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Note

The absolute order on the symmetric group, constructible partially ordered sets and Cohen–Macaulay complexes

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Abstract

The absolute order is a natural partial order on a Coxeter group W. It can be viewed as an analogue of the weak order on W in which the role of the generating set of simple reflections in W is played by the set of all reflections in W. By use of a notion of constructibility for partially ordered sets, it is proved that the absolute order on the symmetric group is homotopy Cohen–Macaulay. This answers in part a question raised by V. Reiner and the first author. The Euler characteristic of the order complex of the proper part of the absolute order on the symmetric group is also computed.

Keywords: Symmetric group; Partial order; Absolute order; Möbius function; Constructible complex; Cohen–Macaulay complex

1. Introduction

Consider a finite Coxeter group W with set of reflections T. Given $w \in W$, let $\ell_T(w)$ denote the smallest integer k such that w can be written as a product of k reflections in T. The *absolute order*, or *reflection length order*, is the partial order on W denoted by $\preccurlyeq$ and defined by letting

$$u \preccurlyeq v \quad \text{if and only if} \quad \ell_T(u) + \ell_T\big(u^{-1}v\big) = \ell_T(v)$$

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for $u, v \in W$. Equivalently, $\preccurlyeq$ is the partial order on W with cover relations $w \prec wt$, where $w \in W$ and $t \in T$ are such that $\ell_T(w) < \ell_T(wt)$. We refer to [2, Section 2.4] for elementary properties of the absolute order and related historical remarks and mention that the pair $(W, \preccurlyeq)$ is a graded poset having the identity $e \in W$ as its unique minimal element and rank function ℓ_T.

The significance of the absolute order in combinatorics, group theory, invariant theory and representation theory stems from the following facts. First, it can be viewed as an analogue of the weak order [7, Chapter 3] on W (this order can be defined by replacing the generating set of all reflections in W, in the definition of the absolute order, with the set of simple reflections). Second, the maximal chains in intervals of the form $[e, w]$ correspond to reduced words of w with respect to the alphabet T and are relevant in the study of conjugacy classes in W [11]. Third, the rank-generating polynomial of $(W, \preccurlyeq)$ is given by

$$\sum_{w\in W} q^{\ell_T(w)} = \prod_{i=1}^{\ell} (1 + e_i q),$$

where $e_1, e_2, \ldots, e_\ell$ are the exponents [13, Section 3.20] of W and ℓ is its rank. Furthermore, if c denotes a Coxeter element of W, then the combinatorial structure of the intervals in $(W, \preccurlyeq)$ of the form $[e, c]$, known as *noncrossing partition lattices*, plays an important role in the construction of new monoid structures and $K(\pi, 1)$ spaces for Artin groups associated with W; see for instance [4,9,10].

When c is a Coxeter element, the intervals $[e, c]$ in the absolute order have pleasant combinatorial and topological properties. In particular, they were shown to be shellable in [3]. The question of determining the topology of $(W \setminus \{e\}, \preccurlyeq)$ was raised by Reiner [15], [1, Problem 3.1] and the first author (unpublished) and was also posed by Wachs [20, Problem 3.3.7]. In this paper we focus on the case of the symmetric group $\mathcal{S}_n$ (the case of other Coxeter groups will be treated in [14]). We will denote by P_n the partially ordered set $(\mathcal{S}_n, \preccurlyeq)$ and by $\bar{P}_n$ its proper part $(\mathcal{S}_n \setminus \{e\}, \preccurlyeq)$. Before we state our main results, let us describe the poset P_n more explicitly. Given a cycle $c = (i_1\ i_2\ \cdots\ i_r) \in \mathcal{S}_n$ and indices $1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant r$, we say that the cycle $(i_{j_1}\ i_{j_2}\ \cdots\ i_{j_s}) \in \mathcal{S}_n$ can be obtained from c by deleting elements. Given two disjoint cycles $a, b \in \mathcal{S}_n$ each of which can be obtained from c by deleting elements, we say that a and b are *noncrossing* with respect to c if there does not exist a cycle $(i\ j\ k\ l)$ of length four which can be obtained from c by deleting elements, such that i, k are elements of a and j, l are elements of b. For instance, if $n = 9$ and $c = (3\,5\,1\,9\,2\,6\,4)$ then the cycles $(3\,6\,4)$ and $(5\,9\,2)$ are noncrossing with respect to c but $(3\,2\,4)$ and $(5\,9\,6)$ are not. It can be checked [9, Section 2] that for $u, v \in \mathcal{S}_n$ we have $u \preccurlyeq v$ if and only if

- every cycle in the cycle decomposition for u can be obtained from some cycle in the cycle decomposition for v by deleting elements and
- any two cycles of u which can be obtained from the same cycle c of v by deleting elements are noncrossing with respect to c.

