

# CUBICAL SUBDIVISIONS AND LOCAL $h$ -VECTORS

CHRISTOS A. ATHANASIADIS

ABSTRACT. Face numbers of triangulations of simplicial complexes were studied by Stanley by use of his concept of a local  $h$ -vector. It is shown that a parallel theory exists for cubical subdivisions of cubical complexes, in which the role of the  $h$ -vector of a simplicial complex is played by the (short or long) cubical  $h$ -vector of a cubical complex, defined by Adin, and the role of the local  $h$ -vector of a triangulation of a simplex is played by the (short or long) cubical local  $h$ -vector of a cubical subdivision of a cube. The cubical local  $h$ -vectors are defined in this paper and are shown to share many of the properties of their simplicial counterparts. Generalizations to subdivisions of locally Eulerian posets are also discussed.

## 1. INTRODUCTION

Simplicial subdivisions (or triangulations) of simplicial complexes were studied from an enumerative point of view by Stanley [17]. Specifically, the paper [17] is concerned with the way in which the face enumeration of a simplicial complex  $\Delta$ , presented in the form of the  $h$ -vector (equivalently, of the  $h$ -polynomial) of  $\Delta$ , changes under simplicial subdivisions of various types; see [19, Chapter II] for the importance of  $h$ -vectors in the combinatorics of simplicial complexes.

A key result in this study is a formula [17, Equation (2)] which expresses the  $h$ -polynomial of a simplicial subdivision  $\Delta'$  of a pure simplicial complex  $\Delta$  as the sum of the  $h$ -polynomial of  $\Delta$  and other terms, one for each nonempty face  $F$  of  $\Delta$ . The term corresponding to  $F$  is a product of two polynomials, one of which depends only on the local combinatorics of  $\Delta$  at  $F$  and the other only on the restriction of  $\Delta'$  to  $F$ . The latter is determined by the local  $h$ -vector of  $\Delta'$  at  $F$ , a concept which is introduced and studied in [17]. It is shown that the local  $h$ -vector of a simplicial subdivision of a simplex is symmetric and that for quasi-geometric subdivisions (a kind of topological subdivision which includes all geometric subdivisions) it is nonnegative. From these results it is deduced that the  $h$ -vector of a Cohen-Macaulay simplicial complex increases under quasi-geometric subdivision. Other properties of local  $h$ -vectors are also established in [17] and a generalization of the theory to formal subdivisions of lower Eulerian posets is developed.

This paper investigates similar questions for cubical subdivisions of cubical complexes. A well-behaved analogue of the  $h$ -polynomial was introduced for cubical complexes (in two

---

*Date:* August 25, 2010; Revised, January 28, 2011.

*Key words and phrases.* Cubical complex, cubical subdivision, face enumeration, cubical  $h$ -vector, cubical local  $h$ -vector, locally Eulerian poset, formal subdivision.

2000 *Mathematics Subject Classification.* Primary 52B05; Secondary 05E45, 06A07, 55U10.

forms, short and long) by Adin [1]. For both types, it is shown that formulas analogous to that of [17] hold for the cubical  $h$ -polynomial of a cubical subdivision of a pure cubical complex (Theorems 4.1 and 6.6), when one suitably defines the (short or long) cubical local  $h$ -vector of a cubical subdivision of a cube (Definitions 3.1 and 6.1). The cubical local  $h$ -vectors are shown to have symmetric coefficients (Theorem 4.2 and Corollary 6.5). For the short type, they are shown to have nonnegative coefficients for a class of subdivisions which are called locally quasi-geometric in this paper and which include all geometric cubical subdivisions. A monotonicity property, analogous to that of [17] in the simplicial case, is deduced for the short cubical  $h$ -vectors of locally quasi-geometric cubical subdivisions (Corollary 4.6). At present, nonnegativity of cubical local  $h$ -vectors and monotonicity of cubical  $h$ -vectors for (say, geometric) cubical subdivisions seem to be out of reach for the long type in high dimensions.

The theory of short cubical local  $h$ -vectors is generalized to formal subdivisions of locally Eulerian posets in Section 7 by suitably modifying the approach of [17, Part II]. This includes other kinds of  $h$ -vectors for simplicial subdivisions, such as the short simplicial  $h$ -vector of Hersh and Novik [8] (see Examples 7.5 and 7.9), as well as cubical subdivisions of complexes more general than regular CW-complexes (see Section 7.2), to which the theory of [17] can be extended.

This paper is organized as follows. Preliminaries on (simplicial and) cubical complexes are given in Section 2. Short and long cubical local  $h$ -vectors are defined in Sections 3 and 6, respectively, where several examples and elementary properties also appear. The main properties of short cubical local  $h$ -vectors are stated and proven in Sections 4 and 5. Section 6 extends some of these properties to the long cubical local  $h$ -vector by observing that the two types of cubical local  $h$ -vectors are related in a simple way. Section 7 is devoted to generalizations of the theory of short cubical local  $h$ -vectors to subdivisions of locally Eulerian posets. The methods used in this paper are similar to those of [17], but extensions and variations are occasionally needed.

## 2. FACE ENUMERATION AND SUBDIVISIONS OF CUBICAL COMPLEXES

This section reviews background material on the face enumeration and subdivisions of simplicial and cubical complexes. We refer the reader to [19] for any undefined terminology and for more information on the algebraic, enumerative and homological properties of simplicial complexes. Basic background on partially ordered sets and convex polytopes can be found in [15, Chapter 3] and [7, 21], respectively. We denote by  $|S|$  the cardinality of a finite set  $S$ .

**2.1. Simplicial complexes.** Let  $\Delta$  be a (finite, abstract) simplicial complex of dimension  $d-1$ . The fundamental enumerative invariant of  $\Delta$  for us will be the  $h$ -polynomial, defined by

$$(2.1) \quad h(\Delta, x) = \sum_{F \in \Delta} x^{|F|} (1-x)^{d-|F|}.$$

This polynomial has nonnegative coefficients if  $\Delta$  is Cohen-Macaulay over some field. If  $\Delta$  is a homology ball over some field, we let  $\text{int}(\Delta)$  denote the set of (necessarily nonempty) faces of  $\Delta$  which are not contained in the boundary  $\partial\Delta$  of  $\Delta$  (observe that  $\text{int}(\Delta)$  is *not* a subcomplex of  $\Delta$ ; for instance, if  $\Delta$  is the simplex  $2^V$  on the vertex set  $V$ , then  $\text{int}(\Delta) = \{V\}$ ). If  $\Delta$  is a homology sphere, we set  $\text{int}(\Delta) := \Delta$  (for a discussion of homology balls and spheres see, for instance, [6, Section 2] or [20, Section 4]). In either case, we set

$$(2.2) \quad h(\text{int}(\Delta), x) = \sum_{F \in \text{int}(\Delta)} x^{|F|} (1-x)^{d-|F|}.$$

The next proposition follows from Theorem 7.1 in [19, Chapter II] (see also [13, Theorem 2], [16, Lemma 6.2] and [18, Lemma 2.3]).

**Proposition 2.1.** *Let  $\Delta$  be a  $(d-1)$ -dimensional simplicial complex. If  $\Delta$  is either a homology ball or a homology sphere over some field, then*

$$(2.3) \quad x^d h(\Delta, 1/x) = h(\text{int}(\Delta), x).$$

**2.2. Cubical complexes.** An (abstract)  $d$ -dimensional cube for us will be any poset which is isomorphic to the poset of faces of the standard  $d$ -dimensional cube  $[0, 1]^d \subseteq \mathbb{R}^d$ . A (finite, abstract) *cubical complex* is a finite poset  $K$  with a minimum element, denoted by  $\emptyset$  and referred to as the empty face, having the following properties: (i) the interval  $[\emptyset, F]$  in  $K$  is an abstract cube for every  $F \in K$ ; and (ii)  $K$  is a meet-semilattice (meaning that any two elements of  $K$  have a greatest lower bound). The elements of  $K$  are called *faces* and have well-defined dimensions. We will refer to the meet (greatest lower bound) of two faces of  $K$  as their *intersection* and will say that a face  $F$  of  $K$  is *contained* in a face  $G$  if  $F \leq G$  holds in  $K$ . We will refer to a nonempty order ideal of  $K$  as a *subcomplex*. The complex  $K$  is *pure* if all its maximal elements have the same dimension. Throughout this paper we will denote by  $\mathcal{F}(K)$  the subposet of nonempty faces of  $K$ . Any algebraic or topological properties of  $K$  we consider will refer to those of the simplicial complex of chains (order complex) of  $\mathcal{F}(K)$ ; see [4].

A geometric cube is any polytope which is combinatorially equivalent to a standard cube. A (finite) *geometric cubical complex* is a finite collection  $K$  of geometric cubes in some space  $\mathbb{R}^N$ , called faces, which has the properties: (i) every face of an element of  $K$  also belongs to  $K$ ; and (ii) the intersection of any two elements of  $K$  is a face of both. The set of faces of any geometric cubical complex, ordered by inclusion, is an (abstract) cubical complex but the converse is not always true; see [9, Section 1]. However, every cubical complex  $K$  is isomorphic to the poset of faces of a regular CW-complex (forced to have the intersection property), and that complex is determined from  $K$  up to homeomorphism; see [3] [5, Section 4.7] and Section 7.2.

The short cubical  $h$ -vector of a cubical complex  $K$  is defined in [1] as

$$(2.4) \quad h^{(sc)}(K, x) = \sum_{F \in K \setminus \{\emptyset\}} (2x)^{\dim(F)} (1-x)^{d-\dim(F)},$$

where  $d = \dim(K)$  is the maximum dimension of a face of  $K$ . The link  $\text{lk}_K(F)$  of any nonempty face  $F$  of  $K$  is naturally an abstract simplicial complex and we have the fundamental observation [1, Theorem 9] (due to G. Heteyi) that if  $K$  is pure, then

$$(2.5) \quad h^{(sc)}(K, x) = \sum_{v \in \text{vert}(K)} h(\text{lk}_K(v), x),$$

where  $\text{vert}(K)$  denotes the set of vertices (zero-dimensional faces) of  $K$ .

Given a cubical complex  $K$  of dimension  $d$  and a cubical complex  $K'$  of dimension  $d'$ , the direct product  $\mathcal{F}(K) \times \mathcal{F}(K')$  is isomorphic to the poset  $\mathcal{F}(K \times K')$  of nonempty faces of a cubical complex  $K \times K'$  of dimension  $d + d'$  and we have  $\dim(F \times F') = \dim(F) + \dim(F')$  for all  $F \in \mathcal{F}(K)$  and  $F' \in \mathcal{F}(K')$ . Hence, equation (2.4) yields

$$(2.6) \quad h^{(sc)}(K \times K', x) = h^{(sc)}(K, x) h^{(sc)}(K', x).$$

By analogy with (2.2), we define

$$(2.7) \quad h^{(sc)}(\text{int}(K), x) = \sum_{F \in \text{int}(K)} (2x)^{\dim(F)} (1-x)^{d-\dim(F)}$$

for every cubical complex  $K$  which is homeomorphic to a  $d$ -dimensional manifold with nonempty boundary  $\partial K$ , where  $\text{int}(K) = K \setminus \partial K$ . The next statement has appeared in [12]. We include a proof, so that it can be easily adapted in other situations (see Example 7.9). The statement (and proof) is valid for manifolds without boundary as well (see part (i) of [1, Theorem 5]), provided one sets  $\text{int}(K) = K$ . The corresponding statement for the long cubical  $h$ -vector (defined in the sequel) of a cubical ball appeared in [2, Proposition 4.1].

