcal{A}(Q)} \widetilde{H}_{d-3}(\Delta(Q_{>x}); \mathbf{k}) \;\to\; \widetilde{H}_{d-3}(\Delta(\bar{R}); \mathbf{k}). \tag{13}$$

We observe that both maps in this sequence are surjective, the one on the left since every map ρ_x is surjective by Corollary 3.4, and the one on the right by Lemma 2.6 (applied to the dual of $\bar{Q}$). For $y \in \mathcal{B}(P)$, let us denote by $P(y)$ the poset which is obtained from the open interval $(\hat{0}, y)$ of P by removing all its minimal elements. Since there are exactly $\alpha(y)$ such elements and $P(y)$ is Cohen-Macaulay of rank $d - 3$, we have

$$\dim_{\mathbf{k}} \widetilde{H}_{d-3}(\Delta(P(y)); \mathbf{k}) \;=\; \alpha(y) - 1$$

by (10) and Lemma 2.3. Clearly, for $x \in \mathcal{A}(P)$ and $y \in \mathcal{B}(P)$ we may have $\rho_x(\omega(y)) \neq 0$ only if $x \leq_P y$. This implies that the image of the composition of the two maps in (13) is contained in

(the image of the map induced by the inclusion of) the sum of the spaces $\widetilde{H}_{d-3}(\Delta(P(y)); \mathbf{k})$ for $y \in \mathcal{B}(P)$. This fact and the surjectivity of the maps in (13) imply that

$$|\widetilde{\chi}(R)| \;=\; \dim_{\mathbf{k}} \widetilde{H}_{d-3}(\Delta(\bar{R}); \mathbf{k}) \;\leq\; \sum_{y \in \mathcal{B}(P)} \dim_{\mathbf{k}} \widetilde{H}_{d-3}(\Delta(P(y)); \mathbf{k}) \;=\; \sum_{y \in \mathcal{B}(P)} (\alpha(y) - 1).$$

This shows (12) and completes the proof of the theorem. $\qquad\square$

5 Applications

This section deduces Corollaries 1.2 and 1.3 from Theorem 1.1.

5.1 The toric h-vector

We first recall from [13] the definition of the toric h-vector of a (finite) lower Eulerian poset P. We denote by $\hat{0}$ the minimum element of P, as usual, and by d the maximum rank $\rho(y)$ of an element $y \in P$. For $y \in P$ we define two polynomials $f(P, y; q)$ and $g(P, y; q)$ by the following rules:

(a) $f(P, \hat{0}; q) = g(P, \hat{0}; q) = 1$.

(b) If $y \in \bar{P}$ and $f(P, y; q) = k_0 + k_1 q + \cdots$, then

$$g(P, y; q) \;=\; k_0 + \sum_{i=1}^{m} (k_i - k_{i-1}) q^i, \tag{14}$$

where $m = \lfloor (\rho(y) - 1)/2 \rfloor$.

(c) If $z \in \bar{P}$, then

$$f(P, z; q) \;=\; \sum_{y <_P z} g(P, y; q)(q - 1)^{\rho(y,z)-1}. \tag{15}$$

The toric h-vector of P is the sequence $h(P) = (h_0(P), h_1(P), \ldots, h_d(P))$ defined by

$$h_d(P) + h_{d-1}(P)q + \cdots + h_0(P)q^d \;=\; \sum_{y \in P} g(P, y; q)(q - 1)^{d-\rho(y)}. \tag{16}$$

For instance, we have $h_0(P) = 1$, $h_1(P) = f_0(P) - d$ and $h_d(P) = (-1)^{d-1}\widetilde{\chi}(P)$, where $f_0(P)$ is the number of atoms of P (see [13, p. 198]). These formulas imply that $h_1(P) \geq 0$, if P is a meet-semilattice (meaning that any two elements of P have a greatest lower bound) and $h_d(P) \geq 0$, if P is Cohen-Macaulay over $\mathbf{k}$. We will write $h(P) \geq 0$ if we have $h_i(P) \geq 0$ for $0 \leq i \leq d$. The following conjecture is part of [13, Conjecture 4.2].

Conjecture 5.1 ([13, Conjecture 4.2 (b)]) *For every lower Eulerian Cohen-Macaulay meet-semilattice P we have $h(P) \geq 0$.*

The generalized Dehn-Somerville equations [13, Theorem 2.4] assert that the polynomial $f(P, z; q)$ is symmetric of degree $\rho(z) - 1$ for every $z \in \bar{P}$. Thus, writing $f(P, z; q) = \sum_{i=0}^{r} k_i q^i$ with $r = \rho(z) - 1$, we have $k_i = k_{r-i}$ for $0 \leq i \leq r$.