Fig. 1 depicts the Hasse diagram of P_n for $n = 4$. We note that the rank of an element w of P_n is equal to $n - p$, where p is the number of cycles in the cycle decomposition for w. In particular, P_n has rank $n - 1$ and its maximal elements are the cycles in $\mathcal{S}_n$ of length n.

The main results of this paper are as follows.

Theorem 1.1. *The poset $\bar{P}_n$ is homotopy Cohen–Macaulay for all $n \geqslant 1$. In particular, it is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres and Cohen–Macaulay over $\mathbb{Z}$.*

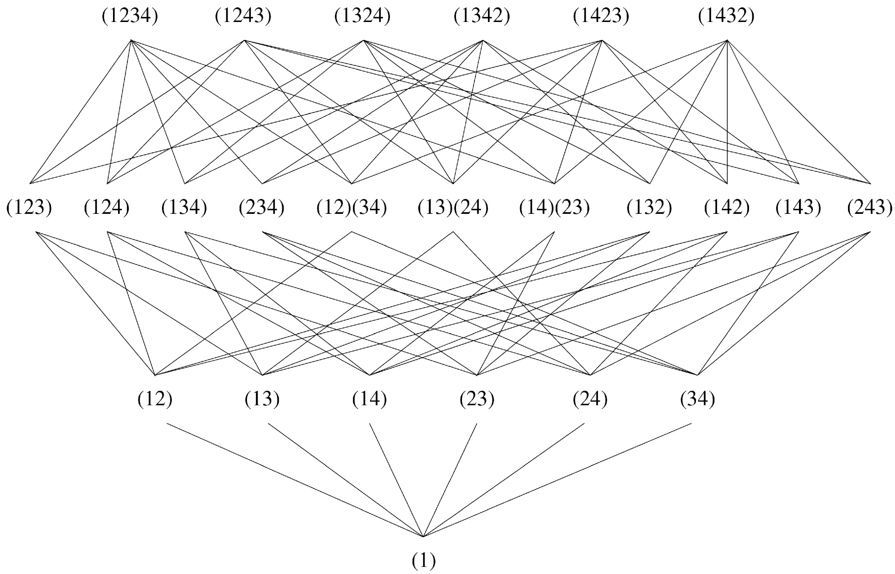

Fig. 1. The absolute order on the symmetric group $\mathcal{S}_4$.

Theorem 1.2. *The reduced Euler characteristic of the order complex $\Delta(\bar{P}_n)$ satisfies*

$$\sum_{n\geqslant 1}(-1)^n\tilde{\chi}\big(\Delta(\bar{P}_n)\big)\frac{t^n}{n!}=1-C(t)\exp\big\{-2tC(t)\big\}, \tag{1}$$

where $C(t)=\frac{1}{2t}(1-\sqrt{1-4t})$ is the ordinary generating function for the Catalan numbers.

Theorems 1.1 and 1.2 are proved in Sections 4 and 5, respectively. Theorem 1.1 is proved by showing that P_n has a property which we call strong constructibility. This notion is motivated by the notion of constructibility for simplicial complexes [12] (see also [16]) and is introduced and studied in Section 3. Section 2 discusses briefly some of the background from topological combinatorics needed to understand Theorems 1.1 and 1.2.

2. Preliminaries

In this section we fix notation, terminology and conventions related to simplicial complexes and partially ordered sets (posets) and recall some fundamental definitions and facts. For more information on these topics we refer the interested reader to [6], [17, Chapter II], [18, Chapter 3] and [20]. Throughout this paper we use the notation $[n]=\{1,2,\dots,n\}$.

