ON THE FLAG f-VECTOR OF A GRADED LATTICE WITH NONTRIVIAL HOMOLOGY

CHRISTOS A. ATHANASIADIS

Abstract. It is proved that the Boolean algebra of rank n minimizes the flag f-vector among all graded lattices of rank n, whose proper part has nontrivial top-dimensional homology. The analogous statement for the flag h-vector is conjectured in the Cohen-Macaulay case.

1. Introduction

Let P be a finite graded poset of rank $n \geq 1$, having a minimum element $\hat{0}$, maximum element $\hat{1}$ and rank function $\rho : P \to \mathbb{N}$ (we refer to [12, Chapter 3] for any undefined terminology on partially ordered sets). Given $S \subseteq [n-1] := \{1, 2, \ldots, n-1\}$, the number of chains $\mathcal{C} \subseteq P \smallsetminus \{\hat{0}, \hat{1}\}$ such that $\{\rho(x) : x \in \mathcal{C}\} = S$ will be denoted by $f_P(S)$. For instance, $f_P(S)$ is equal to the number of elements of P of rank k, if $S = \{k\} \subseteq [n-1]$, and to the number of maximal chains of P, if $S = [n-1]$. The function which maps S to $f_P(S)$ for every $S \subseteq [n-1]$ is an important enumerative invariant of P, known as the *flag f-vector*; see, for instance, [4].

The present note is partly motivated by the results of [2, 6]. There it is proven that the Boolean algebra of rank n minimizes the **cd**-index, an invariant which refines the flag f-vector, among all face lattices of convex polytopes and, more generally, Gorenstein* lattices, of rank n. It is natural to consider lattices which are not necessarily Eulerian, in this context. To state our main result, we fix some more notation as follows. We denote by $\Delta(Q)$ the simplicial complex consisting of all chains in a finite poset Q, known as the *order complex* [5] of Q, and by $\widetilde{H}_*(\Delta; \mathbf{k})$ the reduced simplicial homology over $\mathbf{k}$ of an abstract simplicial complex Δ, where $\mathbf{k}$ is a fixed field or $\mathbb{Z}$. We denote by B_n the Boolean algebra of rank n (meaning, the lattice of subsets of the set $[n]$, partially ordered by inclusion) and recall that if $S = \{s_1 < s_2 < \cdots < s_l\} \subseteq [n-1]$, then $f_{B_n}(S)$ is equal to the multinomial coefficient $\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, \ldots, n - s_l}$.

Theorem 1.1. *Let L be a finite graded lattice of rank n, with minimum element $\hat{0}$ and maximum element $\hat{1}$, and let $\bar{L} = L \smallsetminus \{\hat{0}, \hat{1}\}$ be the proper part of L. If $\widetilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0$, then*

$$f_L(S) \geq \alpha_n(S) \tag{1.1}$$

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for every $S \subseteq [n-1]$. In other words, the Boolean algebra of rank n minimizes the flag f-vector among all finite graded lattices of rank n whose proper part has nontrivial top-dimensional reduced homology over $\mathbf{k}$.

A similar statement, asserting that the Boolean algebra of rank n has the smallest number of elements among all finite lattices L satisfying $\widetilde{H}_{n-2}(\Delta(\bar{L}); \mathbb{Z}) \neq 0$, was proved by Meshulam [10]. The proof of Theorem 1.1, given in Section 2, is elementary and similar in spirit to (but somewhat more involved than) the proof of the result of [10]. A different (but less elementary) proof may be given using the methods of [6, Section 2]. In the remainder of this section we discuss some consequences of Theorem 1.1 and a related open problem.

The *f-vector* of a simplicial complex Δ is defined as the sequence $f(\Delta) = (f_0, f_1, \dots)$, where f_i is the number of i-dimensional faces of Δ. We recall that the order complex $\Delta(\bar{B}_n)$ is isomorphic to the barycentric subdivision of the $(n-1)$-dimensional simplex. The next statement follows from this observation, Theorem 1.1 and the fact (see, for instance, [13, p. 95]) that each entry of the f-vector of the order complex $\Delta(\bar{L})$ can be expressed as a sum of entries of the flag f-vector of L.

Corollary 1.2. *The barycentric subdivision of the $(n-1)$-dimensional simplex has the smallest possible f-vector among all order complexes of the form $\Delta(\bar{L})$, where L is a finite graded lattice of rank n satisfying $\widetilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0$.*

Analogous results for the class of flag simplicial complexes have appeared in [1, 7, 9, 11].