**Corollary 2.2.** ([12, Proposition 4.4]) *For every cubical complex  $K$  which is homeomorphic to a  $d$ -dimensional manifold with nonempty boundary we have*

$$(2.8) \quad x^d h^{(sc)}(K, 1/x) = h^{(sc)}(\text{int}(K), x).$$

*Proof.* We observe that for every vertex  $v$  of  $K$ , the simplicial complex  $\text{lk}_K(v)$  is either a homology sphere or a homology ball, depending on whether  $v$  lies in  $\text{int}(K)$  or not, of dimension  $d - 1$ . Thus, using (2.5) and Proposition 2.1, we compute that

$$\begin{aligned} x^d h^{(sc)}(K, 1/x) &= \sum_{v \in \text{vert}(K)} x^d h(\text{lk}_K(v), 1/x) = \sum_{v \in \text{vert}(K)} h(\text{int}(\text{lk}_K(v)), x) \\ &= \sum_{v \in \text{vert}(K)} \sum_{E \in \text{int}(\text{lk}_K(v))} x^{|E|} (1-x)^{d-|E|} \\ &= \sum_{F \in K} n(F) x^{\dim(F)} (1-x)^{d-\dim(F)}, \end{aligned}$$

where  $n(F)$  denotes the number of vertices  $v$  of  $F$  for which the face of  $\text{lk}_K(v)$  which corresponds to  $F$  is an interior face of  $\text{lk}_K(v)$ . Equation (2.8) follows by noting that

$$n(F) = \begin{cases} 2^{\dim(F)}, & \text{if } F \in \text{int}(K) \\ 0, & \text{otherwise} \end{cases}$$

for  $F \in K$ . □

The (long) cubical  $h$ -polynomial of a  $d$ -dimensional cubical complex  $K$  may be defined [1] by the equation

$$(2.9) \quad (x+1)h^{(c)}(K, x) = 2^d + xh^{(sc)}(K, x) + (-2)^d \tilde{\chi}(K)x^{d+2},$$

where

$$\tilde{\chi}(K) = \sum_{F \in K} (-1)^{\dim(F)}$$

is the reduced Euler characteristic of  $K$ . Since  $h^{(sc)}(K, -1) = 2^d + 2^d \tilde{\chi}(K)$  [1, Lemma 1 (ii)], the function  $h^{(c)}(K, x)$  is indeed a polynomial in  $x$  of degree at most  $d+1$ . For use in Section 6, we note (see also [1, Lemma 2]) that

$$(2.10) \quad h_0^{(c)}(K) = 2^d$$

and

$$(2.11) \quad h_{d+1}^{(c)}(K) = (-2)^d \tilde{\chi}(K),$$

where  $h^{(c)}(K, x) = \sum_{i=0}^{d+1} h_i^{(c)}(K)x^i$ .

**2.3. Cubical subdivisions.** The notion of cubical subdivision of a cubical complex we will adopt is analogous to that of a simplicial subdivision of an abstract simplicial complex, introduced in [17, Section 2].

A (finite, topological) *cubical subdivision* of a cubical complex  $K$  is a cubical complex  $K'$ , together with a map  $\sigma : \mathcal{F}(K') \rightarrow \mathcal{F}(K)$  (the associated subdivision map), having the following properties:

- (a) For every  $F \in \mathcal{F}(K)$ , the set  $K'_F := \sigma^{-1}(\mathcal{F}(F)) \cup \{\emptyset\}$  is a subcomplex of  $K'$  which is homeomorphic to a ball of dimension  $\dim(F)$ .
- (b)  $\sigma^{-1}(F)$  consists of the interior faces of the ball  $K'_F$ .

The subcomplex  $K'_F$  is called the *restriction* of  $K'$  to  $F$ . For  $G \in \mathcal{F}(K')$ , the face  $\sigma(G)$  of  $K$  is called the *carrier* of  $G$ . It follows from the defining properties that  $\sigma$  is surjective and that  $\dim(G) \leq \dim(\sigma(G))$  for every  $G \in \mathcal{F}(K')$ . Several examples of these concepts appear in the following sections.

Let  $C$  be a geometric cube. A cubical subdivision  $\Gamma$  of  $C$  is said to be *geometric* if it can be realized by a geometric cubical complex  $\Gamma'$  which subdivides  $C$  (meaning, the union of faces of  $\Gamma'$  is equal to  $C$ ). In that case, the carrier of  $G \in \Gamma' \setminus \{\emptyset\}$  is the smallest face of  $C$  which contains  $G$ .

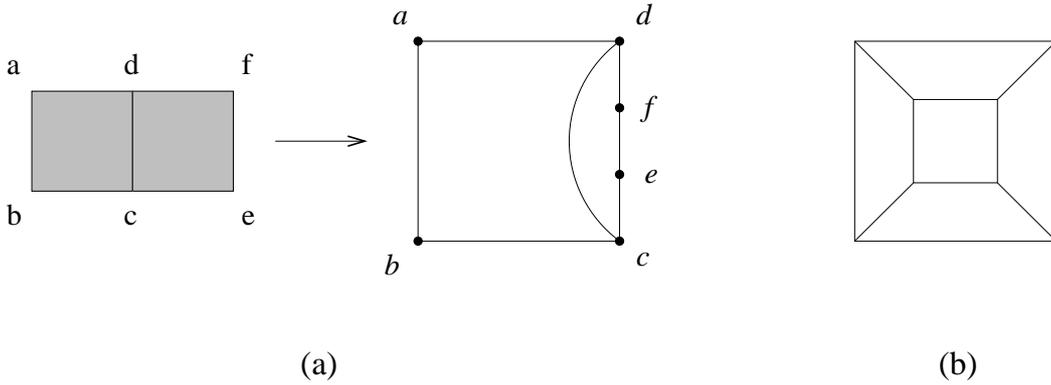


FIGURE 1. Two cubical subdivisions of a square.

### 3. THE SHORT CUBICAL LOCAL $h$ -VECTOR

This section includes definitions, basic examples and elementary properties of short cubical local  $h$ -vectors.

**Definition 3.1.** Let  $C$  be a  $d$ -dimensional cube. For any cubical subdivision  $\Gamma$  of  $C$ , we define a polynomial  $\ell_C(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_d x^d$  by

$$(3.1) \quad \ell_C(\Gamma, x) = \sum_{F \in \mathcal{F}(C)} (-1)^{d-\dim(F)} h^{(sc)}(\Gamma_F, x).$$

We call  $\ell_C(\Gamma, x)$  the *short cubical local  $h$ -polynomial* of  $\Gamma$  (with respect to  $C$ ). We call  $\ell_C(\Gamma) = (\ell_0, \ell_1, \dots, \ell_d)$  the *short cubical local  $h$ -vector* of  $\Gamma$  (with respect to  $C$ ).

**Example 3.2.** (a) For the trivial subdivision  $\Gamma = C$  of the  $d$ -dimensional cube  $C$  we have

$$\ell_C(\Gamma, x) = \begin{cases} 1, & \text{if } d = 0 \\ 0, & \text{if } d \geq 1. \end{cases}$$

This follows from (3.1) since for  $0 \leq k \leq d$ , there are  $2^{d-k} \binom{d}{k}$  faces  $F \in \mathcal{F}(C)$  of dimension  $k$  and we have  $h^{(sc)}(\Gamma_F, x) = 2^k$  for every such face.

(b) Suppose that  $\dim(C) = 1$ . We have  $h^{(sc)}(\Gamma, x) = tx + t + 2$  and  $\ell_C(\Gamma, x) = t(x + 1)$ , where  $t$  is the number of interior vertices of  $\Gamma$ .

(c) Part (a) of Figure 1 shows a nongeometric cubical subdivision  $\Gamma$  of a square  $C$  ( $\Gamma$  itself is shown as a geometric cubical complex). We have  $h^{(sc)}(\Gamma, x) = 2x + 6$  and  $\ell_C(\Gamma, x) = 0$ .

(d) Let  $C$  be a facet ( $d$ -dimensional face) of a  $(d+1)$ -dimensional cube  $D$ . The complex of faces of  $D$  other than  $C$  and  $D$  defines a cubical subdivision  $\Gamma$  of  $C$  (this is the subdivision defined by the Schlegel diagram of  $D$  with respect to  $C$ ; see [21, Section 5.2]). The special case  $d = 2$  is shown in part (b) of Figure 1. Since for the boundary complex (complex of proper faces)  $\partial D$  of  $D$  we have  $h^{(sc)}(\partial D, x) = 2^{d+1}(1 + x + \cdots + x^d)$  (see, for instance, [1, Lemma 1 (iv)]) and  $\Gamma_F$  is the trivial subdivision of  $F$  for every  $F \in \mathcal{F}(C) \setminus \{C\}$ , we may

conclude that

$$h^{(sc)}(\Gamma_F, x) = \begin{cases} 2^d (2 + 2x + \cdots + 2x^{d-1} + x^d), & \text{if } F = C \\ 2^{\dim(F)}, & \text{otherwise} \end{cases}$$

for  $F \in \mathcal{F}(C)$ . A simple calculation shows that  $\ell_C(\Gamma, x) = 2^d(1+x)(1+x+\cdots+x^{d-1})$ . In the special case  $d = 2$  we have  $h^{(sc)}(\Gamma, x) = 4x^2 + 8x + 8$  and  $\ell_C(\Gamma, x) = 4(x+1)^2$ .

(e) Let  $\Gamma$  be the complex of faces of two 3-dimensional cubes  $C$  and  $C'$ , intersecting on a common facet  $G$ . Let  $\sigma : \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(C)$  be the subdivision map which pushes  $C'$  into  $C$ , so that the facet of  $C'$  opposite to  $G$  ends up in the relative interior of  $G$ . Formally, we have  $\sigma(G) = \sigma(C') = C$ ,  $\sigma(F) = G$  for every proper face of  $C'$  not contained in  $G$  and  $\sigma(F) = F$  for all other nonempty faces of  $\Gamma$ . Thus, the restriction of this subdivision to the facet  $G$  of  $C$  is the one shown in part (b) of Figure 1, while the restriction to any other proper face of  $C$  is the trivial subdivision. We have  $h^{(sc)}(\Gamma, x) = 4(x+3)$  and  $\ell_C(\Gamma, x) = -4x(x+1)$ . The subdivision of part (a) of Figure 1 is a two-dimensional version of this construction.  $\square$

**Remark 3.3.** Let  $\Gamma$  be a cubical subdivision of a cube  $C$  of positive dimension  $d$ . We recall [1, Lemma 1 (ii)] that  $h^{(sc)}(K, -1) = 2^k \chi(K)$  for every cubical complex  $K$  of dimension  $k$ , where  $\chi(K) = 1 + \tilde{\chi}(K)$  is the Euler characteristic of  $K$ . Thus  $h^{(sc)}(\Gamma_F, -1) = 2^k$  for every face  $F \in \mathcal{F}(C)$  of dimension  $k$ . It follows from (3.1) that  $\ell_C(\Gamma, -1) = 0$  and hence the polynomial  $\ell_C(\Gamma, x)$  is divisible by  $x+1$ .  $\square$

Let  $\Gamma$  be a cubical subdivision of a cube  $C$  with associated subdivision map  $\sigma$ . The *excess* of a face  $F \in \Gamma \setminus \{\emptyset\}$  is defined as  $e(F) := \dim \sigma(F) - \dim(F)$ . The following statement is the analogue of [17, Proposition 2.2] in our setting.

**Proposition 3.4.** *For every cubical subdivision  $\Gamma$  of a  $d$ -dimensional cube  $C$  we have*

$$(3.2) \quad \ell_C(\Gamma, x) = (-1)^d \sum_{F \in \Gamma \setminus \{\emptyset\}} (-2)^{\dim(F)} x^{d-e(F)} (x-1)^{e(F)},$$

where  $e(F)$  is the excess of a nonempty face  $F$  of  $\Gamma$ .

*Proof.* We compute that

$$\begin{aligned} \ell_C(\Gamma, x) &= \sum_{G \in \mathcal{F}(C)} (-1)^{d-\dim(G)} h^{(sc)}(\Gamma_G, x) \\ &= \sum_{G \in \mathcal{F}(C)} (-1)^{d-\dim(G)} \left( \sum_{F \in \Gamma_G \setminus \{\emptyset\}} (2x)^{\dim(F)} (1-x)^{\dim(G)-\dim(F)} \right) \\ &= \sum_{F \in \Gamma \setminus \{\emptyset\}} (-1)^d \left( \frac{2x}{1-x} \right)^{\dim(F)} \left( \sum_{G \in \mathcal{F}(C): \sigma(F) \subseteq G} (x-1)^{\dim(G)} \right) \end{aligned}$$

and observe that the inner sum in the last expression is equal to  $(x-1)^{\dim \sigma(F)} x^{d-\dim \sigma(F)}$ , since the interval  $[\sigma(F), C]$  in  $\mathcal{F}(C)$  is a Boolean lattice of rank  $d - \dim \sigma(F)$ . The proposed formula follows.  $\square$

Next we consider products of cubical subdivisions. Let  $\Gamma$  be a cubical subdivision of a  $d$ -dimensional cube  $C$  with associated subdivision map  $\sigma$  and  $\Gamma'$  be a cubical subdivision of a  $d'$ -dimensional cube  $C'$  with associated subdivision map  $\sigma'$ . Then  $\Gamma \times \Gamma'$  is a cubical subdivision of the  $(d + d')$ -dimensional cube  $C \times C'$ , if one defines the carrier of a face  $G \times G'$  of  $\Gamma \times \Gamma'$  as the product  $\sigma(G) \times \sigma'(G')$ .