All simplicial complexes and posets we will consider in this paper are finite. All topological properties of an abstract simplicial complex Δ we mention will refer to those of its geometric realization X (see [6, Section 9]). For instance, Δ is *k-connected* if the homotopy groups $\pi_i(X,x)$ vanish for all $0\leqslant i\leqslant k$ and $x\in X$. The elements of an abstract simplicial complex Δ are called *faces*. The *link* of a face $F\in\Delta$ is defined as $\mathrm{link}_\Delta(F)=\{G\setminus F\colon\ G\in\Delta,\ F\subseteq G\}$. The complex Δ is said to be *Cohen–Macaulay* (over $\mathbb{Z}$) if

$$\tilde{H}_i\big(\mathrm{link}_\Delta(F),\mathbb{Z}\big)=0$$

for all $F \in \Delta$ and $i < \dim \mathrm{link}_\Delta(F)$ and *homotopy Cohen–Macaulay* if $\mathrm{link}_\Delta(F)$ is $(\dim \mathrm{link}_\Delta(F) - 1)$-connected for all $F \in \Delta$. A d-dimensional simplicial complex Δ is said to be *pure* if all facets (faces which are maximal with respect to inclusion) of Δ have dimension d. A pure d-dimensional simplicial complex Δ is (*pure*) *shellable* if there exists a total ordering $G_1, G_2, \dots, G_m$ of the set of facets of Δ such that for all $1 < i \leqslant m$, the intersection of $G_1 \cup \cdots \cup G_{i-1}$ with G_i is pure of dimension $d - 1$. We have the hierarchy of properties

$$\text{pure shellable} \Rightarrow \text{homotopy Cohen–Macaulay} \Rightarrow \text{Cohen–Macaulay} \Rightarrow \text{pure}$$

for a simplicial complex (in Section 3 we will insert constructibility between the first and second property). Moreover, any d-dimensional (finite) homotopy Cohen–Macaulay simplicial complex is $(d-1)$-connected and hence homotopy equivalent to a wedge of d-dimensional spheres.

The order complex, denoted by $\Delta(P)$, of a poset P is the abstract simplicial complex with vertex set P and faces the chains (totally ordered subsets) of P. All topological properties of a poset P we mention will refer to those of (the geometric realization of) $\Delta(P)$. The *rank* of P is defined as the dimension of $\Delta(P)$, in other words as one less than the largest cardinality of a chain in P. We say that P is *bounded* if it has a minimum and a maximum element, *graded* if $\Delta(P)$ is pure and *pure shellable* if so is $\Delta(P)$. A subset I of P is called an (order) *ideal* if we have $x \in I$ whenever $x \leqslant y$ holds in P and $y \in I$.

3. Constructible complexes and posets

In this section we introduce the notion of strong constructibility for partially ordered sets and discuss some of its features which will be important for us. We will use the following variation of the notion of constructibility for simplicial complexes [12,16], [6, Section 11.2].

Definition 3.1. A d-dimensional simplicial complex Δ is constructible if it is a simplex or it can be written as $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1, Δ_2 are d-dimensional constructible simplicial complexes such that $\Delta_1 \cap \Delta_2$ is constructible of dimension at least $d - 1$.

The classical notion of constructibility differs in that, in the previous definition, the dimension of $\Delta_1 \cap \Delta_2$ has to equal $d - 1$. It is well known that pure shellability implies constructibility (in the classical sense). We do not know whether our notion of constructibility coincides with the (possibly more restrictive) classical notion. Observe, however, that constructible simplicial complexes, in the sense of Definition 3.1, are pure and that they enjoy the properties listed in the following lemma and corollary.

Lemma 3.2.