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We now deduce from Theorem 1.1 that $h_{d-1}(P) \geq 0$ for every lower Eulerian Cohen-Macaulay meet-semilattice P of rank d.

Proof of Corollary 1.3. We note that for every maximal element $y \in P$, the interval $[\hat{0}, y]$ in P is an Eulerian lattice of rank d and therefore has at least d atoms (this holds, more generally, for all graded lattices of rank d with nowhere vanishing Möbius function). Thus, in the notation of Theorem 1.1, we have $\alpha_P \geq d$. We claim that

$$h_{d-1}(P) = \sum_{x \in \mathcal{A}(P)} |\widetilde{\chi}(P_{\geq x})| - d\,|\widetilde{\chi}(P)|, \qquad (17)$$

where $P_{\geq x} := \{y \in P : x \leq_P y\}$ is considered as a subposet of P. In view of (1) and the inequality $\alpha_P \geq d$, it suffices to prove the claim. For $y \in P$, let $\alpha(y)$ denote the number of atoms of P in the interval $[\hat{0}, y]$. The constant term of $g(P, y; q)$ is equal to 1 for every $y \in P$. Furthermore, it follows from equations (14) and (15) and from the symmetry of the polynomials $f(P, z; q)$ that the coefficient of q in $g(P, y; q)$ is equal to $\alpha(y) - \rho(y)$. Thus, equation (16) implies that

$$h_{d-1}(P) = \sum_{y \in P} (-1)^{d-\rho(y)}\,(\alpha(y) - d).$$

An easy computation, essentially already carried out in the proof of Lemma 4.1, shows that the right-hand side of the previous equation is equal to the right-hand side of (17). This completes the proof. □

5.2 The cubical h-vector

Cubical posets form an important class of lower Eulerian posets. As mentioned in the introduction, a cubical poset is a (finite) poset having a minimum element $\hat{0}$, such that for every $x \in P$, the interval $[\hat{0}, x]$ in P is isomorphic to the poset of faces of a cube (the dimension of which has to equal one less than the rank of x). We assume that P is graded of rank d and denote by $f_{i-1}(P)$ the number of elements of P of rank i for $0 \leq i \leq d$, as usual. The *cubical h-polynomial* of P can be defined (see [1]) by the formula

$$(1 + q)h^{(c)}(P, q) = 2^{d-1} + q\,h^{(sc)}(P, q) + (-2)^{d-1}\,\widetilde{\chi}(P)\,q^{d+1}, \qquad (18)$$

where

$$h^{(sc)}(P, q) = \sum_{i=0}^{d-1} f_i(P)\,(2q)^i(1-q)^{d-i-1} \qquad (19)$$

is the short cubical h-polynomial of P and $\widetilde{\chi}(P) = \psi(P) = \mu_{\hat{P}}(\hat{0}, \hat{1})$ (see Section 2.3). The function $h^{(c)}(P, q)$ is a polynomial in q of degree at most d. The *cubical h-vector* of P is defined as the sequence $(h_0^{(c)}(P), h_1^{(c)}(P), \ldots, h_d^{(c)}(P))$, where

$$h^{(c)}(P, q) = \sum_{i=0}^{d} h_i^{(c)}(P)q^i.$$

Clearly, (18) implies that $h_d^{(c)}(P) = (-2)^{d-1}\widetilde{\chi}(P)$ and hence we have $h_d^{(c)}(P) \geq 0$, if P is Cohen-Macaulay over **k**. By direct computation we also find that

$$h_{d-1}^{(c)}(P) = (-2)^{d-1} + \sum_{i=1}^{d} (-1)^{d-i-1}\,(2^{d-1} - 2^{i-1})f_{i-1}(P). \qquad (20)$$

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For instance, we have

$$h_{d-1}^{(c)}(P) = \begin{cases} f_0(P) - 2, & \text{if } d = 2 \\ 2f_1(P) - 3f_0(P) + 4, & \text{if } d = 3 \\ 4f_2(P) - 6f_1(P) + 7f_0(P) - 8, & \text{if } d = 4. \end{cases}$$

We now deduce from Theorem 1.1 that $h_{d-1}^{(c)}(P) \geq 0$ for every Cohen-Macaulay cubical poset P of rank d.