Let P be a graded poset of rank n, as in the beginning of this section. The *flag h-vector* of P is the function assigning to each $S \subseteq [n-1]$ the integer

$$h_P(S) \;=\; \sum_{T \subseteq S} (-1)^{|S \smallsetminus T|} \, f_P(T). \tag{1.2}$$

Equivalently, we have

$$f_P(S) \;=\; \sum_{T \subseteq S} h_P(T) \tag{1.3}$$

for every $S \subseteq [n-1]$. We write $\beta_n(S)$ for the entry $h_{B_n}(S)$ of the flag h-vector of the Boolean algebra of rank n and recall [12, Corollary 3.12.2] that $\beta_n(S)$ is equal to the number of permutations of $[n]$ with descent set S.

It is known that if P is Cohen-Macaulay over $\mathbf{k}$ (see [5, Section 11] or [12, Section 3.8] for the definition), then $h_P(S) \geq 0$ for every $S \subseteq [n-1]$. Moreover, in this case $\Delta(\bar{L})$ has nontrivial top-dimensional reduced homology over $\mathbf{k}$ if and only if $\mu_P(\hat{0}, \hat{1}) \neq 0$, where μ_P is the Möbius function of P. Hence, Theorem 1.1 implies that the Boolean algebra of rank n minimizes the flag f-vector among all Cohen-Macaulay lattices of rank n with nonzero Möbius number. In view of (1.3), the following conjecture provides a natural strengthening of this statement.

Conjecture 1.3. *Let L be a finite lattice of rank n, with minimum element $\hat{0}$ and maximum element $\hat{1}$. If L is Cohen-Macaulay over $\mathbf{k}$ and $\mu_L(\hat{0},\hat{1}) \neq 0$, then*

$$h_L(S) \geq \beta_n(S) \tag{1.4}$$

for every $S \subseteq [n-1]$. In other words, the Boolean algebra of rank n minimizes the flag h-vector among all Cohen-Macaulay lattices of rank n with nonzero Möbius number.

This conjecture was initially stated by the author under the assumption that $\mu_L(x,y) \neq 0$ holds for all $x, y \in L$ with $x \leq_L y$ and took its present form after a question raised by R. Stanley [14], asking whether this condition could be relaxed to $\mu_L(\hat{0},\hat{1}) \neq 0$. It would imply that among all Cohen-Macaulay order complexes of the form $\Delta(\bar{L})$, where L is a lattice of rank n satisfying $\mu_L(\hat{0},\hat{1}) \neq 0$, the barycentric subdivision of the $(n-1)$-dimensional simplex has the smallest possible h-vector (the entries of the h-vector of this subdivision are the Eulerian numbers, counting permutations of the set $[n]$ by the number of descents). Conjecture 1.3 is known to hold for Gorenstein* lattices (in this case it follows from the stronger result [6, Corollary 1.3], mentioned earlier, on the **cd**-index of such a lattice) and for geometric lattices [3, Proposition 7.4].

2. Proof of Theorem 1.1

Throughout this section, L is a lattice as in Theorem 1.1. For $a, b \in L$ with $a \leq_L b$, we denote by $\Delta(a,b)$ (respectively, by $\Delta(a,b]$) the order complex of the open interval (a,b) (respectively, half-open interval $(a,b]$) in L. We say that an element $x \in L$ is *good* if $x = \hat{0}$ or $\widetilde{H}_{k-2}(\Delta(\hat{0},x);\mathbf{k}) \neq 0$, where k is the rank of x in L, and otherwise that x is *bad*.

The proof of Theorem 1.1 will follow from the next proposition.

Proposition 2.1. *Under the assumptions of Theorem 1.1, the lattice L has at least $\binom{n}{k}$ good elements of rank k for every $k \in \{0, 1, \ldots, n\}$.*

Proof. We proceed in several steps.

Step 1: We show that L has at least one good coatom. Suppose, by the way of contradiction, that no such coatom exists. Suppose further that L has the minimum possible number of coatoms among all lattices of rank n which satisfy the assumptions of Theorem 1.1 and have no good coatom. Since $\Delta(\bar{L})$ is non-acyclic over $\mathbf{k}$, the order complex $\Delta(\bar{L})$ cannot be a cone and hence L must have at least two coatoms. Let c be one of them and consider the complexes $\Delta(\bar{L} \smallsetminus \{c\})$ and $\Delta(\hat{0},c]$. The union of these complexes is equal to $\Delta(\bar{L})$ and their intersection is equal to $\Delta(\hat{0},c)$. Since $\Delta(\hat{0},c]$ is a cone, hence contractible, and since $\widetilde{H}_{n-3}(\Delta(\hat{0},c);\mathbf{k}) = 0$ by assumption, it follows from the Mayer-Vietoris long exact sequence in homology for $\Delta(\bar{L} \smallsetminus \{c\})$ and $\Delta(\hat{0},c]$ that

$$\widetilde{H}_{n-2}(\Delta(\bar{L} \smallsetminus \{c\});\mathbf{k}) \cong \widetilde{H}_{n-2}(\Delta(\bar{L});\mathbf{k}) \neq 0. \tag{2.1}$$