**Proposition 3.5.** *For cubical subdivisions  $\Gamma$  and  $\Gamma'$  of cubes  $C$  and  $C'$ , respectively, we have  $\ell_{C \times C'}(\Gamma \times \Gamma', x) = \ell_C(\Gamma, x)\ell_{C'}(\Gamma', x)$ .*

*Proof.* By construction, the restriction of  $\Gamma \times \Gamma'$  to a face  $F \times F'$  is equal to  $\Gamma_F \times \Gamma'_{F'}$ . Thus, recalling that  $\mathcal{F}(C \times C') = \mathcal{F}(C) \times \mathcal{F}(C')$  and using the defining equation (3.1) and (2.6), we find that

$$\begin{aligned} \ell_{C \times C'}(\Gamma \times \Gamma', x) &= \sum_{F \times F' \in \mathcal{F}(C \times C')} (-1)^{\dim(C \times C') - \dim(F \times F')} h^{(sc)}((\Gamma \times \Gamma')_{F \times F'}, x) \\ &= \sum_{F \times F' \in \mathcal{F}(C) \times \mathcal{F}(C')} (-1)^{d + d' - \dim(F) - \dim(F')} h^{(sc)}(\Gamma_F \times \Gamma'_{F'}, x) \\ &= \sum_{F \in \mathcal{F}(C), F' \in \mathcal{F}(C')} (-1)^{d - \dim(F)} h^{(sc)}(\Gamma_F, x) (-1)^{d' - \dim(F')} h^{(sc)}(\Gamma'_{F'}, x) \\ &= \ell_C(\Gamma, x) \ell_{C'}(\Gamma', x), \end{aligned}$$

as promised. □

We end this section with some more examples and remarks.

**Example 3.6.** (a) Let  $\ell_C(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_d x^d$ , where  $\Gamma$  and  $C$  are as in Definition 3.1. Since for  $F \in \Gamma \setminus \{\emptyset\}$  we have  $e(F) = d$  if and only if  $F$  is a vertex in the interior of  $C$ , it follows from Proposition 3.4 that  $\ell_0$  is equal to the number of interior vertices of  $\Gamma$ . Alternatively, this follows from the equality

$$\ell_0 = \sum_{F \in \mathcal{F}(C)} (-1)^{d - \dim(F)} h^{(sc)}(\Gamma_F, 0)$$

by a Möbius inversion argument (see [15, Proposition 3.7.1]), since  $h^{(sc)}(K, 0)$  is equal to the number of vertices of  $K$  for every cubical complex  $K$ . The coefficient  $\ell_d$  is equal to the coefficient of  $x^d$  in  $h^{(sc)}(\Gamma, x)$ . By setting  $x = 0$  in

$$x^d h^{(sc)}(\Gamma, 1/x) = h^{(sc)}(\text{int}(\Gamma), x)$$

(see Corollary 2.2), we deduce that  $\ell_d$  is also equal to the number of interior vertices of  $\Gamma$ . Alternatively, this follows from our previous remark on  $\ell_0$  and the symmetry of  $\ell_C(\Gamma, x)$  (Theorem 4.2).

(b) Similarly, from (3.2) we deduce the formula  $\ell_1 = -df_0^\circ + 2f_1^\circ - \tilde{f}_0$  for the coefficient of  $x$  in  $\ell_C(\Gamma, x)$ , where (as in part (g) of [17, Example 2.3])  $f_0^\circ$  and  $f_1^\circ$  are the numbers of interior vertices and edges of  $\Gamma$ , respectively, and  $\tilde{f}_0$  is the number of vertices which lie in the relative interior of a facet of  $C$ .

(c) Combining part (a) with Remark 3.3 we find that  $\ell_C(\Gamma, x) = t(x+1)^2$  for every cubical subdivision  $\Gamma$  of a square  $C$  (the case  $d = 2$ ), where  $t$  is the number of interior vertices of  $\Gamma$ .

(d) Part (c) and Example 3.2 (b) imply that for  $d \leq 2$ , the polynomial  $\ell_C(\Gamma, x)$  depends only on the face poset (in fact, on the short cubical  $h$ -polynomial) of the complex  $\Gamma$  (and not on  $\sigma$ ). This is no longer true for  $d \geq 3$ . For instance, the complex  $\Gamma$  of Example 3.2 (e) gives rise to a subdivision of the 3-dimensional cube  $C$  which is the product of the trivial subdivision of a square and the subdivision of a line segment with one interior point. For this subdivision we have  $\ell_C(\Gamma, x) = 0$ .  $\square$

#### 4. MAIN PROPERTIES OF SHORT CUBICAL LOCAL $h$ -VECTORS

This section develops the main properties of short cubical local  $h$ -vectors. From these properties, locality (Theorem 4.1), symmetry (Theorem 4.2) and monotonicity (Corollary 4.6) are proved in this section, while the proof of nonnegativity (Theorem 4.3) is deferred to Section 5.

The proof of our first result is a variation of that of Theorem 3.2 in [17, Section 3].

**Theorem 4.1.** *Let  $K$  be a pure cubical complex. For every cubical subdivision  $K'$  of  $K$  we have*

$$(4.1) \quad h^{(sc)}(K', x) = \sum_{F \in \mathcal{F}(K)} \ell_F(K'_F, x) h(\mathrm{lk}_K(F), x).$$

*Proof.* Denoting by  $R(K', x)$  the right-hand side of (4.1) and setting  $P = \mathcal{F}(K)$ , we compute that

$$\begin{aligned} R(K', x) &= \sum_{G \in P} \ell_G(K'_G, x) h(\mathrm{lk}_K(G), x) \\ &= \sum_{G \in P} \left( \sum_{F \leq_P G} (-1)^{\dim(G) - \dim(F)} h^{(sc)}(K'_F, x) \right) h(\mathrm{lk}_K(G), x) \\ &= \sum_{F \in P} h^{(sc)}(K'_F, x) \left( \sum_{F \leq_P G} (-1)^{\dim(G) - \dim(F)} h(\mathrm{lk}_K(G), x) \right) \\ &= - \sum_{F \in P} \left( \sum_{E \in K'_F \setminus \{\emptyset\}} (2x)^{\dim(E)} (1-x)^{\dim(K) - \dim(E)} \right) \tilde{\chi}(\mathrm{lk}_K(F)) \\ &= - \sum_{E \in K' \setminus \{\emptyset\}} (2x)^{\dim(E)} (1-x)^{\dim(K) - \dim(E)} \sum_{\sigma(E) \leq_P F} \tilde{\chi}(\mathrm{lk}_K(F)). \end{aligned}$$

The fourth of the previous equalities follows from the defining equation

$$h^{(sc)}(K'_F, x) = \sum_{E \in K'_F \setminus \{\emptyset\}} (2x)^{\dim(E)} (1-x)^{\dim(F) - \dim(E)}$$

and the equality

$$\sum_{F \leq_P G} (-1)^{\dim(G) - \dim(F)} h(\mathrm{lk}_K(G), x) = -(1-x)^{\dim(K) - \dim(F)} \tilde{\chi}(\mathrm{lk}_K(F)),$$

for given  $F \in P$ . The latter is equivalent to

$$\sum_{F \leq_P G} (-1)^{\dim(K) - \dim(G)} h(\mathrm{lk}_K(G), x) = -(x-1)^{\dim(K) - \dim(F)} \tilde{\chi}(\mathrm{lk}_K(F)),$$

which in turn follows by applying [17, Lemma 3.1] to  $\mathrm{lk}_K(F)$ , which is a pure simplicial complex of dimension  $\dim(K) - \dim(F) - 1$ . As in the proof of Theorem 3.2 in [17, p. 813] we find that

$$\sum_{\sigma(E) \leq_P F} \tilde{\chi}(\mathrm{lk}_K(F)) = -1$$

and hence

$$R(K', x) = \sum_{E \in K' \setminus \{\emptyset\}} (2x)^{\dim(E)} (1-x)^{\dim(K) - \dim(E)} = h^{(sc)}(K', x),$$

as claimed in the statement of the theorem. □

Our second result on short cubical local  $h$ -vectors is as follows.

**Theorem 4.2.** *For every cubical subdivision  $\Gamma$  of a  $d$ -dimensional cube  $C$  we have*

$$(4.2) \quad x^d \ell_C(\Gamma, 1/x) = \ell_C(\Gamma, x).$$

*Equivalently, we have  $\ell_i = \ell_{d-i}$  for  $0 \leq i \leq d$ , where  $\ell_C(\Gamma) = (\ell_0, \ell_1, \dots, \ell_d)$ .*

*Proof.* Setting  $P = \mathcal{F}(C)$  and using (3.1) and Corollary 2.2 we find that

$$\begin{aligned} x^d \ell_C(\Gamma, 1/x) &= \sum_{G \in P} (-x)^{d - \dim(G)} x^{\dim(G)} h^{(sc)}(\Gamma_G, 1/x) \\ &= \sum_{G \in P} (-x)^{d - \dim(G)} h^{(sc)}(\mathrm{int}(\Gamma_G), x). \end{aligned}$$

We claim that

$$(4.3) \quad h^{(sc)}(\mathrm{int}(\Gamma_G), x) = \sum_{F \leq_P G} (x-1)^{\dim(G) - \dim(F)} h^{(sc)}(\Gamma_F, x)$$

for every  $G \in P$ . Given the claim, we may conclude that

$$\begin{aligned}
 x^d \ell_C(\Gamma, 1/x) &= \sum_{G \in P} \sum_{F \leq_P G} (-x)^{d-\dim(G)} (x-1)^{\dim(G)-\dim(F)} h^{(sc)}(\Gamma_F, x) \\
 &= \sum_{F \in P} (-x)^{d-\dim(F)} h^{(sc)}(\Gamma_F, x) \sum_{F \leq_P G \leq_P C} \left( \frac{1-x}{x} \right)^{\dim(G)-\dim(F)} \\
 &= \sum_{F \in P} (-x)^{d-\dim(F)} h^{(sc)}(\Gamma_F, x) (1/x)^{d-\dim(F)} \\
 &= \ell_C(\Gamma, x)
 \end{aligned}$$

where, in the third equality, we have used the fact that the interval  $[F, C]$  in  $P$  is a Boolean lattice of rank  $d - \dim(F)$ . To verify (4.3) we note that, in view of (2.4) and (2.7), we may write

$$(4.4) \quad (1-x)^{-\dim(G)} h^{(sc)}(\Gamma_G, x) = \sum_{E \in \Gamma_G \setminus \{\emptyset\}} \left( \frac{2x}{1-x} \right)^{\dim(E)}$$

and

$$(4.5) \quad (1-x)^{-\dim(G)} h^{(sc)}(\text{int}(\Gamma_G), x) = \sum_{E \in \text{int}(\Gamma_G)} \left( \frac{2x}{1-x} \right)^{\dim(E)}$$

for  $G \in P$ . Denoting by  $\beta(G)$  and  $\alpha(G)$  the right-hand sides of (4.4) and (4.5), respectively, we have

$$\beta(G) = \sum_{F \leq_P G} \alpha(F)$$

for every  $G \in P$ . By Möbius inversion [15, Proposition 3.7.1] on  $P$  we get

$$\alpha(G) = \sum_{F \leq_P G} (-1)^{\dim(G)-\dim(F)} \beta(F)$$

for every  $G \in P$ . Replacing  $\alpha(G)$  and  $\beta(F)$  with the appropriate expressions from the left-hand sides of (4.5) and (4.4) shows that the last equality is equivalent to (4.3).  $\square$

We recall [17, Section 4] that a simplicial subdivision  $\Delta'$  of a simplicial complex  $\Delta$  is called *quasi-geometric* if no face of  $\Delta'$  has all its vertices lying in a face of  $\Delta$  of smaller dimension. A cubical subdivision  $\Gamma$  of a cube  $C$  will be called *locally quasi-geometric* if there are no faces  $F \in \mathcal{F}(C)$  and  $G \in \mathcal{F}(\Gamma)$  and vertex  $v$  of  $G$ , satisfying the following: (a)  $F$  has smaller dimension than  $G$ ; and (b) for every edge  $e$  of  $G$  which contains  $v$ , the carrier of  $e$  is contained in  $F$ . Clearly, every geometric cubical subdivision of (a geometric cube)  $C$  is locally quasi-geometric. Cubical subdivisions which are not locally quasi-geometric are given in parts (c) and (e) of Example 3.2. A cubical subdivision  $K'$  of a cubical complex  $K$  will be called *locally quasi-geometric* if for every  $F \in K \setminus \{\emptyset\}$ , the restriction  $K'_F$  is a

locally quasi-geometric subdivision of  $F$ . Part (e) of Example 3.2 shows that the following statement fails to hold for all cubical subdivisions of cubes.

**Theorem 4.3.** *For every locally quasi-geometric cubical subdivision  $\Gamma$  of a  $d$ -dimensional cube  $C$  we have  $\ell_i \geq 0$  for  $0 \leq i \leq d$ , where  $\ell_C(\Gamma) = (\ell_0, \ell_1, \dots, \ell_d)$ .*

The proof of Theorem 4.3 is given in Section 5.

