The symmetric group action on rank-selected posets of injective words

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Abstract

The symmetric group $\mathfrak{S}_n$ acts naturally on the poset of injective words over the alphabet $\{1, 2, \ldots, n\}$. The induced representation on the homology of this poset has been computed by Reiner and Webb. We generalize their result by computing the representation of $\mathfrak{S}_n$ on the homology of all rank-selected subposets, in the sense of Stanley. A further generalization to the poset of r-colored injective words is given.

Keywords: Symmetric group, representation, poset, injective word, rank-selection, homology.

1   Introduction and results

Let P be a finite partially ordered set (poset, for short) and G be a subgroup of the group of automorphisms of P. We assume that P has a minimum element $\hat{0}$ and a maximum element $\hat{1}$ and that it is graded of rank $n+1$, with rank function $\rho : P \to \{0, 1, \ldots, n+1\}$ (basic definitions on posets can be found in [18, Chapter 3]). Then G defines a permutation representation $\alpha_P(S)$, induced by the action of G on the set of maximal chains of the rank-selected subposet
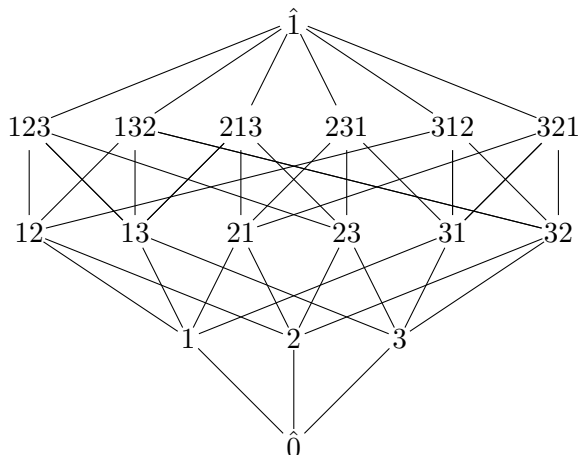
$$P_S \ = \ \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}, \tag{1}$$

for every $S \subseteq \{1, 2, \ldots, n\}$ and one can consider the virtual G-representation

$$\beta_P(S) \ = \ \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T). \tag{2}$$

When P is Cohen–Macaulay, $\beta_P(S)$ coincides with the non-virtual G-representation induced on the top homology group of (the order complex of) $\bar{P}_S := P_S \smallsetminus \{\hat{0}, \hat{1}\}$. The relations of the representations $\alpha_P(S)$ and $\beta_P(S)$ with one another and with the structure of P were investigated by Stanley in his seminal work [16], where several interesting examples (such as the symmetric group action on Boolean, subspace and partition lattices and the hyperoctahedral group action on the face lattice of the cross-polytope) were analyzed.

This paper aims to analyze the symmetric group action on another interesting poset, namely the poset of injective words. We denote by P_n the set of words over the alphabet $\{1, 2, \ldots, n\}$ with no repeated letter, partially ordered by setting $u \le v$ if u is a subword of v, along with an artificial maximum element $\hat{1}$ (the empty word is the minimum element $\hat{0}$). The poset P_n is the augmented face poset of a Boolean regular cell complex; see, for instance, [14, Section 1]. It was introduced by Farmer [9], was shown to be Cohen–Macaulay by Björner and Wachs [2, Section 6] and was further studied, among other places, in [10, 12, 13, 14].

The symmetric group $\mathfrak{S}_n$ acts on P_n, by the natural action of a permutation $w \in \mathfrak{S}_n$ on each letter of a word over the alphabet $\{1, 2, \ldots, n\}$, as a group of automorphisms. The decomposition of the $\mathfrak{S}_n$-representation on the top homology of $\bar{P}_n$ into irreducibles was computed, using the Hopf trace formula, by Reiner and Webb [14] and was further refined by Hanlon and Hersh [10], who also discovered interesting connections with the spectrum of a certain Markov chain on $\mathfrak{S}_n$; see [8] for recent developments. Recall that the set $\mathrm{Des}(Q)$ of descents of a standard Young tableau Q of size n consists of all integers $i \in \{1, 2, \ldots, n-1\}$ for which $i+1$ lies in a lower row that i. The result of Reiner and Webb can be stated as follows.

Theorem 1.1 ([14, Proposition 2.3]) *For $T = \{1, 2, \ldots, n\}$, the multiplicity of the irreducible $\mathfrak{S}_n$-representation corresponding to $\lambda \vdash n$ in $\beta_{P_n}(T)$ is equal to the number of standard Young tableaux Q of shape λ for which the smallest element of $\mathrm{Des}(Q) \cup \{n\}$ is even.*