(i) *If Δ is a d-dimensional constructible simplicial complex then Δ is $(d-1)$-connected.*
(ii) *If Δ is constructible then so is the link of any face of Δ.*

Proof. Part (i) follows from the fact [6, Lemma 10.3(ii)] that if Δ_1, Δ_2 are k-connected and $\Delta_1 \cap \Delta_2$ is $(k-1)$-connected then $\Delta_1 \cup \Delta_2$ is k-connected. Part (ii) follows from the observation that if F is a face of $\Delta_1 \cup \Delta_2$ then $\mathrm{link}_{\Delta_1 \cup \Delta_2}(F) = \mathrm{link}_{\Delta_1}(F) \cup \mathrm{link}_{\Delta_2}(F)$ and $\mathrm{link}_{\Delta_1}(F) \cap \mathrm{link}_{\Delta_2}(F) = \mathrm{link}_{\Delta_1 \cap \Delta_2}(F)$. □

Corollary 3.3. *If Δ is a constructible simplicial complex then Δ is homotopy Cohen–Macaulay.*

Proof. This follows from Lemma 3.2. □

Lemma 3.4. *Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be d-dimensional constructible simplicial complexes.*

(i) *If the intersection of any two or more of $\Delta_1, \Delta_2, \ldots, \Delta_k$ is constructible of dimension d, then their union is also constructible.*
(ii) *If the intersection of any two or more of $\Delta_1, \Delta_2, \ldots, \Delta_k$ is constructible of dimension $d-1$, then their union is also constructible.*

Proof. We proceed by induction on k. The case $k=1$ is trivial and the case $k=2$ is clear by definition, so we assume that $k \geqslant 3$. The complexes $\Delta_1 \cap \Delta_k, \ldots, \Delta_{k-1} \cap \Delta_k$ have dimension d or $d-1$ in the cases of parts (i) and (ii), respectively, and satisfy the hypothesis of part (i). Hence, by our induction hypothesis, their union $(\Delta_1 \cup \cdots \cup \Delta_{k-1}) \cap \Delta_k$ is constructible of dimension d or $d-1$, respectively. Since, by induction, $\Delta_1 \cup \cdots \cup \Delta_{k-1}$ is constructible of dimension d and, by assumption, so is Δ_k, it follows that $\Delta_1 \cup \cdots \cup \Delta_k$ is constructible as well. □

We now consider the class of finite posets with a minimum element and define the notion of strong constructibility as follows.

Definition 3.5. A finite poset P of rank d with a minimum element is strongly constructible if it is bounded and pure shellable or it can be written as a union $P = I_1 \cup I_2$ of two strongly constructible proper ideals I_1, I_2 of rank d, such that $I_1 \cap I_2$ is strongly constructible of rank at least $d-1$.

Note that any strongly constructible poset is graded.

Proposition 3.6. *The order complex of any strongly constructible poset is constructible.*

Proof. Let P be a strongly constructible poset of rank d. To show that $\Delta(P)$ is constructible we will use induction on the cardinality of P. If P is pure shellable then $\Delta(P)$ is pure shellable and hence constructible. Otherwise P is the union of two strongly constructible proper ideals I_1, I_2 of rank d, such that $I_1 \cap I_2$ is strongly constructible of rank at least $d-1$. Clearly we have $\Delta(P) = \Delta(I_1) \cup \Delta(I_2)$ and $\Delta(I_1) \cap \Delta(I_2) = \Delta(I_1 \cap I_2)$. Since, by the induction hypothesis, $\Delta(I_1)$ and $\Delta(I_2)$ are constructible of dimension d and $\Delta(I_1 \cap I_2)$ is constructible of dimension at least $d-1$, it follows that $\Delta(P)$ is constructible as well. This completes the induction and the proof of the proposition. □

The next lemma asserts that our notion of strong constructibility for posets behaves well under direct products.

Lemma 3.7. *The direct product of two strongly constructible posets is strongly constructible.*

Proof. Let P, Q be two strongly constructible posets of ranks d and e, respectively. We proceed by induction on the sum of the cardinalities of P and Q. If P and Q are both bounded and pure shellable then their direct product $P \times Q$ is also (bounded and) pure shellable [8, Theorem 8.3] and hence strongly constructible. If not then one of them, say P, can be written as a union $P = I_1 \cup I_2$ of two strongly constructible proper ideals I_1, I_2 of rank d, such that $I_1 \cap I_2$ is strongly constructible of rank at least $d-1$. Then $P \times Q$ is the union of its proper ideals $I_1 \times Q$ and $I_2 \times Q$, each of rank $d+e$. By our induction hypothesis, these products are strongly constructible and so is their intersection $(I_1 \cap I_2) \times Q$, which has rank at least $d+e-1$. As a result, $P \times Q$ is strongly constructible as well. □

The proof of the following lemma is analogous to that of Lemma 3.4(ii) and is omitted.