Proof of Corollary 1.2. Since the poset P is cubical (and graded) of rank d, in the notation of Theorem 1.1 we have $\alpha_P = 2^{d-1}$. Thus, in view of (1) and since P is Cohen-Macaulay over $\mathbf{k}$, it suffices to verify (4). Comparing the coefficients of x^d in the two sides of (18) we get

$$h_{d-1}^{(c)}(P) = h_{d-1}^{(sc)}(P) - h_d^{(c)}(P) = h_{d-1}^{(sc)}(P) - (-2)^{d-1}\,\widetilde{\chi}(P), \tag{21}$$

where $h_{d-1}^{(sc)}(P)$ denotes the coefficient of x^{d-1} in $h^{(sc)}(P, q)$. We next recall that for $x \in \bar{P}$, the subposet $P_{\geq x} := \{y \in P : x \leq_P y\}$ of P is a simplicial poset and that, by a fundamental observation of Hetyei (see [1, Theorem 9]), we have

$$h^{(sc)}(P, q) = \sum_{x \in \mathcal{A}(P)} h(P_{\geq x}, q), \tag{22}$$

where $h(Q, q)$ stands for the simplicial h-vector of a simplicial poset Q (as defined, for instance, by the right-hand side of (2)). Equation (22) implies that

$$h_{d-1}^{(sc)}(P) = \sum_{x \in \mathcal{A}(P)} (-1)^d \widetilde{\chi}(P_{\geq x}). \tag{23}$$

Finally, (21) and (23) imply (4) and the proof follows. $\square$

Adin [1, Question 1] raised the question whether $h^{(c)}(P, q) \geq 0$ holds for every Cohen-Macaulay cubical meet-semilattice P (where the inequality is meant to hold coefficientwise). In view of Corollary 1.2, it is natural to extend this question as follows.

Question 5.2 *Does $h^{(c)}(P, q) \geq 0$ hold for every Cohen-Macaulay cubical poset P?*

Acknowledgments

References

[1] R.M. Adin, *A new cubical h-vector*, Discrete Math. **157** (1996), 3–14.

[2] C.A. Athanasiadis and V. Welker, *Buchsbaum* complexes*, Math. Z. (to appear).

[3] E.K. Babson, L.J. Billera and C.S. Chan, *Neighborly cubical spheres and a cubical lower bound conjecture*, Israel J. Math. **102** (1997), 297–315.

[4] A. Björner, *Topological methods*, in *Handbook of Combinatorics* (R.L. Graham, M. Grötschel and L. Lovász, eds.), North Holland, Amsterdam, 1995, pp. 1819–1872.

[5] J. Browder and S. Klee, *Lower bounds for Buchsbaum* complexes*, European J. Combin. **32** (2011), 146–153.

[6] G. Fløystad, *Cohen-Macaulay cell complexes*, in *Algebraic and Geometric Combinatorics*, Contemporary Mathematics **423**, American Mathematical Society, Providence, RI, 2007, pp. 205–220.

[7] G. Hetyei, *Invariants des complexes cubiques*, Ann. Sci. Math. Québec **20** (1996), 35–52.

[8] J.R. Munkres, Elements of Algebraic Topology, Addison-Wesley, Reading, MA, 1984.

[9] R.P. Stanley, *The Upper Bound Conjecture and Cohen–Macaulay rings*, Studies in Appl. Math. **54** (1975), 135–142.

[10] R.P. Stanley, *Cohen–Macaulay complexes*, in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht and Boston, 1977, pp. 51–62.

[11] R.P. Stanley, *Balanced Cohen-Macaulay complexes*, Trans. Amer. Math. Soc. **249** (1979), 139–157.

[12] R.P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, Cambridge, 1997.

[13] R.P. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties and related results*, in *Commutative Algebra and Combinatorics* (N. Nagata and H. Matsumura, eds.), Adv. Stud. Pure Math. **11**, Kinokuniya, Tokyo and North-Holland, Amsterdam-New York, 1987, pp. 187–213.

[14] R.P. Stanley, *f-vectors and h-vectors of simplicial posets*, J. Pure Appl. Algebra **71** (1991), 319–331.

[15] R.P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhäuser, Basel, 1996.

[16] R.P. Stanley, *Positivity problems and conjectures in algebraic combinatorics*, in *Mathematics: Frontiers and Perspectives* (V. Arnold, M. Atiyah, P. Lax and B. Mazur, eds.), Amer. Math. Society, Providence, RI, 2000, pp. 295–319.

[17] J. Walker, *Canonical homeomorphisms of posets*, European J. Combin. **9** (1988), 97–107.

[18] K. Yanagawa, *Higher Cohen-Macaulay property of squarefree modules and simplicial posets*, Proc. Amer. Math. Soc. **139** (2011), 3057–3066.