To state our main result, we introduce the following notation and terminology. Recall that the descent set of a permutation $w \in \mathfrak{S}_n$ consists of those integers $i \in \{1, 2, \ldots, n-1\}$ for which $w(i) > w(i+1)$. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a subset of $\{1, 2, \ldots, n-1\}$ with $s_1 < s_2 < \cdots < s_k$ and denote by w_S the element of $\mathfrak{S}_n$ of largest possible Coxeter length (number of inversions) with descent set equal to S. For example, if $n = 8$ and $S = \{2, 3, 6\}$ then, written in one-line notation, $w_S = (7, 8, 6, 3, 4, 5, 1, 2)$. For a permutation $w \in \mathfrak{S}_n$ with descent set S and standard Young tableaux Q of size n, let $\tau(w, Q)$ denote the largest $i \in \{0, 1, \ldots, k+1\}$ for which: (a) $w(x) = w_S(x)$ for $x > s_{k-i+1}$; and (b) no descent of Q is smaller than $n - s_{k-i+1}$, where we have set $s_0 = 0$ and $s_{k+1} = n$. Note that these conditions are trivially satisfied for $i = 0$ and that for $S = \{1, 2, \ldots, n-1\}$ (in which case $w = w_S$ is the only element of $\mathfrak{S}_n$ with descent set equal to S), the integer $\tau(w, Q)$ is exactly the smallest element of $\mathrm{Des}(Q) \cup \{n\}$. Thus, the following statement generalizes Theorem 1.1.

Theorem 1.2 *For $T \subseteq \{1, 2, \ldots, n\}$, denote by $b_\lambda(T)$ the multiplicity of the irreducible $\mathfrak{S}_n$-representation corresponding to $\lambda \vdash n$ in $\beta_{P_n}(T)$. Then, for $S \subseteq \{1, 2, \ldots, n-1\}$ and $\lambda \vdash n$,*

(a) *$b_\lambda(S)$ is equal to the number of pairs (w, Q) of permutations $w \in \mathfrak{S}_n$ with descent set S and standard Young tableaux Q of shape λ for which $\tau(w, Q)$ is odd; and*

(b) *$b_\lambda(S \cup \{n\})$ is equal to the number of pairs (w, Q) of permutations $w \in \mathfrak{S}_n$ with descent set S and standard Young tableaux Q of shape λ for which $\tau(w, Q)$ is even.*

The structure and other results of this paper are as follows. Section 2 fixes notation and terminology. Section 3 obtains a formula (Equation 6) and a partial result (Equation 7) for the representations $\beta_{P_n}(S)$ which are valid for actions more general than that of the symmetric group on the poset of injective words and discusses examples. Section 4 proves Theorem 1.2, generalized to the action of the symmetric group on the poset of r-colored injective words, and discusses some consequences and special cases.

2 Preliminaries

This section fixes some basic notation and terminology which will be used in the sequel. For all positive integers n, we set $[n] =: \{1, 2, \ldots, n\}$.

Our notation on posets follows mostly that of [18, Chapter 3], where basic background on this topic can also be found. In particular, we denote by B_n the boolean lattice of all subsets of the set $[n]$, partially ordered by inclusion. A finite poset P is *graded* of rank $n + 1$ if every maximal chain in P has length $n+1$ (meaning, cardinality $n+2$). The rank $\rho(x)$ of $x \in P$ is then defined as the length of the longest chain in P with maximum element equal to x. Assume that such a poset P has a minimum and a maximum element. For $S \subseteq [n]$, we denote by $a_P(S)$ the number of maximal chains of the rank-selected subposet P_S of P defined in (1). For instance, $a_P(\varnothing) = 1$ and $a_P([n])$ is the number of maximal chains of P. The numbers $b_P(S)$ are defined by setting

$$b_P(S) \;=\; \sum_{T \subseteq S} (-1)^{|S-T|} a_P(T) \tag{3}$$

for $S \subseteq [n]$ or, equivalently, by

$$a_P(S) \;=\; \sum_{T \subseteq S} b_P(T) \tag{4}$$

for $S \subseteq [n]$.

An *automorphism* of a poset P is a bijective map $\varphi : P \to P$ such that $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$ for $x, y \in P$. The automorphisms of P form a group under the law of composition of maps.

For basic background on the representation theory of the symmetric group and the combinatorics of Young tableaux, we refer the reader to [15] [19, Chapter 7] [20, Lecture 2]. All representations we consider are defined over the field of complex numbers $\mathbb{C}$. The trivial representation of a group G is denoted by $\mathbb{1}_G$. We follow the English notation to describe Young tableaux.

Let P be a finite, graded poset of rank $n + 1$ with a minimum and a maximum element and let G be a subgroup of the group of automorphisms of P. As discussed in the introduction, for $S \subseteq [n]$, we denote by $\alpha_P(S)$ the permutation G-representation induced by the action of G on

the set of maximal chains of P_S and by $\beta_P(S)$ the virtual G-representation defined by (2) or, equivalently, by

$$\alpha_P(S) \;=\; \sum_{T \subseteq S} \beta_P(T) \tag{5}$$

for $S \subseteq [n]$. We note that the dimension of $\alpha_P(S)$ is equal to $a_P(S)$. If P is Cohen–Macaulay over $\mathbb{C}$, then $\beta_P(S)$ is a non-virtual G-representation [16, Theorem 1.2] whose dimension is equal to $b_P(S)$; see [16] [20, Section 3.4] for more information on these representations.