**Question 4.4.** Is  $\ell_C(\Gamma)$  unimodal for every locally quasi-geometric cubical subdivision  $\Gamma$  of a cube  $C$ ?

**Remark 4.5.** In analogy with the simplicial case, we may call a cubical subdivision  $\Gamma$  of  $C$  *quasi-geometric* if no face of  $\Gamma$  has all its vertices lying in a face of  $C$  of smaller dimension. Consider a square  $C$  with vertices  $a, b, c, d$  and edges  $\{a, b\}$ ,  $\{b, c\}$ ,  $\{c, d\}$ ,  $\{a, d\}$ . Insert two points  $e$  and  $f$  in the relative interior of the edge  $\{b, c\}$  and two points  $g$  and  $h$  in the relative interior of the edge  $\{c, d\}$ , so that  $e$  lies in the relative interior of  $\{b, f\}$  and  $g$  lies in the relative interior of  $\{c, h\}$ . The subdivision of  $C$  with three two-dimensional faces, having vertex sets  $\{a, b, d, e\}$ ,  $\{d, e, g, h\}$  and  $\{c, e, f, g\}$ , is quasi-geometric but not locally quasi-geometric. We leave it to the reader to construct a two-dimensional example of a locally quasi-geometric cubical subdivision which is not quasi-geometric.  $\square$

The following statement is a short cubical analogue of [17, Theorem 4.10]. All inequalities of the form  $p(x) \geq q(x)$  which appear in the sequel, where  $p(x)$  and  $q(x)$  are real polynomials, will be meant to hold coefficientwise.

**Corollary 4.6.** *Let  $K$  be a cubical complex such that  $\text{lk}_K(F)$  is Cohen-Macaulay, over some field, for every face  $F \in K$  with  $\dim(F) \geq 1$ . Then we have  $h^{(sc)}(K', x) \geq h^{(sc)}(K, x)$  for every locally quasi-geometric cubical subdivision  $K'$  of  $K$ .*

*Proof.* In view of equation (2.5), we may rewrite (4.1) as

$$(4.6) \quad h^{(sc)}(K', x) = h^{(sc)}(K, x) + \sum_{F \in K: \dim(F) \geq 1} \ell_F(K'_F, x) h(\text{lk}_K(F), x).$$

The result now follows from Theorem 4.3 and our assumptions on  $K$ .  $\square$

**Question 4.7.** Let  $K$  be a Buchsbaum cubical complex, over some field. Does the conclusion of Corollary 4.6 hold for every cubical subdivision  $K'$  of  $K$ ?

We end this section with two specific kinds of cubical subdivision, to which Theorem 4.1 can be applied. The second one (see Example 4.9) provided some of the motivation for this paper.

**Example 4.8.** Let  $K$  be a cubical complex of positive dimension  $d$  and let  $K'$  be a cubical subdivision of  $K$  for which the restriction  $K'_F$  is the one in part (d) of Example 3.2, for a given  $d$ -dimensional face  $F$  of  $K$ , and the trivial subdivision for every other nonempty face of  $K$ . This subdivision, considered in [11] in the context of the lower bound problem for cubical polytopes, may be thought of as a cubical analogue of the stellar subdivision on a maximal face of a simplicial complex. The computations in Example 3.2 and (4.6) imply that  $h^{(sc)}(K', x) = h^{(sc)}(K, x) + 2^d(1+x)(1+x+\dots+x^{d-1})$ .  $\square$

**Example 4.9.** Let  $K$  be a  $d$ -dimensional cubical complex and  $t$  be a positive integer. Let  $K'$  be the cubical subdivision of  $K$  which has the following property: the restriction  $K'_F$  to any nonempty face  $F$  of  $K$  is combinatorially isomorphic to the product of as many as  $\dim(F)$  copies of the subdivision of a line segment with  $t$  interior vertices. For  $t = 1$ , the face poset  $\mathcal{F}(K')$  is isomorphic to the set of nonempty closed intervals in  $\mathcal{F}(K)$ , ordered by inclusion, and  $K'$  is the *cubical barycentric subdivision* (called the barycentric cover in [2]) of  $K$ , studied in [14] (see also Example 7.12). We refer the reader to [2, Section 2.3] and [14] for more information on this special case.

We can express the short cubical  $h$ -vector of  $K'$  in terms of that of  $K$  as follows. By Example 3.2 (b) and Proposition 3.5, we have  $\ell_F(K'_F, x) = t^{\dim(F)}(x + 1)^{\dim(F)}$  for every  $F \in \mathcal{F}(K)$ . Thus, setting  $P = \mathcal{F}(K)$  and using Theorem 4.1, we compute that

$$\begin{aligned}
 h^{(sc)}(K', x) &= \sum_{F \in P} (tx + t)^{\dim(F)} h(\mathrm{lk}_K(F), x) \\
 &= \sum_{F \in P} (tx + t)^{\dim(F)} \left( \sum_{F \leq_P G} x^{\dim(G) - \dim(F)} (1 - x)^{d - \dim(G)} \right) \\
 &= (1 - x)^d \sum_{G \in P} \left( \frac{x}{1 - x} \right)^{\dim(G)} \sum_{F \leq_P G} \left( \frac{tx + t}{x} \right)^{\dim(F)} \\
 &= (1 - x)^d \sum_{G \in P} \left( \frac{x}{1 - x} \right)^{\dim(G)} \left( \frac{(t + 2)x + t}{x} \right)^{\dim(G)} \\
 &= (1 - x)^d \sum_{G \in P} \left( \frac{(t + 2)x + t}{1 - x} \right)^{\dim(G)} \\
 &= (1 - x)^d f_K \left( \frac{(t + 2)x + t}{1 - x} \right),
 \end{aligned}$$

where  $f_K(x) = \sum_{F \in \mathcal{F}(K)} x^{\dim(F)}$  is the  $f$ -polynomial of  $K$  and for the fourth equality, we have used the fact that there are  $2^{k-i} \binom{k}{i}$  faces of  $G$  of dimension  $i$  for  $0 \leq i \leq k$ , where  $k = \dim(G)$ . Expressing  $f_K(x)$  in terms of  $h^{(sc)}(K, x)$  (see, for instance, [14, Equation (3)]), we find that

$$(4.7) \quad h^{(sc)}(K', x) = \left( \frac{tx + t + 2}{2} \right)^d h^{(sc)} \left( K, \frac{(t + 2)x + t}{tx + t + 2} \right).$$

The previous equality agrees with [14, Theorem 3.2] in the special case  $t = 1$  and thus recovers one of the main results of [14].  $\square$

## 5. NONNEGATIVITY OF SHORT CUBICAL LOCAL $h$ -VECTORS

This section proves the nonnegativity of short cubical local  $h$ -vectors for locally quasi-geometric subdivisions (Theorem 4.3). Our method follows that employed by Stanley to

prove [17, Theorem 4.6]. Thus we will assume familiarity with [17, Section 4] and omit some of the details of the proof, giving emphasis to those points in which the arguments in [17] need to be adapted or generalized.

We first observe that nonnegativity of the short cubical local  $h$ -vector is a consequence of the following statement.

**Theorem 5.1.** *For every locally quasi-geometric cubical subdivision  $\Gamma$  of a  $d$ -dimensional cube  $C$  and every vertex  $v$  of  $\Gamma$  we have*

$$(5.1) \quad \sum_{F \in \mathcal{F}(C), v \in F} (-1)^{d-\dim(F)} h(\text{lk}_{\Gamma_F}(v), x) \geq 0.$$

*Proof of Theorem 4.3.* Using (2.5), we compute that

$$\begin{aligned} \ell_C(\Gamma, x) &= \sum_{F \in \mathcal{F}(C)} (-1)^{d-\dim(F)} h^{(sc)}(\Gamma_F, x) \\ &= \sum_{F \in \mathcal{F}(C)} (-1)^{d-\dim(F)} \left( \sum_{v \in \text{vert}(\Gamma_F)} h(\text{lk}_{\Gamma_F}(v), x) \right) \\ &= \sum_{v \in \text{vert}(\Gamma)} \sum_{F \in \mathcal{F}(C), v \in F} (-1)^{d-\dim(F)} h(\text{lk}_{\Gamma_F}(v), x). \end{aligned}$$

Thus, the result follows from Theorem 5.1.  $\square$

For the remainder of this section we let  $\sigma : \mathcal{F}(\Gamma) \rightarrow P = \mathcal{F}(C)$  be the subdivision map associated with  $\Gamma$ . We also write  $\dim(\sigma(v)) = d - e$ , where  $d = \dim(C)$  and  $0 \leq e \leq d$ . To motivate our approach towards proving Theorem 5.1, we consider two special cases. Suppose first that  $e = d$ , so that  $v$  is a vertex of  $C$ . Assume further that  $\text{lk}_{\Gamma}(v)$  is a topological subdivision of the  $(d-1)$ -dimensional simplex  $\text{lk}_C(v)$  (this happens, for instance, if  $\Gamma$  is a geometric subdivision). Then  $\text{lk}_{\Gamma}(v)$  is a quasi-geometric subdivision of  $\text{lk}_C(v)$ , in the sense of [17, Section 4], and the left-hand side of (5.1) is equal to the simplicial local  $h$ -vector of this subdivision [17, Definition 2.1]. Therefore, Theorem 5.1 follows in this case from [17, Corollary 4.7]. On the other extreme, if  $e = 0$ , so that  $v$  is an interior vertex of  $\Gamma$ , then the left-hand side of (5.1) consists of the single term  $h(\text{lk}_{\Gamma}(v), x)$ . This polynomial is nonnegative since  $\text{lk}_{\Gamma}(v)$  is Cohen-Macaulay (in fact, a homology sphere) over any field. The proof which follows generalizes that in [17, Section 4] and interpolates between these two special cases.

*Proof of Theorem 5.1.* We fix a vertex  $v$  of  $\Gamma$  and set  $\Delta = \text{lk}_{\Gamma}(v)$ . We observe that the poset of faces  $F \in \mathcal{F}(C)$  with  $v \in F$  is equal to the closed interval  $[\sigma(v), C]$  in  $\mathcal{F}(C)$ . This interval is a Boolean lattice of rank  $e$ . We denote (in the case  $e \geq 1$ ) by  $\Delta_1, \Delta_2, \dots, \Delta_e$  the subcomplexes of  $\Delta$  of the form  $\text{lk}_{\Gamma_G}(v)$ , where  $G$  runs through the codimension one faces of  $C$  in the interval  $[\sigma(v), C]$ . We also write  $\Delta_S = \bigcap_{i \in S} \Delta_i$  for  $S \subseteq [e] := \{1, 2, \dots, e\}$ ,

where  $\Delta_{\emptyset} = \Delta$ . Then the subcomplexes  $\text{lk}_{\Gamma_F}(v)$  of  $\Delta$  which appear in the left-hand side of (5.1) are precisely the complexes  $\Delta_S$  for  $S \subseteq [e]$ . Thus (5.1) can be rewritten as

$$(5.2) \quad \sum_{S \subseteq [e]} (-1)^{|S|} h(\Delta_S, x) \geq 0.$$

Moreover, the complexes  $\Gamma_F$  are Cohen-Macaulay over all fields (as they are homeomorphic to balls) for  $F \in \mathcal{F}(C)$  and therefore, so are the links of their faces and hence the complexes  $\Delta_S$ , for  $S \subseteq [e]$ .

We fix an infinite field  $\mathbb{K}$  and consider the face ring (or Stanley-Reisner ring)  $\mathbb{K}[\Delta]$  of  $\Delta$ ; see [19, Chapter II]. Given a linear system of parameters  $\Theta = \{\Theta_1, \dots, \Theta_d\}$  for  $\mathbb{K}[\Delta]$ , we set  $\mathbb{K}(\Delta) := \mathbb{K}[\Delta]/(\Theta)$  and denote by  $L_v(\Gamma)$  the image in  $\mathbb{K}(\Delta)$  of the ideal of  $\mathbb{K}[\Delta]$  generated by the face monomials  $x^F$ , where  $F \in \Delta$  runs through all faces which do not belong to any of the  $\Delta_i$ . It follows from [17, Theorem 4.6] that, in the special case  $e = d$ , the system  $\Theta$  can be chosen so that

$$(5.3) \quad \sum_{S \subseteq [e]} (-1)^{|S|} h(\Delta_S, x) = \sum_{i=0}^d \dim_{\mathbb{K}} L_v(\Gamma)_i x^i,$$

where  $L_v(\Gamma)_i$  denotes the homogeneous part of  $L_v(\Gamma)$  of degree  $i$ . We note that (5.3) also holds when  $e = 0$  (for every  $\Theta$ ), since then  $L_v(\Gamma) = \mathbb{K}(\Delta)$ , the left-hand side of (5.3) is equal to  $h(\Delta, x)$  and Cohen-Macaulayness of  $\Delta$  over  $\mathbb{K}$  implies that  $\dim_{\mathbb{K}} \mathbb{K}(\Delta)_i = h_i(\Delta)$  for all  $0 \leq i \leq d$ .