Since $L \smallsetminus \{c\}$ may not be graded, we consider the subposet $M = J \cup \{\hat{1}\}$ of L, where J stands for the order ideal of L generated by all coatoms other than c. The poset M is a finite meet-semilattice with a maximum element and hence it is a lattice by [12, Proposition

3.3.1]. Since L is graded of rank n, so is M and the set of $(n-1)$-element chains of $\Delta(\bar{M})$ coincides with that of $\Delta(\bar{L} \smallsetminus \{c\})$, where $\bar{M} = M \smallsetminus \{\hat{0}, \hat{1}\}$ is the proper part of M. The last statement and (2.1) imply that

$$\widetilde{H}_{n-2}(\Delta(\bar{M}); \mathbf{k}) \cong \widetilde{H}_{n-2}(\Delta(\bar{L} \smallsetminus \{c\}); \mathbf{k}) \neq 0.$$

Clearly, all coatoms of M are bad. Since M has one coatom less than L, we have arrived at the desired contradiction.

Step 2: Assume that $n \geq 2$ and let b be any coatom of L. We show that there exists an atom a of L which is not comparable to b and satisfies $\widetilde{H}_{n-3}(\Delta(a, \hat{1}); \mathbf{k}) \neq 0$. Arguing by contradiction, once again, suppose that no such atom exists. Suppose further that the number of atoms of L which do not belong to the interval $[\hat{0}, b]$ is as small as possible for a graded lattice L of rank n and coatom b which have this property and satisfy the assumptions of Theorem 1.1. Since $\Delta(\bar{L})$ is non-acyclic over $\mathbf{k}$, the Crosscut Theorem of Rota [5, Theorem 10.8] implies that there exists at least one atom of L which does not belong to the interval $[\hat{0}, b]$. Let a be any such atom and let M be the subposet of L consisting of $\hat{0}$ and the elements of the dual order ideal of L generated by the atoms other than a. The arguments in Step 1, applied to the dual of L, show that M is a graded lattice of rank n which satisfies $\widetilde{H}_{n-2}(\Delta(\bar{M}); \mathbf{k}) \cong \widetilde{H}_{n-2}(\Delta(\bar{L}); \mathbf{k}) \neq 0$. Since $M \smallsetminus (\hat{0}, b]$ has one atom less than $L \smallsetminus (\hat{0}, b]$, this contradicts our assumptions on L and b.

Step 3: We now show that L has at least n good coatoms by induction on n. The statement is trivial for $n = 1$, so suppose that $n \geq 2$. By replacing $L \smallsetminus \{\hat{1}\}$ with its order ideal generated by the good coatoms, as in Step 1, we may assume that all coatoms of L are good. Let b be any coatom of L. By Step 2, there exists an atom a of L which is not comparable to b and satisfies $\widetilde{H}_{n-3}(\Delta(a, \hat{1}); \mathbf{k}) \neq 0$. The interval $[a, \hat{1}]$ in L is a graded lattice of rank $n-1$ to which the induction hypothesis applies. Therefore, it has at least $n-1$ coatoms and all of these are different from b. It follows that L has at least n coatoms, all of which are good.

Step 4: We prove the following: Given any integers $0 \leq r \leq k \leq n$ and any order ideal I of $L \smallsetminus \{\hat{1}\}$ generated by at most r elements, there exist at least $\binom{n-r}{k-r}$ good elements of L of rank k which do not belong to I. The special case $r = 0$ of this statement, in which I is the empty ideal, is equivalent to the proposition. Thus, it suffices to prove the statement.