Lemma 3.8. *Let P be a finite poset of rank d with a minimum element. If P is the union of strongly constructible ideals $I_1, I_2, \ldots, I_k$ of P of rank d and the intersection of any two or more of these ideals is strongly constructible of rank $d-1$, then P is strongly constructible.*

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by showing that P_n is strongly constructible. We will in fact prove a more general statement. For that reason, we introduce the following notation. Let $\tau_0, \tau_1, \ldots, \tau_k$ be pairwise disjoint subsets of $[n]$, such that $\tau_1, \ldots, \tau_k$ are nonempty. Let also σ be a nonempty sequence of distinct elements of $[n]$, none of which belongs to any of the sets τ_i. We set $R = (\sigma, \tau_0, \ldots, \tau_k)$ and denote by $\mathcal{S}_n(R)$ the set of permutations $w \in \mathcal{S}_n$ which have exactly $k+1$ cycles $c_0, c_1, \ldots, c_k$ in their cycle decomposition, such that

(a) the elements of σ appear consecutively in the cycle c_0 in the order in which they appear in σ and
(b) the elements of τ_i appear in the cycle c_i for $0 \leqslant i \leqslant k$.

Example 4.1. Suppose $k = 0$ and $\sigma = (1, 2, \ldots, r)$. Then $\mathcal{S}_n(R)$ is the set of cycles $w \in \mathcal{S}_n$ of length n for which $w(i) = i+1$ for $1 \leqslant i \leqslant r-1$. In particular, if $r = 1$ then $\mathcal{S}_n(R)$ is the set of all maximal elements of P_n.

The following proposition is the main result in this section.

Proposition 4.2. *If R is as above then the order ideal of P_n generated by $\mathcal{S}_n(R)$ is strongly constructible.*

The next remark will be used in the proof of the following technical lemma, which will be used in turn in the proof of Proposition 4.2.

Remark 4.3. Suppose that $w \preccurlyeq c$ holds in $\mathcal{S}_n$, where $c = (a_1\ a_2\ \cdots\ a_n)$ is a cycle of length n, and let $1 \leqslant p \leqslant n$. Suppose further that w has a cycle containing no a_i with $1 \leqslant i \leqslant p$. Then there exists a permutation in $\mathcal{S}_n$ which has exactly two cycles u, v in its cycle decomposition, such that $u(a_i) = a_{i+1}$ for $1 \leqslant i \leqslant p-1$, the elements appearing in u are exactly the elements which appear in those cycles of w containing $a_1, a_2, \ldots, a_p$, and $w \preccurlyeq uv$. This statement follows easily from the description of the absolute order on $\mathcal{S}_n$ given in Section 1. One constructs the cycle u by merging appropriately the cycles of w in which the elements $a_1, a_2, \ldots, a_p$ appear. The cycle v can be constructed by merging appropriately the remaining cycles of w. The details are left to the reader.

Lemma 4.4. *Let $1 \leqslant r \leqslant n-2$ and J be a subset of $\{r+1, \dots, n\}$ with at least two elements. Suppose that $w \in \mathcal{S}_n$ is such that for all $j \in J$ there exists a cycle c in $\mathcal{S}_n$ of length n satisfying $c(i) = i+1$ for $1 \leqslant i \leqslant r-1$, $c(r) = j$ and $w \preccurlyeq c$.*

(i) *There exists at most one $j \in J$ such that i and j are elements of the same cycle in the cycle decomposition for w for some $1 \leqslant i \leqslant r$.*
(ii) *There exists a permutation in $\mathcal{S}_n$ which has exactly two cycles u, v in its cycle decomposition such that $w \preccurlyeq uv$, $u(i) = i+1$ for $1 \leqslant i \leqslant r-1$ and one of the following holds*: (a) *all elements of J appear in v, or* (b) *there exists $j \in J$ with $u(r) = j$ and all other elements of J appear in v.*