To establish (5.2) in the general case, it suffices to show that  $\Theta$  can always be chosen so that (5.3) holds. We call a sequence  $\{\Theta_1, \dots, \Theta_d\}$  of linear forms on  $\mathbb{K}[\Delta]$  *special* if  $\Theta_{d-e+i}$  is a linear combination of vertices of  $\Delta$  which do not belong to  $\Delta_i$ , for each  $1 \leq i \leq e$ . Since  $\Gamma$  is locally quasi-geometric, there is no face  $F$  of  $\Delta$  and set  $S \subseteq [e]$  of cardinality exceeding  $d - \dim(F)$ , such that all vertices of  $F$  belong to  $\Delta_S$ . Thus, by choosing  $\{\Theta_1, \dots, \Theta_d\}$  as generically as possible, subject to the condition of being special, it follows as in the proof of [17, Corollary 4.4] that a special linear system of parameters for  $\mathbb{K}[\Delta]$  exists. We will show that (5.3) holds for such  $\Theta$ . We consider the natural complex of  $\mathbb{K}[\Delta]$ -modules

$$(5.4) \quad \mathbb{K}[\Delta] \rightarrow \bigoplus_{i=1}^e \mathbb{K}[\Delta]/N_i \rightarrow \bigoplus_{1 \leq i < j \leq e} \mathbb{K}[\Delta]/(N_i + N_j) \rightarrow \dots \rightarrow \mathbb{K}[\Delta]/(N_1 + \dots + N_e) \rightarrow 0,$$

where  $N_i$  is the ideal of  $\mathbb{K}[\Delta]$  generated by all face monomials  $x^F$  for which  $F \in \Delta$  does not belong to  $\Delta_i$ . The complex (5.4) naturally generalizes that in [17, Lemma 4.9]. Clearly, for  $1 \leq i_1 < \dots < i_r \leq e$  we have

$$(5.5) \quad \mathbb{K}[\Delta]/(N_{i_1} + \dots + N_{i_r}) \cong \mathbb{K}[\Delta_S]$$

as  $\mathbb{K}[\Delta]$ -modules, where  $S = [e] \setminus \{i_1, \dots, i_r\}$ . Therefore (5.4) can be rewritten as

$$(5.6) \quad \mathbb{K}[\Delta] \rightarrow \bigoplus_{i=1}^e \mathbb{K}[\Delta_i] \rightarrow \dots \rightarrow \bigoplus_{S \subseteq [e]: |S|=k} \mathbb{K}[\Delta_S] \rightarrow \dots \rightarrow \mathbb{K}[\Delta_{[e]}] \rightarrow 0.$$

We claim that taking the quotient of (5.6) by  $(\Theta)$  produces an exact complex, namely

$$(5.7) \quad \mathbb{K}(\Delta) \rightarrow \bigoplus_{i=1}^e \mathbb{K}(\Delta_i) \rightarrow \cdots \rightarrow \bigoplus_{S \subseteq [e]: |S|=k} \mathbb{K}(\Delta_S) \rightarrow \cdots \rightarrow \mathbb{K}(\Delta_{[e]}) \rightarrow 0,$$

where  $\mathbb{K}(\Delta_S) := \mathbb{K}[\Delta_S]/(\Theta)$  for  $S \subseteq [e]$ . Given the claim, the proof concludes as follows. Except for the irrelevant terms  $\Theta_i$  which annihilate  $\mathbb{K}[\Delta_S]$  (specifically, those with  $i \in S$ ), the sequence  $\Theta$  is a linear system of parameters for  $\mathbb{K}[\Delta_S]$ . Since each complex  $\Delta_S$  is Cohen-Macaulay over  $\mathbb{K}$ , we have

$$\sum_{i=0}^d \dim_{\mathbb{K}} \mathbb{K}(\Delta_S)_i x^i = h(\Delta_S, x)$$

for every  $S \subseteq [e]$ . Moreover, if  $\delta$  is the leftmost map in (5.7), then  $\ker(\delta) = L_v(\Gamma)$  and hence (5.3) follows by computing the Hilbert series of  $\ker(\delta)$  from the exact sequence (5.7).

To prove the claim we show by induction on  $r$ , as in the proof of [17, Lemma 4.9], that the sequence

$$(5.8) \quad \overline{\mathbb{K}[\Delta]} \rightarrow \bigoplus_{i=1}^e \overline{\mathbb{K}[\Delta_i]} \rightarrow \cdots \rightarrow \bigoplus_{S \subseteq [e]: |S|=k} \overline{\mathbb{K}[\Delta_S]} \rightarrow \cdots \rightarrow \overline{\mathbb{K}[\Delta_{[e]}}] \rightarrow 0,$$

obtained by taking the quotient of (5.6) by the ideal  $(\Theta_1, \dots, \Theta_r)$ , is exact. For  $r = 0$ , this means that the sequence (5.6) itself is exact and the argument given on [17, p. 817] works (with the understanding that the dimension of the simplex  $\Sigma$ , mentioned there, is one less than the number of indices  $1 \leq i \leq e$  with  $F \in \Delta_i$ ). For the inductive step, we distinguish two cases. First, if  $0 \leq r < d - e$ , then  $\Theta_{r+1}$  is a nonzero-divisor on each term of (5.8) and hence all kernels  $B^k$  of the maps which are given by multiplication with  $\Theta_{r+1}$  on these terms are identically zero. Otherwise, we write  $r + 1 = d - e + j$  for some  $1 \leq j \leq e$  and note that  $\Theta_{r+1}$  either annihilates  $\overline{\mathbb{K}[\Delta_S]}$  or is a nonzero-divisor on it, depending on whether  $j \in S$  or not. Thus we have

$$B^k = \bigoplus_{\substack{S \subseteq [e]: |S|=k, \\ j \in S}} \overline{\mathbb{K}[\Delta_S]}$$

for  $0 \leq k \leq d$  (for instance,  $B^0 = 0$  is the kernel of  $\Theta_{r+1}$  on  $\overline{\mathbb{K}[\Delta]}$ ) and the argument given in the proof of [17, Lemma 4.9] to show that  $B^1 \rightarrow \cdots \rightarrow B^d \rightarrow 0$  is exact goes through, as does the last step of the proof, completing the induction.  $\square$

**Remark 5.2.** Using (2.5), one can show as in the proof of Theorem 4.2 that each contribution (5.1) to  $\ell_C(\Gamma, x)$  is a polynomial with symmetric coefficients. Moreover, it follows as in [17, Corollary 4.19] that  $L_v(\Gamma)$  is a Gorenstein  $\mathbb{K}[\Delta]$ -module for every vertex  $v$  of  $\Gamma$ . It might be worth studying further properties and examples of the modules  $L_v(\Gamma)$ .  $\square$

**Remark 5.3.** It was shown by Stanley [17, Theorem 5.2] that the simplicial local  $h$ -vector is unimodal for every *regular* (hence geometric) triangulation of a simplex. It is plausible that the techniques of [17, Section 5] can be applied to show that if  $\Gamma$  is a regular cubical

subdivision of the geometric cube  $C$ , then each contribution (5.1) to  $\ell_C(\Gamma, x)$  is unimodal. This would provide an affirmative answer to Question 4.4 in the special case of regular cubical subdivisions.  $\square$

### 6. THE LONG CUBICAL LOCAL $h$ -VECTOR

This section defines (long) cubical local  $h$ -vectors, studies their elementary properties and lists some open questions related to them.

**Definition 6.1.** Let  $C$  be a  $d$ -dimensional cube. For any cubical subdivision  $\Gamma$  of  $C$ , we define a polynomial  $L_C(\Gamma, x) = L_0 + L_1x + \cdots + L_{d+1}x^{d+1}$  by

$$(6.1) \quad L_C(\Gamma, x) = \sum_{F \in \mathcal{F}(C)} (-1)^{d - \dim(F)} h^{(c)}(\Gamma_F, x).$$

We call  $L_C(\Gamma, x)$  the (long) *cubical local  $h$ -polynomial* of  $\Gamma$  (with respect to  $C$ ). We call  $L_C(\Gamma) = (L_0, L_1, \dots, L_{d+1})$  the (long) *cubical local  $h$ -vector* of  $\Gamma$  (with respect to  $C$ ).

**Remark 6.2.** (i) It follows from (2.11) that the coefficient  $L_{d+1}$  of  $x^{d+1}$  in the right-hand side of (6.1) is equal to zero. Hence the degree of  $L_C(\Gamma, x)$  cannot exceed  $d$ .

(ii) Suppose that  $d \geq 1$ . It follows from (2.10) that the constant term  $L_0$  of the right-hand side of (6.1) is equal to zero. Hence  $L_C(\Gamma, x)$  is divisible by  $x$ .  $\square$

The two kinds of cubical local  $h$ -vectors we have defined are related in a simple way, as the following proposition shows.

**Proposition 6.3.** *For every cubical subdivision  $\Gamma$  of a cube  $C$  of dimension  $d \geq 1$  we have*

$$(6.2) \quad x \ell_C(\Gamma, x) = (x + 1) L_C(\Gamma, x).$$

*Thus we have  $L_0 = L_{d+1} = 0$  and  $L_{i+1} = \ell_i - \ell_{i-1} + \cdots + (-1)^i \ell_0$  for  $0 \leq i \leq d - 1$ , where  $L_C(\Gamma) = (L_0, L_1, \dots, L_{d+1})$  and  $\ell_C(\Gamma) = (\ell_0, \ell_1, \dots, \ell_d)$ .*

*Proof.* Since  $\tilde{\chi}(\Gamma_F) = 0$  for  $F \in \mathcal{F}(C)$ , applying (2.9) to  $\Gamma_F$  we get

$$(x + 1) h^{(c)}(\Gamma_F, x) = 2^{\dim(F)} + x h^{(sc)}(\Gamma_F, x).$$

The result follows by multiplying (6.1) by  $x + 1$  and using the previous equality.  $\square$

The statements in the next example follow from Proposition 6.3 and the computations in Examples 3.2 and 3.6.

**Example 6.4.** (a) For the trivial subdivision of the  $d$ -dimensional cube  $C$ , the cubical local  $h$ -polynomial is equal to 1, if  $d = 0$ , and vanishes otherwise. For a general cubical subdivision  $\Gamma$  of  $C$  we have  $L_C(\Gamma, x) = tx$ , if  $d = 1$ , and  $L_C(\Gamma, x) = tx(x + 1)$ , if  $d = 2$ , where  $t$  is the number of interior vertices of  $\Gamma$ . For the subdivision in part (d) of Example 3.2 we have  $L_C(\Gamma, x) = 2^d(x + x^2 + \cdots + x^d)$ .

(b) Let  $L_C(\Gamma) = (L_0, L_1, \dots, L_{d+1})$ , where  $\Gamma$  and  $C$  are as in Definition 6.1. Then  $L_1 = L_d$  is equal to the number of interior vertices of  $\Gamma$  and  $L_2 = -(d + 1)f_0^\circ + 2f_1^\circ - \tilde{f}_0$ , where the notation is as in part (b) of Example 3.6.  $\square$

**Corollary 6.5.** *For every cubical subdivision  $\Gamma$  of a cube  $C$  of dimension  $d \geq 1$  we have*

$$(6.3) \quad x^{d+1} L_C(\Gamma, 1/x) = L_C(\Gamma, x).$$

*Equivalently, we have  $L_i = L_{d+1-i}$  for  $0 \leq i \leq d+1$ , where  $L_C(\Gamma) = (L_0, L_1, \dots, L_{d+1})$ .*

*Proof.* This follows from Theorem 4.2 and Proposition 6.3.  $\square$

The following statement is the analogue of Theorem 4.1 for the long cubical  $h$ -vector.