We proceed by induction on n and $n - r$, in this order. The statement is trivial for $n = 1$ and for $r = n$, so we assume that $n \geq 2$ and $0 \leq r \leq n-1$. Consider an order ideal I of $L \smallsetminus \{\hat{1}\}$ generated by at most r elements and let k be an integer in the range $r \leq k \leq n$. Since I contains at most $r \leq n-1$ coatoms of L, Step 3 implies that there exists a good coatom, say b, of L which does not belong to I. The interval $[\hat{0}, b]$ of L is a graded lattice of rank $n-1$ whose proper part has nontrivial top-dimensional reduced homology over $\mathbf{k}$. Moreover, the intersection $I \cap [\hat{0}, b]$ is an order ideal of $[\hat{0}, b)$ which is generated by at most r elements, namely the meets of b with the maximal elements of I. By our induction on n, there exist at least $\binom{n-r-1}{k-r}$ good elements of $[\hat{0}, b]$ of rank k which do not belong to I. The union $J = I \cup [\hat{0}, b]$ is an order ideal of $L \smallsetminus \{\hat{1}\}$ which is generated by at most $r+1$ elements. By our induction on $n-r$, there exist at least $\binom{n-r-1}{k-r-1}$ good elements of L of rank k which do not belong to J. We conclude that there exist at least $\binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1} = \binom{n-r}{k-r}$ good elements of L of rank k which do not belong to I. This completes the inductive step and the proof of the statement. □

Proof of Theorem 1.1. We proceed by induction on n. The result is trivial for $n = 1$ and for $S = \varnothing$, so we assume that $n \geq 2$ and choose a nonempty subset S of $[n-1]$. We denote by k the largest element of S and observe that $f_L(S)$ is equal to the number of pairs $(x, \mathcal{C})$, where x is an element of L of rank k and $\mathcal{C}$ is a chain in the interval $[\hat{0}, x]$, such that the set of ranks of the elements of $\mathcal{C}$ is equal to $S \smallsetminus \{k\}$. By Proposition 2.1, there are at least $\binom{n}{k}$ good elements x of rank k in L and each of the intervals $[\hat{0}, x]$ is a graded lattice of rank k whose proper part has nontrivial top-dimensional reduced homology over $\mathbf{k}$. Thus, the induction hypothesis applies to these intervals and we may conclude that

$$f_L(S) \ \geq \ \binom{n}{k} \alpha_k(S \smallsetminus \{k\}) \ = \ \alpha_n(S).$$

This completes the induction and the proof of the theorem. □

Note added in revision. It was shown in [10] that every lattice L which has 2^n elements and satisfies $\widetilde{H}_{n-2}(\Delta(\bar{L}); \mathbb{Z}) \neq 0$ must be isomorphic to the Boolean algebra B_n. As a result, if equality holds in (1.1) for every singleton $S \subseteq [n-1]$, then L is isomorphic to B_n. Using the arguments in this section, as well as induction on n and k, the following statement has been verified by Kolins and Klee [8]: if L satisfies the assumptions of Theorem 1.1 and for some $k \in \{1, 2, \ldots, n-1\}$ equality holds in (1.1) for every subset S of $[n-1]$ of cardinality k, then L is isomorphic to the Boolean algebra of rank n.

References

[1] C.A. Athanasiadis, *Some combinatorial properties of flag simplicial pseudomanifolds and spheres*, Ark. Mat. **49** (2011), 17–29.

[2] L.J. Billera and R. Ehrenborg, *Monotonicity of the cd-index for polytopes*, Math. Z. **233** (2000), 421–441.

[3] L.J. Billera, R. Ehrenborg and M. Readdy, *The* **c-2d**-*index of oriented matroids*, J. Combin. Theory Series A **80** (1997), 79–105.

[4] L.J. Billera and G. Hetyei, *Linear inequalities for flags in graded partially ordered sets*, J. Combin. Theory Series A **89** (2000), 77–104.

[5] A. Björner, *Topological methods*, in *Handbook of Combinatorics* (R.L. Graham, M. Grötschel and L. Lovász, eds.), North Holland, Amsterdam, 1995, pp. 1819–1872.

[6] R. Ehrenborg and K. Karu, *Decomposition theorem for the cd-index of Gorenstein* posets*, J. Algebraic Combin. **26** (2007), 225–251.

[7] M. Goff, S. Klee and I. Novik, *Balanced complexes and complexes without large missing faces*, Ark. Mat. **49** (2011), 335–350.

[8] S.R. Kolins and S. Klee, Personal communication, February 7, 2011.

[9] R. Meshulam, *Domination numbers and homology*, J. Combin. Theory Series A **102** (2003), 321–330.

[10] R. Meshulam, *On the homological dimension of lattices*, Order **25** (2008), 153–155.

[11] E. Nevo, *Remarks on missing faces and generalized lower bounds on face numbers*, Electron. J. Combin. **16** (2009), no. 2, Research Paper 8, 11pp (electronic).

[12] R.P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, Cambridge, 1997.

[13] R.P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhäuser, Basel, 1996.

[14] R.P. Stanley, Personal communication, April 22, 2009.

Department of Mathematics (Division of Algebra-Geometry), University of Athens, Panepistimioupolis, Athens 15784, Hellas (Greece)

E-mail address: caath@math.uoa.gr