Proof. Part (i) is once again an easy consequence of the description of the absolute order on $\mathcal{S}_n$ given in Section 1. Part (ii) follows from part (i) and Remark 4.3 (the latter is applied either for $p = r$ to w and a cycle c of length n satisfying $c(i) = i+1$ for $1 \leqslant i \leqslant r-1$, if no element of J appears in the same cycle of w with some $1 \leqslant i \leqslant r$, or for $p = r+1$ and a cycle c of length n satisfying $c(i) = i+1$ for $1 \leqslant i \leqslant r-1$ and $c(r) = j$, if $j \in J$ appears in the same cycle of w with some $1 \leqslant i \leqslant r$). □

Proof of Proposition 4.2. We denote by $\mathcal{I}_n(R)$ the order ideal of P_n generated by $\mathcal{S}_n(R)$, so that $\mathcal{I}_n(R)$ is a graded poset of rank $n-k-1$, and by m the number of elements of $[n]$ not appearing in $R = (\sigma, \tau_0, \dots, \tau_k)$. We proceed by induction on n, $n-k$ and m, in this order.

We assume $n \geqslant 3$, the result being trivial otherwise. We first treat the case $k \geqslant 1$. For $m = 0$, the poset $\mathcal{I}_n(R)$ is isomorphic to the direct product $\mathcal{I}_r(S) \times P_{r_1} \times \cdots \times P_{r_k}$ where $S = (\sigma, \tau_0)$, r is the number of elements of $[n]$ appearing in S and r_i is the cardinality of τ_i for $1 \leqslant i \leqslant k$. Since $\mathcal{I}_r(S)$ and P_{r_i} are strongly constructible by our induction hypothesis on n, the poset $\mathcal{I}_n(R)$ is strongly constructible by Lemma 3.7. Suppose now that $m \geqslant 1$ and let j be an element of $[n]$ which does not appear in R. Clearly we have

$$\mathcal{I}_n(R) = \bigcup_{i=0}^{k} \mathcal{I}_n(R_i),$$

where R_i is obtained from R by inserting j in the set τ_i. Each ideal $\mathcal{I}_n(R_i)$ has rank $n-k-1$ and, by our induction hypothesis on m, it is strongly constructible. Moreover, the intersection of any two or more of these ideals is equal to $\mathcal{I}_n(S)$, where $S = (\sigma, \tau_0, \dots, \tau_{k+1})$ with $\tau_{k+1} = \{j\}$. Since $\mathcal{I}_n(S)$ has rank $n-k-2$, it is strongly constructible by our induction hypothesis on $n-k$. It follows from Lemma 3.8 that $\mathcal{I}_n(R)$ is strongly constructible as well.

Finally, suppose that $k = 0$. Since the elements of τ_0 are irrelevant in this case, we may assume that τ_0 is empty. Clearly, the isomorphism type of $\mathcal{I}_n(R)$ depends only on the length of σ. Thus, for convenience with the notation, we will also assume that $\sigma = (1, 2, \dots, r)$ for some $1 \leqslant r \leqslant n$. If $m = 0$, so that $r = n$, then $\mathcal{S}_n(R)$ consists of a single cycle of length n and $\mathcal{I}_n(R)$ is isomorphic to the lattice of noncrossing partitions of $[n]$. Thus $\mathcal{I}_n(R)$ is bounded and pure shellable [5, Example 2.9] and, in particular, strongly constructible. Suppose that $m \geqslant 1$, so that $r \leqslant n-1$. For $r+1 \leqslant j \leqslant n$ we set $R_j = (\sigma_j, \emptyset)$, where $\sigma_j = (1, \dots, r, j)$ is obtained from σ by attaching j at the end. Each ideal $\mathcal{I}_n(R_j)$ has rank $n-1$ and, by our induction hypothesis on m, it is strongly constructible. Moreover, we have

$$\mathcal{I}_n(R) = \bigcup_{j=r+1}^{n} \mathcal{I}_n(R_j).$$

In view of Lemma 3.8, to prove that $\mathcal{I}_n(R)$ is strongly constructible it suffices to show that the intersection of any two or more of the ideals $\mathcal{I}_n(R_j)$ is strongly constructible of rank $n-2$. Let J be any subset of $\{r+1,\dots,n\}$ with at least two elements. We claim that

$$\bigcap_{j\in J}\mathcal{I}_n(R_j)=\mathcal{I}_n(S_0)\cup\left(\bigcup_{j\in J}\mathcal{I}_n(S_j)\right), \tag{2}$$