**Theorem 6.6.** *Let  $K$  be a pure cubical complex. For every cubical subdivision  $K'$  of  $K$  we have*

$$(6.4) \quad h^{(c)}(K', x) = h^{(c)}(K, x) + \sum_{F \in K: \dim(F) \geq 1} L_F(K'_F, x) h(\text{lk}_K(F), x).$$

*Proof.* We multiply (4.1) by  $x$  and use Proposition 6.3, equation (2.5) and the fact that  $\ell_F(K'_F, x) = 1$  for each zero-dimensional face  $F \in K$  to get

$$\begin{aligned} x h^{(sc)}(K', x) &= (x+1) \sum_{F \in K: \dim(F) \geq 1} L_F(K'_F, x) h(\text{lk}_K(F), x) + x \sum_{v \in \text{vert}(K)} h(\text{lk}_K(v), x) \\ &= (x+1) \sum_{F \in K: \dim(F) \geq 1} L_F(K'_F, x) h(\text{lk}_K(F), x) + x h^{(sc)}(K, x). \end{aligned}$$

Applying (2.9) to  $K$  and  $K'$  and noting that  $\tilde{\chi}(K) = \tilde{\chi}(K')$  we get

$$(x+1) h^{(c)}(K', x) - x h^{(sc)}(K', x) = (x+1) h^{(c)}(K, x) - x h^{(sc)}(K, x).$$

The result follows by adding the previous two equalities and dividing by  $x+1$ .  $\square$

**Example 6.7.** For the cubical stellar subdivision  $K'$  of Example 4.8 we have  $h^{(c)}(K', x) = h^{(c)}(K, x) + 2^d(x + x^2 + \dots + x^d)$ .  $\square$

The following are the main open questions on cubical local  $h$ -vectors. In view of Theorem 6.6, a positive answer to the first question would imply a positive answer to the third one for locally quasi-geometric subdivisions. Since we have  $L_C(\Gamma, x) = -4x^2$  for the subdivision in part (e) of Example 3.2, the first question has a negative answer for general subdivisions.

**Question 6.8.** Does  $L_C(\Gamma, x) \geq 0$  hold for every locally quasi-geometric cubical subdivision  $\Gamma$  of a cube  $C$ ?

**Question 6.9.** Is  $L_C(\Gamma)$  unimodal for every locally quasi-geometric cubical subdivision  $\Gamma$  of a cube  $C$ ?

**Question 6.10.** Does  $h^{(c)}(K', x) \geq h^{(c)}(K, x)$  hold for every cubical subdivision  $K'$  of a Buchsbaum cubical complex  $K$ ?

Example 6.4 and Theorem 6.6 make it clear that Question 6.10 has an affirmative answer for any cubical subdivision  $K'$  of any cubical complex  $K$  of dimension at most two. A partial result related to Question 6.8 is the fact (which follows from Theorem 4.3 and Proposition 6.3) that  $(x+1) L_C(\Gamma, x) \geq 0$  holds for every locally quasi-geometric cubical subdivision  $\Gamma$  of a cube  $C$ .

7. GENERALIZATIONS

This section explains how the theory of short cubical local  $h$ -vectors can be extended to more general types of subdivisions, in the spirit of [17, Part II], namely to subdivisions of locally Eulerian posets. Basic examples are provided by CW-regular subdivisions of a family of complexes which includes all regular CW-complexes. The motivating special case is that of a cubical regular CW-complex which subdivides a cube and does not necessarily have the intersection property. As a byproduct of this generalization, it is shown that the theory of [17, Part I] applies to other types of  $h$ -vectors of simplicial complexes, such as the short simplicial  $h$ -vector of [8] and certain natural generalizations.

Throughout this section, we will assume familiarity with Sections 6 and 7 of [17]. The proofs of the results in this section are straightforward generalizations of proofs of corresponding results in [17] and thus, most of them will be omitted.

**7.1. Formal subdivisions.** We will call a poset  $P$  *lower graded* if the principal order ideal  $P_{\leq t} = \{s \in P : s \leq_P t\}$  is finite and graded for every  $t \in P$ . Contrary to the convention of [17], such a poset need not have a minimum element. We denote by  $\rho(t)$  the rank (common length of all maximal chains) of  $P_{\leq t}$ . Our primary example of a lower graded poset will be the poset of nonempty faces of a (cubical) regular CW-complex.

As in [17, Section 6], we will work with the space  $\mathbb{K}[x]^P$  of functions  $f : P \rightarrow \mathbb{K}[x]$ , where  $\mathbb{K}$  is a fixed field, and with the incidence algebra  $I(P)$  of functions  $g : \text{Int}(P) \rightarrow \mathbb{K}[x]$ , where  $\text{Int}(P)$  stands for the set of all (nonempty) closed intervals of  $P$ . We will write  $f_t(x)$  and  $g_{st}(x)$  for the value of  $f \in \mathbb{K}[x]^P$  and  $g \in I(P)$ , respectively, on an element  $t \in P$  and interval  $[s, t] \in \text{Int}(P)$ . The (convolution) product on  $I(P)$  is defined whenever  $P$  is locally finite (meaning that each element of  $\text{Int}(P)$  is finite). For such  $P$ , a function  $\kappa \in I(P)$  is called *unitary* if  $\kappa_{tt} = 1$  for every  $t \in P$ .

Assuming that  $P$  is lower graded, we write, as in [17],

$$\overline{f}_t(x) = x^{\rho(t)} f_t(1/x), \quad t \in P$$

for every  $f \in \mathbb{K}[x]^P$  for which  $\deg f_t(x) \leq \rho(t)$  holds for all  $t \in P$ . Similarly, assuming that  $P$  is locally graded (meaning that each element of  $\text{Int}(P)$  is finite and graded), we write  $\overline{g}_{st}(x) = x^{\rho(s,t)} g_{st}(1/x)$  for every  $g \in I(P)$  for which  $\deg g_{st}(x) \leq \rho(s, t)$  holds for all  $[s, t] \in \text{Int}(P)$ , where  $\rho(s, t)$  denotes the rank of the interval  $[s, t]$ . The following definition is [17, Definition 6.2], where the assumption that  $P$  has a minimum element has been removed from part (a).

**Definition 7.1.** Let  $P$  be a locally graded poset and  $\kappa \in I(P)$  be unitary.

- (a) Assume that  $P$  is lower graded. A function  $f : P \rightarrow \mathbb{K}[x]$  is called  $\kappa$ -*acceptable* if  $f\kappa = \overline{f}$ , i.e., if

$$(7.1) \quad \sum_{s \leq_P t} f_s(x) \kappa_{st}(x) = \overline{f}_t(x)$$

for every  $t \in P$ .

(b) A function  $g \in I(P)$  is called  $\kappa$ -totally acceptable if  $g\kappa = \bar{g}$ , i.e., if

$$(7.2) \quad \sum_{s \leq_P t \leq_P u} g_{st}(x) \kappa_{tu}(x) = \bar{g}_{su}(x)$$

for all  $s \leq_P t$ .

Given a locally finite poset  $P$ , we recall (see [17, Theorem 6.5]) that a unitary function  $\kappa \in I(P)$  is a  $P$ -kernel if and only if  $\bar{\kappa} = \kappa^{-1}$ . Part (a) of the following statement generalizes part (a) of [17, Corollary 6.7].

**Proposition 7.2.** *Let  $P$  be a locally graded poset with  $P$ -kernel  $\kappa$ .*

- (a) *Assume that  $P$  is lower graded. Then there exists a unique  $\kappa$ -acceptable function  $\gamma = \gamma(P, \kappa) : P \rightarrow \mathbb{K}[x]$  satisfying: (i)  $\gamma_a(x) = 1$  for every minimal element  $a \in P$ , and (ii)  $\deg \gamma_t(x) \leq \lfloor (\rho(t) - 1)/2 \rfloor$  for every  $t \in P$  with  $\rho(t) \geq 1$ .*
- (b) *([17, Corollary 6.7 (b)]) There exists a unique  $\kappa$ -totally acceptable function  $\xi = \xi(P, \kappa) \in I(P)$  satisfying: (i)  $\xi_{tt}(x) = 1$  for every  $t \in P$ , and (ii)  $\deg \xi_{st}(x) \leq \lfloor (\rho(s, t) - 1)/2 \rfloor$  for all  $s <_P t$ .*
- (c) *Assume that  $P$  is lower graded. Then the functions  $\gamma = \gamma(P, \kappa)$  and  $\xi = \xi(P, \kappa)$  are related by the equation*

$$(7.3) \quad \gamma_t(x) = \sum_{a \in \min(P_{\leq t})} \xi_{at}(x)$$

for  $t \in P$ , where  $\min(P_{\leq t})$  stands for the set of minimal elements of  $P_{\leq t}$ .

*Proof.* The proof of part (a) of [17, Corollary 6.7] (as well as those of Lemma 6.4 and Proposition 6.6 in [17], on which this proof is based) does not use the assumption that  $P$  has a minimum element. Thus part (a) of the proposition holds as well. Part (b) is identical to part (b) of [17, Corollary 6.7]. For part (c), let  $\delta_t(x)$  denote the right-hand side of (7.3). Clearly, we have  $\delta_a(x) = 1$  for every minimal element  $a \in P$  and  $\deg \delta_t(x) \leq \lfloor (\rho(t) - 1)/2 \rfloor$  for every  $t \in P$  with  $\rho(t) \geq 1$ . Moreover, for  $t \in P$  we have

$$\begin{aligned} \sum_{s \leq_P t} \delta_s(x) \kappa_{st}(x) &= \sum_{s \leq_P t} \sum_{a \in \min(P_{\leq s})} \xi_{as}(x) \kappa_{st}(x) = \sum_{a \in \min(P_{\leq t})} \sum_{a \leq_P s \leq_P t} \xi_{as}(x) \kappa_{st}(x) \\ &= \sum_{a \in \min(P_{\leq t})} \bar{\xi}_{at}(x) = \bar{\delta}_t(x). \end{aligned}$$

Thus  $\delta : P \rightarrow \mathbb{K}[x]$  is  $\kappa$ -acceptable and the result follows from the uniqueness statement of part (a).  $\square$

For the remainder of this section we consider a lower graded, locally Eulerian poset  $P$  (so that the Möbius function of  $P$  satisfies  $\mu_P(s, t) = (-1)^{\rho(s, t)}$  for  $s \leq_P t$ ) and fix the  $P$ -kernel  $\lambda \in I(P)$  with  $\lambda(s, t) = (x - 1)^{\rho(s, t)}$  for  $s \leq_P t$  (see [17, Proposition 7.1]).

**Example 7.3.** Suppose that every closed interval in  $P$  is isomorphic to a Boolean lattice. Then we have  $\xi_{st}(x) = 1$  for all  $s \leq_P t$  [16, Proposition 2.1] [15, Example 3.14.8] and hence  $\gamma_t(x)$  is equal to the number of minimal elements of  $P_{\leq t}$  by (7.3). Thus, if  $P_{\leq u}$  is isomorphic

to the poset of nonempty faces of a simplex for some  $u \in P$ , then  $\gamma_t(x) = \rho(t) + 1$  for  $t \leq_P u$ . Similarly, if  $P_{\leq u}$  is isomorphic to the poset of nonempty faces of a cube, then  $\gamma_t(x) = 2^{\rho(t)}$  for  $t \leq_P u$ .  $\square$

We will refer to the maximum length of a chain in a finite poset  $P$  as the *length* of  $P$ . The following definition generalizes that of the  $h$ -vector of a finite lower Eulerian poset in [17, Example 7.2].

**Definition 7.4.** Let  $P$  be a finite, lower graded and locally Eulerian poset of length  $d$  and let  $\gamma = \gamma(P, \lambda) : P \rightarrow \mathbb{K}[x]$ . The polynomial  $h(P, x)$  defined by

$$(7.4) \quad \bar{h}(P, x) = x^d h(P, 1/x) = \sum_{t \in P} \gamma_t(x) (x-1)^{d-\rho(t)}$$

is called the  $h$ -polynomial of  $P$ . The  $h$ -vector of  $P$  is the sequence  $(h_0, h_1, \dots, h_d)$ , where  $h(P, x) = h_0 + h_1x + \dots + h_dx^d$ .

Since  $\gamma$  is  $\lambda$ -acceptable, we have  $h(P, x) = \gamma_{\hat{1}}(x)$  when  $P$  has a maximum element  $\hat{1}$ . The following example lists several known notions of  $h$ -vectors of complexes and posets which are captured by Definition 7.4.

**Example 7.5.** (a) Suppose that  $P$  is finite of length  $d$  and has a minimum element. Then  $\gamma$  is as in [17, Section 7] and  $h(P, x)$  coincides with the generalized  $h$ -polynomial of  $P$ , introduced in [16, Section 4] (see also [17, Example 7.2]). If, in addition, every principal order ideal of  $P$  is isomorphic to a Boolean lattice (so that  $P$  is a simplicial poset, in the sense of [15, p. 135] [19, p. 112]), then  $\gamma_t(x) = 1$  for every  $t \in P$  and hence (see [16, Corollary 2.2])

$$(7.5) \quad h(P, x) = x^d \sum_{t \in P} \gamma_t(1/x) \left(\frac{1}{x} - 1\right)^{d-\rho(t)} = \sum_{t \in P} x^{\rho(t)} (1-x)^{d-\rho(t)}$$

is the  $h$ -polynomial of the simplicial poset  $P$ . We refer the reader to [19, Section III.6] for further information on  $h$ -polynomials (equivalently, on  $h$ -vectors) of simplicial posets.

(b) Suppose that  $P$  is finite of length  $d-1$  and that every principal order ideal of  $P$  is isomorphic to the poset of nonempty faces of a simplex. In view of Example 7.3, we have

$$h(P, x) = x^{d-1} \sum_{t \in P} \gamma_t(1/x) \left(\frac{1}{x} - 1\right)^{d-\rho(t)-1} = \sum_{t \in P} (\rho(t) + 1) x^{\rho(t)} (1-x)^{d-\rho(t)-1}.$$

Thus, if  $P$  is the poset of nonempty faces of a pure simplicial complex  $\Delta$  of dimension  $d-1$ , then

$$\begin{aligned} h(P, x) &= \sum_{F \in \Delta \setminus \{\emptyset\}} |F| \cdot x^{|F|-1} (1-x)^{d-|F|} = \sum_{v \in \text{vert}(\Delta)} \sum_{E \in \text{lk}_{\Delta}(v)} x^{|E|} \cdot (1-x)^{d-|E|-1} \\ &= \sum_{v \in \text{vert}(\Delta)} h(\text{lk}_{\Delta}(v), x) \end{aligned}$$

is the short simplicial  $h$ -polynomial of  $\Delta$ , introduced by Hersh and Novik [8]. More generally, if  $m \in \mathbb{N}$  is fixed and  $P$  is the poset of faces of  $\Delta$  of dimension  $m$  or higher, then  $h(P, x)$  is equal to the sum of the  $h$ -polynomials of the links of all  $m$ -dimensional faces of  $\Delta$ , introduced in [20, Definition 4.9] as a generalization of the short simplicial  $h$ -polynomial of  $\Delta$ .

(c) Suppose that  $P$  is finite of length  $d$  and that every principal order ideal of  $P$  is isomorphic to the poset of nonempty faces of a cube. Using Example 7.3, we find similarly that

$$h(P, x) = \sum_{t \in P} (2x)^{\rho(t)} (1-x)^{d-\rho(t)}.$$

Thus we have  $h(P, x) = h^{(sc)}(\mathbb{K}, x)$  when  $P = \mathcal{F}(\mathbb{K})$  for some cubical complex  $\mathbb{K}$ . More generally, if  $m \in \mathbb{N}$  is fixed and  $P$  is the poset of faces of a pure cubical complex  $\mathbb{K}$  of dimension  $m$  or higher, then  $h(P, x)$  is equal to the sum of the  $h$ -polynomials of the links of all  $m$ -dimensional faces of  $\mathbb{K}$ .  $\square$

We now recall the following key definition from [17], extended to lower graded, locally Eulerian posets.

**Definition 7.6.** ([17, Definition 7.4]) Let  $P$  be a lower graded, locally Eulerian poset. A *formal subdivision* of  $P$  consists of a lower graded, locally Eulerian poset  $Q$  and a surjective map  $\sigma : Q \rightarrow P$  which have the following properties:

- (i)  $\rho(t) \leq \rho(\sigma(t))$  for every  $t \in Q$ .
- (ii) For every  $u \in P$ , the inverse image  $Q_u := \sigma^{-1}(P_{\leq u})$  is an order ideal of  $Q$  of length  $\rho(u)$ , called the restriction of  $Q$  to  $u$ .
- (iii) For every  $u \in P$  we have

$$(7.6) \quad x^{\rho(u)} h(Q_u, 1/x) = h(\text{int}(Q_u), x),$$

where  $\text{int}(Q_u) := \sigma^{-1}(u)$  is the interior of  $Q_u$ , the polynomial  $h(Q_u, x)$  is defined by (7.4); i.e.,

$$x^{\rho(u)} h(Q_u, 1/x) = \sum_{t \in Q_u} \gamma_t(x) (x-1)^{\rho(u)-\rho(t)}$$

and  $h(\text{int}(Q_u), x)$  is defined by

$$x^{\rho(u)} h(\text{int}(Q_u), 1/x) = \sum_{t \in \text{int}(Q_u)} \gamma_t(x) (x-1)^{\rho(u)-\rho(t)},$$

where  $\gamma = \gamma(Q, \lambda) : Q \rightarrow \mathbb{K}[x]$ .

We now summarize the main properties of  $h$ -vectors and local  $h$ -vectors of formal subdivisions. The proofs are straightforward generalizations of the proofs of Theorem 7.5, Corollary 7.7 and Theorem 7.8 in [17] and will thus be omitted.

**Theorem 7.7.** *Let  $\sigma : Q \rightarrow P$  be a formal subdivision of a lower graded, locally Eulerian poset  $P$ .*

- (a) *The function  $f : P \rightarrow \mathbb{K}[x]$  defined by  $f_u(x) = h(Q_u, x)$  is  $\lambda$ -acceptable.*

(b) Assume that  $P$  has a maximum element  $\hat{1}$  and define

$$(7.7) \quad \ell_P(Q, x) = \sum_{u \in P} h(Q_u, x) \xi_{u\hat{1}}^{-1}(x),$$

where  $\xi = \xi(P, \lambda) \in I(P)$ . Then

$$(7.8) \quad x^d \ell_P(Q, 1/x) = \ell_P(Q, x),$$

where  $d$  is the rank of  $P$ .

(c) Assume that  $P$  is graded of rank  $d$ . Then

$$(7.9) \quad h(Q, x) = \sum_{u \in P} \ell_{P_{\leq u}}(Q_u, x) h(P_{\geq u}, x),$$

where  $h$  is defined by (7.4).

Parts (b) and (c) of Theorem 7.7 generalize Theorems 4.2 and 4.1, respectively. Indeed, let  $C$  be a  $d$ -dimensional cube and let  $P = \mathcal{F}(C)$ . For each  $u \in P$ , the interval  $[u, C]$  in  $P$  is a Boolean lattice and hence  $\xi_{st}(x) = 1$  for all  $s \leq_P t$ . Thus, in view of Example 7.5 (c), if  $Q$  is a cubical subdivision  $\Gamma$  of  $C$ , then the right-hand side of (7.7) is equal to  $\ell_C(\Gamma, x)$  and (7.8) reduces to (4.2). For similar reasons, (7.9) reduces to (4.1) in the case of cubical subdivisions of cubical complexes.

**Remark 7.8.** For a cubical subdivision  $\Gamma$  of a cube  $C$ , part (a) of Theorem 7.7 asserts that

$$\sum_{F \leq_P G} h^{(sc)}(\Gamma_F, x) (x-1)^{\dim(G)-\dim(F)} = x^{\dim(G)} h^{(sc)}(\Gamma_G, 1/x)$$

holds for every  $G \in P = \mathcal{F}(C)$  (this indeed follows from (4.3) and Corollary 2.2). As the subdivision of the line segment with one interior vertex shows already, the corresponding property fails for the long cubical  $h$ -vector.  $\square$

**Example 7.9.** (a) Let  $\Delta'$  be a simplicial subdivision of a simplicial complex  $\Delta$ , in the sense of [17, Part I], with associated subdivision map  $\sigma : \Delta' \rightarrow \Delta$ . The map  $\sigma$  induces a surjective map  $\sigma : Q \rightarrow P$ , where  $P$  and  $Q$  are the posets of nonempty faces of  $\Delta$  and  $\Delta'$ , respectively. Endowed with this map, the poset  $Q$  is a formal subdivision of  $P$ . Indeed, conditions (i) and (ii) of Definition 7.6 are clear. By Example 7.5 (b), the polynomial  $h(Q_F, x)$  is equal to the short simplicial  $h$ -polynomial of the restriction  $\Delta'_F$  of  $\Delta'$  to  $F \in \Delta \setminus \{\emptyset\}$ . Thus, condition (iii) can be verified by an argument analogous to the one in the proof of Corollary 2.2. We note that if  $\Delta$  (and hence  $\Delta'$ ) is pure, then  $h(Q, x)$  is equal to the short simplicial  $h$ -polynomial of  $\Delta'$ , discussed in Example 7.5 (b).

More generally, if  $m \in \mathbb{N}$  is fixed and  $P$  and  $Q$  are the posets of faces of  $\Delta$  and  $\Delta'$ , respectively, of dimension  $m$  or higher, then there is a formal subdivision  $\sigma : Q \rightarrow P$  of  $P$  in which  $h(Q_F, x)$  is equal to the sum of the  $h$ -polynomials of the links of all  $m$ -dimensional faces of  $\Delta'_F$ , for every  $F \in P$  (condition (iii) of Definition 7.6 can be verified again as in the proof of Corollary 2.2). Moreover, if  $\Delta$  is pure, then  $h(Q, x)$  is equal to the sum of the  $h$ -polynomials of the links of all  $m$ -dimensional faces of  $\Delta'$ .

(b) Similar remarks hold for the induced map  $\sigma : Q \rightarrow P$ , when  $P$  and  $Q$  are the posets of faces of dimension  $m$  or higher of two cubical complexes  $K$  and  $K'$ , respectively, such that  $K'$  is a cubical subdivision of  $K$ .  $\square$

**7.2. CW-regular subdivisions.** We now consider a class of complexes which generalize regular CW-complexes. A triangulable space for us will be any topological space which is homeomorphic to the geometric realization of a finite simplicial complex. Let  $C$  be a finite collection of subspaces of a Hausdorff space  $X$ , called *faces*, such that each element of  $C$  is homeomorphic either to the zero-dimensional ball or to a triangulable, connected manifold with nonempty boundary. Assume that the following conditions hold:

- $\emptyset \in C$ ,
- the interiors of the nonempty faces of  $C$  partition  $X$  and
- the boundary of any face of  $C$  is a union of faces of  $C$ .

For lack of better terminology, we will refer to such a complex  $C$  as a *regular  $M$ -complex*. Thus,  $C$  will be a regular CW-complex if every face is homeomorphic to a ball. The intersection of any two faces of  $C$  is a union (possibly empty) of faces of  $C$ . We say that  $C$  has the *intersection property* if the intersection of any two faces of  $C$  is also a face of  $C$ . We denote by  $\mathcal{F}(C)$  the set of nonempty faces of  $C$ , partially ordered by inclusion. This poset is lower graded but not necessarily locally Eulerian. Moreover, the homeomorphism type of  $X$  is not necessarily determined by  $\mathcal{F}(C)$ . As was the case with cubical complexes, any topological properties of a regular CW-complex  $C$  we consider will refer to those of the order complex of  $\mathcal{F}(C)$ .

Let  $C$  be a regular  $M$ -complex and set  $P = \mathcal{F}(C)$ . A (topological) *CW-regular subdivision* of  $C$  is a regular CW-complex  $\Gamma$ , together with a (necessarily surjective) map  $\sigma : \Gamma \setminus \{\emptyset\} \rightarrow P$ , satisfying conditions (i) and (ii) of Definition 7.6, as well as the following, instead of condition (iii):

- (iii') For every  $u \in P$ , the subcomplex  $\Gamma_u := \sigma^{-1}(P_{\leq u}) \cup \{\emptyset\}$  of  $\Gamma$  is homeomorphic to the manifold  $u$  and  $\text{int}(\Gamma_u) := \sigma^{-1}(u)$  is equal to the set of interior faces of this manifold.

This specializes to the notion of CW-regular subdivision of [17, p. 839] when  $C$  is a regular CW-complex. The poset  $Q = \mathcal{F}(\Gamma)$  of nonempty faces of  $\Gamma$  is lower graded and locally Eulerian and we have a surjective map  $\sigma : Q \rightarrow P$  which satisfies conditions (i) and (ii) of Definition 7.6. The following statement generalizes [17, Proposition 7.6].