where $S_0=(\sigma,\emptyset,J)$ and $S_j=(\sigma_j,\emptyset,J\setminus\{j\})$ for $j\in J$. Indeed, it should be clear that each ideal $\mathcal{I}_n(S_j)$ for $j\in J\cup\{0\}$ is contained in the intersection in the left-hand side of (2). The reverse inclusion follows from Lemma 4.4(ii). Next, we note that the ideals $\mathcal{I}_n(S_j)$ for $j\in J\cup\{0\}$ have rank $n-2$ and that, by our induction hypothesis on $n-k$, they are strongly constructible. Applying induction on the cardinality of J, to show that the union in the right-hand side of (2) is strongly constructible it suffices to show that for $q\in J$, the intersection

$$\mathcal{I}_n(S_q)\cap\left(\bigcup_{j\in(J\setminus\{q\})\cup\{0\}}\mathcal{I}_n(S_j)\right)$$

is strongly constructible of rank $n-3$. We claim that this intersection is equal to $\mathcal{I}_n(S)$, where $S=(\sigma,\emptyset,J\setminus\{q\},\{q\})$. Indeed, one inclusion follows from the fact that $\mathcal{I}_n(S)\subseteq\mathcal{I}_n(S_q)\cap\mathcal{I}_n(S_0)$. For the reverse inclusion observe that in each permutation in $\mathcal{S}_n(S_q)$, q appears in a cycle containing $1,2,\dots,r$ but no element of $J\setminus\{q\}$ and that for all $j\in(J\setminus\{q\})\cup\{0\}$, in each permutation in $\mathcal{S}_n(S_j)$, q appears in a cycle containing none of $1,2,\dots,r$. Finally, observe that the ideal $\mathcal{I}_n(S)$ has the desired rank $n-3$ and is strongly constructible by our induction hypothesis on $n-k$. This completes the induction and the proof of the proposition. □

Proof of Theorem 1.1. When $k=0$ and σ has length one (see Example 4.1), the ideal $\mathcal{I}_n(R)$ coincides with P_n. Therefore Proposition 4.2 implies that P_n is strongly constructible. It follows from Proposition 3.6 and Corollary 3.3 that P_n is homotopy Cohen–Macaulay. As a result, so is $\bar{P}_n$. □

5. Proof of Theorem 1.2

In this section we denote by $\hat{0}$ the minimum element of P_n and by $\hat{P}_n$ the poset obtained from P_n by adding a maximum element $\hat{1}$.

Proof of Theorem 1.2. From [18, Proposition 3.8.6] we have that $\tilde{\chi}(\Delta(\bar{P}_n))=\mu_n(\hat{0},\hat{1})$, where μ_n stands for the Möbius function of $\hat{P}_n$, and hence that

$$\tilde{\chi}\big(\Delta(\bar{P}_n)\big)=-\sum_{x\in P_n}\mu_n(\hat{0},x). \tag{3}$$

Let $C_m=\frac{1}{m+1}\binom{2m}{m}$ denote the mth Catalan number. It is well known (see, for instance, [18, Exercise 3.68(b)]) that

$$\mu_n(\hat{0},x)=(-1)^{k-1}C_{k-1}$$

if $x\in\mathcal{S}_n$ is a cycle of length k, since in this case the interval $[\hat{0},x]$ in P_n is isomorphic to the lattice of noncrossing partitions of the set $[k]$. Moreover, for any $x\in\mathcal{S}_n$ the interval $[\hat{0},x]$

Table 1
The numbers $(-1)^n \tilde{\chi}(\Delta(\bar{P}_n))$ for $n \leqslant 9$

n	1	2	3	4	5	6	7	8	9
$(-1)^n \tilde{\chi}(\Delta(\bar{P}_n))$	1	0	2	16	192	3008	58 480	1 360 896	36 931 328

is isomorphic to the direct product over the cycles y in the cycle decomposition for x of the intervals $[\hat{0}, y]$. Therefore we have

$$\mu_n(\hat{0}, x) = \prod_{y \in \mathcal{C}(x)} (-1)^{\#y-1} C_{\#y-1}, \tag{4}$$

where $\mathcal{C}(x)$ is the set of cycles in the cycle decomposition for x and $\#y$ is the number of elements (length) of y. Given (3) and (4), the exponential formula [19, Corollary 5.1.9] implies that

$$1 - \sum_{n \geqslant 1} \tilde{\chi}\big(\Delta(\bar{P}_n)\big) \frac{t^n}{n!} = \exp \sum_{n \geqslant 1} (-1)^{n-1} C_{n-1} \frac{t^n}{n}. \tag{5}$$