**Proposition 7.10.** *Let  $C$  be a regular  $M$ -complex and assume that  $P = \mathcal{F}(C)$  is locally Eulerian. Then every CW-regular subdivision of  $C$  is a formal subdivision.*

*Proof.* As in the special case of [17, Proposition 7.6], equation (7.6) can be verified by the computation in the proof of [16, Lemma 6.2]. To facilitate the reader, we give the details as follows. We recall that  $Q_u$  is the poset of nonempty faces of  $\Gamma_u$  and that  $\text{int}(Q_u) =$

$\text{int}(\Gamma_u) = \sigma^{-1}(u)$  for  $u \in P$  and set  $\gamma = \gamma(Q, \lambda) : Q \rightarrow \mathbb{K}[x]$ . For  $u \in P$ , we have

$$\begin{aligned} h(\text{int}(Q_u), x) &= x^{\rho(u)} \sum_{t \in \text{int}(\Gamma_u)} \gamma_t(1/x) \left(\frac{1}{x} - 1\right)^{\rho(u) - \rho(t)} \\ &= \sum_{t \in \text{int}(\Gamma_u)} x^{\rho(t)} \gamma_t(1/x) (1-x)^{\rho(u) - \rho(t)} \\ &= \sum_{t \in \text{int}(\Gamma_u)} (1-x)^{\rho(u) - \rho(t)} \sum_{s \leq_Q t} \gamma_s(x) (x-1)^{\rho(t) - \rho(s)} \\ &= \sum_{s \in \Gamma_u \setminus \{\emptyset\}} \gamma_s(x) (x-1)^{\rho(u) - \rho(s)} \sum_{t \in \text{int}(\Gamma_u) : t \geq_Q s} (-1)^{\rho(u) - \rho(t)}. \end{aligned}$$

Denoting the inner sum by  $\nu(s)$  and noting that  $Q$  is locally Eulerian, we find that

$$\begin{aligned} \nu(s) &= \sum_{t \in \Gamma_u : t \geq_Q s} (-1)^{\rho(u) - \rho(t)} - \sum_{t \in \partial\Gamma_u : t \geq_Q s} (-1)^{\rho(u) - \rho(t)} \\ &= (-1)^{\rho(u) - \rho(s)} \left( -\mu_{\hat{Q}_u}(s, \hat{1}) + \mu_{\partial\hat{Q}_u}(s, \hat{1}) \right), \end{aligned}$$

where  $\partial Q_u$  is the poset of nonempty faces of  $\partial\Gamma_u$  and  $\hat{\Omega}$  denotes a poset  $\Omega$  with a maximum element  $\hat{1}$  adjoined. Since  $\Gamma_u$  is a regular CW-decomposition of a manifold with nonempty boundary and  $s$  is a nonempty face of  $\Gamma_u$ , the various Möbius functions can be computed from [15, Proposition 3.8.9] and we find that  $\nu(s) = 1$  in both cases  $s \in \text{int}(\Gamma_u)$  and  $s \in \partial\Gamma_u$ . Hence

$$h(\text{int}(Q_u), x) = \sum_{s \in Q_u} \gamma_s(x) (x-1)^{\rho(u) - \rho(s)} = x^{\rho(u)} h(Q_u, 1/x)$$

and the proof follows.  $\square$

**Remark 7.11.** A regular CW-complex  $\Gamma$  will be called *cubical* if each face of  $\Gamma$  is combinatorially equivalent to a cube (such a complex may not have the intersection property). Example 7.5 implies that the  $h$ -polynomial of such a complex is equal to its short cubical  $h$ -polynomial, as defined by the right-hand side of (2.4). Thus, we may deduce from Proposition 7.10 and Theorem 7.7 (c) that Theorem 4.1 continues to hold if  $K$  is replaced by a pure cubical regular CW-complex  $C$  and  $K'$  is replaced by a cubical CW-regular subdivision of  $C$ . We should point out that the links of nonempty faces of  $C$  are no longer necessarily simplicial complexes. However, their face posets are simplicial, in the sense of [15, p. 135] [19, p. 112]. Thus the  $h$ -vectors of these links, which appear in (4.1), should be defined by (7.5). Similarly, Theorem 4.2 continues to hold if  $\Gamma$  is replaced by a cubical CW-regular subdivision of a cube  $C$ .  $\square$

**Example 7.12.** Let  $C$  be a regular CW-complex and set  $P = \mathcal{F}(C)$ . Let  $Q$  denote the set of nonempty closed intervals in  $P$ , ordered by inclusion, and consider the map  $\sigma : Q \rightarrow P$  defined by  $\sigma([s, t]) = t$  for all  $s \leq_P t$ . One can easily check that the poset  $Q$  is lower graded

and locally Eulerian (since so is  $P$ ) and that  $\sigma$  is surjective and satisfies conditions (i) and (ii) of Definition 7.6.

Under further assumptions (for instance, if the complex  $C$  is Boolean or cubical), we have  $Q = \mathcal{F}(\Gamma)$  for some (cubical) regular CW-complex  $\Gamma$  and the resulting map  $\sigma : \Gamma \setminus \{\emptyset\} \rightarrow P$  is a CW-regular subdivision of  $C$ ; see [2, Section 2.3]. Let us compute  $\ell_P(Q, x)$  in one interesting special case, namely that in which  $C$  is the simplex  $2^V$  on a  $d$ -element vertex set  $V$ . Then  $\Gamma$  is a cubical complex with  $d$  maximal faces, having a common vertex, and  $\sigma$  is the *cubical barycentric subdivision*, studied in [10], of  $2^V$ . From (7.7) we have

$$(7.10) \quad \ell_P(Q, x) = \sum_{\emptyset \subset F \subseteq V} (-1)^{d-|F|} h^{(sc)}(\Gamma_F, x)$$

where, setting  $k = |F|$ ,

$$h^{(sc)}(\Gamma_F, x) = \sum_{i=0}^{k-1} f_i(\Gamma_F) (2x)^i (1-x)^{k-i-1}.$$

Recall that  $f_i(\Gamma_F)$  counts the number of pairs  $s \subseteq t$  of nonempty subsets of  $F$  such that  $|t \setminus s| = i$ . There are  $\binom{k}{i}$  ways to choose the elements of  $t \setminus s$  and, given any such choice, there are  $2^{k-i} - 1$  ways to choose those of  $s$ . We conclude that  $f_i(\Gamma_F) = \binom{k}{i} (2^{k-i} - 1)$  and hence that

$$(7.11) \quad h^{(sc)}(\Gamma_F, x) = \frac{2^k - (1+x)^k}{1-x}.$$

Using equations (7.10) and (7.11) and the binomial theorem, we get

$$\begin{aligned} \ell_P(Q, x) &= \sum_{k=1}^d (-1)^{d-k} \binom{d}{k} \frac{2^k - (1+x)^k}{1-x} \\ &= 1 + x + x^2 + \cdots + x^{d-1}. \end{aligned}$$

Using the previous formula and (7.9) and doing some more work, one can express the (short or long) cubical  $h$ -vector of the cubical barycentric subdivision of any simplicial (or Boolean) complex  $\Delta$  in terms of the simplicial  $h$ -vector of  $\Delta$  and thus deduce the main results of [10, Section 4]. We leave the details to the interested reader.  $\square$

**Example 7.13.** Figure 2 shows a graded, locally Eulerian poset of rank 2. This poset is isomorphic to the poset of nonempty faces of a two-dimensional regular M-complex  $C$  whose only two-dimensional face is homeomorphic to an annulus with boundary the union of two disjoint triangles. For the cubical CW-regular subdivision  $\Gamma$  of  $C$ , shown on the right of this figure, we compute that  $h^{(sc)}(\Gamma, x) = 6x + 6$  and  $\ell_P(Q, x) = 6x$ .  $\square$

It is an interesting problem to find conditions under which local  $h$ -vectors of CW-regular subdivisions are nonnegative. The proof of Theorem 4.3 works in the following situation. Let  $C$  be a regular M-complex with an inclusionwise maximum face, so that  $P = \mathcal{F}(C)$  has a maximum element. The notion of a *locally quasi-geometric* CW-regular subdivision of  $C$  can be defined as for cubical subdivisions in Section 4.

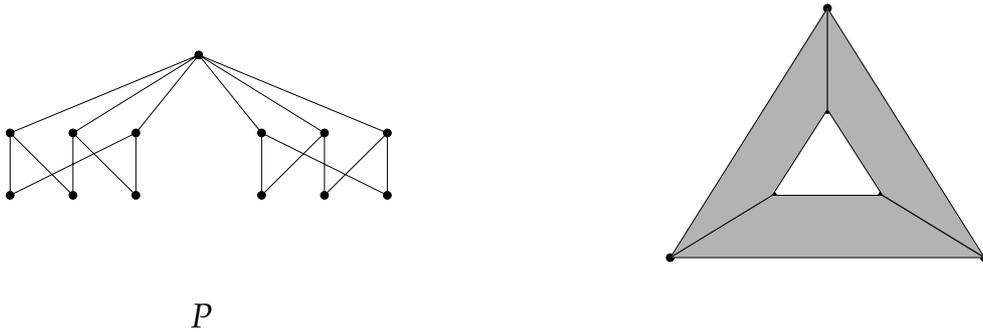


FIGURE 2. Cubical subdivision of an annulus.

**Theorem 7.14.** *Let  $\sigma : \Gamma \setminus \{\emptyset\} \rightarrow P$  be a locally quasi-geometric CW-regular subdivision of  $C$ . If every closed interval in  $Q = \mathcal{F}(\Gamma)$  and  $P = \mathcal{F}(C)$  is isomorphic to a Boolean lattice and  $\Gamma$  has the intersection property, then  $\ell_P(Q, x) \geq 0$ .*

## ACKNOWLEDGMENTS

The author wishes to thank Richard Stanley for helpful comments and the anonymous referees for useful suggestions on the presentation.

## REFERENCES

- [1] R.M. Adin, *A new cubical  $h$ -vector*, Discrete Math. **157** (1996), 3–14.
- [2] E.K. Babson, L.J. Billera and C.S. Chan, *Neighborly cubical spheres and a cubical lower bound conjecture*, Israel J. Math. **102** (1997), 297–315.
- [3] A. Björner, *Posets, regular CW complexes, and Bruhat order*, European J. Combin. **5** (1984), 7–16.
- [4] A. Björner, *Topological methods*, in *Handbook of Combinatorics* (R.L. Graham, M. Grötschel and L. Lovász, eds.), North Holland, Amsterdam, 1995, pp. 1819–1872.
- [5] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G.M. Ziegler, *Oriented Matroids*, Encyclopedia of Mathematics and Its Applications **46**, Cambridge University Press, Cambridge, 1993; second printing, 1999.
- [6] R. Ehrenborg and K. Karu, *Decomposition theorem for the  $cd$ -index of Gorenstein\* posets*, J. Algebraic Combin. **26** (2007), 225–251.
- [7] B. Grünbaum, *Convex Polytopes*, Interscience, London, 1967; second edition, Graduate Texts in Mathematics **221**, Springer, New York, 2003.
- [8] P. Hersh and I. Novik, *A short simplicial  $h$ -vector and the upper bound theorem*, Discrete Comput. Geom. **28** (2002), 283–289.
- [9] G. Hetyei, *On the Stanley ring of a cubical complex*, Discrete Comput. Geom. **14** (1995), 305–330.
- [10] G. Hetyei, *Invariants des complexes cubiques*, Ann. Sci. Math. Québec **20** (1996), 35–52.
- [11] W. Jockusch, *The lower and upper bound problems for cubical polytopes*, Discrete Comput. Geom. **9** (1993), 159–163.
- [12] S. Klee, *Lower bounds for cubical pseudomanifolds*, Discrete Comput. Geom. (to appear).
- [13] P. McMullen and D.W. Walkup, *A generalized lower-bound conjecture for simplicial polytopes*, Mathematika **18** (1971), 264–273.
- [14] C. Savvidou, *Face numbers of cubical barycentric subdivisions*, preprint, 2010, arXiv:1005.4156.

- [15] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986; second printing, *Cambridge Studies in Advanced Mathematics* **49**, Cambridge University Press, Cambridge, 1997.
- [16] R.P. Stanley, *Generalized  $h$ -vectors, intersection cohomology of toric varieties and related results*, in *Commutative Algebra and Combinatorics* (N. Nagata and H. Matsumura, eds.), *Adv. Stud. Pure Math.* **11**, Kinokuniya, Tokyo and North-Holland, Amsterdam-New York, 1987, pp. 187–213.
- [17] R.P. Stanley, *Subdivisions and local  $h$ -vectors*, *J. Amer. Math. Soc.* **5** (1992), 805–851.
- [18] R.P. Stanley, *A monotonicity property of  $h$ -vectors and  $h^*$ -vectors*, *European J. Combin.* **14** (1993), 251–258.
- [19] R.P. Stanley, *Combinatorics and Commutative Algebra*, second edition, Birkhäuser, Basel, 1996.
- [20] E. Swartz,  *$g$ -elements, finite buildings and higher Cohen-Macaulay connectivity*, *J. Combin. Theory Series A* **113** (2006), 1305–1320.
- [21] G.M. Ziegler, *Lectures on Polytopes*, *Graduate Texts in Mathematics* **152**, Springer-Verlag, New York, 1995.

DEPARTMENT OF MATHEMATICS (DIVISION OF ALGEBRA-GEOMETRY), UNIVERSITY OF ATHENS,  
PANEPISTIMIOUPOLIS, ATHENS 15784, HELLAS (GREECE)

*E-mail address:* caath@math.uoa.gr