Integrating the well-known equality

$$\sum_{n \geqslant 1} C_{n-1} t^{n-1} = \frac{1 - \sqrt{1-4t}}{2t},$$

we get

$$\sum_{n \geqslant 1} C_{n-1} \frac{t^n}{n} = 1 - \sqrt{1-4t} + \log(1 + \sqrt{1-4t}\,) - \log 2.$$

Switching t to $-t$ in the previous equality and exponentiating, we get

$$\exp \sum_{n \geqslant 1} (-1)^{n-1} C_{n-1} \frac{t^n}{n} = \frac{\sqrt{1+4t} - 1}{2t} \exp(\sqrt{1+4t} - 1).$$

In view of the previous equality, the result follows by switching t to $-t$ in (5). □

Remark 5.1. It follows from Theorems 1.1 and 1.2 that if $C(t) = \frac{1}{2t}(1 - \sqrt{1-4t}\,)$ then the generating function

$$1 - C(t) \exp\big\{-2tC(t)\big\}$$

has nonnegative coefficients.

Table 1 lists the first few values of $(-1)^n \tilde{\chi}(\Delta(\bar{P}_n))$.

References

[1] D. Armstrong, Braid groups, clusters and free probability: An outline from the AIM Workshop, January 2005, available at http://www.aimath.org/WWN/braidgroups/.

[2] D. Armstrong, Generalized noncrossing partitions and combinatorics of Coxeter groups, preprint, 2007, 156 pp., math.CO/0611106, Mem. Amer. Math. Soc., in press.

[3] C.A. Athanasiadis, T. Brady, C. Watt, Shellability of noncrossing partition lattices, Proc. Amer. Math. Soc. 135 (2007) 939–949.

[4] D. Bessis, The dual braid monoid, Ann. Sci. Ecole Norm. Sup. 36 (2003) 647–683.

[5] A. Björner, Shellable and Cohen–Macaulay partially ordered sets, Trans. Amer. Math. Soc. 260 (1980) 159–183.
[6] A. Björner, Topological methods, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1995, pp. 1819–1872.
[7] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, Grad. Texts in Math., vol. 231, Springer-Verlag, New York, 2005.
[8] A. Björner, M. Wachs, On lexicographically shellable posets, Trans. Amer. Math. Soc. 277 (1983) 323–341.
[9] T. Brady, A partial order on the symmetric group and new $K(\pi, 1)$'s for the braid groups, Adv. Math. 161 (2001) 20–40.
[10] T. Brady, C. Watt, $K(\pi, 1)$'s for Artin groups of finite type, in: Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I, Haifa, 2000, Geom. Dedicata 94 (2002) 225–250.
[11] R.W. Carter, Conjugacy classes in the Weyl group, Compos. Math. 25 (1972) 1–59.
[12] M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, Ann. of Math. 96 (1972) 318–337.
[13] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge University Press, Cambridge, England, 1990.
[14] M. Kallipoliti, Doctoral Dissertation, University of Athens, in preparation.
[15] V. Reiner, Personal communication with the first author, February 2003.
[16] R.P. Stanley, Cohen–Macaulay rings and constructible polytopes, Bull. Amer. Math. Soc. 81 (1975) 133–135.
[17] R.P. Stanley, Combinatorics and Commutative Algebra, first ed., Progr. Math., vol. 41, Birkhäuser, Boston, 1983, second ed., 1996.
[18] R.P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986; second printing, Cambridge Stud. Adv. Math., vol. 49, Cambridge University Press, Cambridge, 1998.
[19] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Stud. Adv. Math., vol. 62, Cambridge University Press, Cambridge, 1999.
[20] M. Wachs, Poset topology: Tools and applications, in: E. Miller, V. Reiner, B. Sturmfels (Eds.), Geometric Combinatorics, in: IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 497–615.