3   Good actions on simplicial posets

This section obtains a formula for the representations $\beta_P(S)$ under certain hypotheses, which are satisfied by the symmetric group action on the poset of injective words. Throughout this section, P is a finite, graded poset of rank $n+1$ with minimum element $\hat{0}$ and maximum element $\hat{1}$ and G is a subgroup of the group of automorphisms of P.

Following [18, Section 3.16], we say that $P \smallsetminus \{\hat{1}\}$ is *simplicial* if the closed interval $[\hat{0}, x]$ of P is isomorphic to a Boolean lattice for every $x \in P \smallsetminus \{\hat{1}\}$. We call the action of G on P *good* if $g \cdot y = y$ for some $g \in G$ and $y \in P \smallsetminus \{\hat{1}\}$ implies that $g \cdot x = x$ for every $x \in P$ with $x \le y$. Let us also denote by $b_n(S)$ the number of permutations $w \in \mathfrak{S}_n$ with descent set S.

Theorem 3.1 *Suppose that the poset $P \smallsetminus \{\hat{1}\}$ is simplicial and that the action of G is good. Then,*

$$\beta_P(S) \;=\; (-1)^k \, \mathbb{1}_G \;+\; \sum_{i=1}^{k} (-1)^{k-i} \, b_{s_i}(\{s_1, s_2, \dots, s_{i-1}\}) \cdot \alpha_P(\{s_i\}) \tag{6}$$

for every $S = \{s_1, s_2, \dots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \cdots < s_k$. In particular,

$$\beta_P(S) \;+\; \beta_P(S \cup \{n\}) \;=\; b_n(S) \cdot \alpha_P(\{n\}) \tag{7}$$

for every $S \subseteq [n-1]$.

Proof. We set $a_n(T) := a_{B_n}(T)$ for $T \subseteq [n-1]$. Thus, $a_n(T)$ is the multinomial coefficient equal to the number of chains c in the Boolean lattice B_n for which the set of cardinalities (ranks) of the elements of c is equal to T. It is well known [18, Corollary 3.13.2] that

$$\sum_{T \subseteq S} (-1)^{|S-T|} \, a_n(T) \;=\; b_n(S) \tag{8}$$

for every $S \subseteq [n-1]$, where $b_n(S)$ is as in the sentence preceding the theorem.

We claim that

$$\alpha_P(S) \;=\; a_m(S \smallsetminus \{m\}) \cdot \alpha_P(\{m\}) \tag{9}$$

for every $S \subseteq [n]$ with maximum element equal to m. Indeed, since the action of G on the poset P is good, a maximal chain c of P_S is fixed by an element $g \in G$ if and only if the maximum element of $c \smallsetminus \{\hat{1}\}$ is fixed by g. As a result, and since $P \smallsetminus \{\hat{1}\}$ is simplicial, the number of maximal chains of P_S fixed by g is equal to the product of the number of elements of P of rank m which are fixed by g with the number $a_m(S \smallsetminus \{m\})$ of chains in the Boolean lattice B_m whose elements have set of ranks equal to S. This shows that the characters of the representations in the two hand sides of (9) are equal and verifies our claim.

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Using (9), for $S = \{s_1, s_2, \ldots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \cdots < s_k$, the defining equation (2) gives

$$\beta_P(S) = (-1)^k \mathbb{1}_G + \sum_{i=1}^{k} \sum_{\substack{T \subseteq S : \\ \max(T) = s_i}} (-1)^{|S-T|} a_{s_i}(T \smallsetminus \{s_i\}) \cdot \alpha_P(\{s_i\})$$

$$= (-1)^k \mathbb{1}_G + \sum_{i=1}^{k} \sum_{T \subseteq \{s_1, \ldots, s_{i-1}\}} (-1)^{k-|T|-1} a_{s_i}(T) \cdot \alpha_P(\{s_i\}).$$

Using (8) to compute the inner sum gives the expression (6) for $\beta_P(S)$. This expression directly implies Equation (7). Alternatively, using the defining equation (2) we find that, for $S \subseteq [n-1]$,

$$\beta_P(S \cup \{n\}) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T \cup \{n\}) - \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T)$$

$$= \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T \cup \{n\}) - \beta_P(S).$$

Using (8) and (9), we conclude that

$$\beta_P(S) + \beta_P(S \cup \{n\}) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T \cup \{n\}) = \sum_{T \subseteq S} (-1)^{|S-T|} a_n(T) \cdot \alpha_P(\{n\})$$

$$= b_n(S) \cdot \alpha_P(\{n\})$$

and the proof follows. □

Example 3.2 Let n, r be positive integers and let $\mathbb{Z}_r$ denote the abelian group of integers modulo r. We consider the cartesian product $[n] \times \mathbb{Z}_r$ as an alphabet and think of the second coordinate of an element of this set as one of r possible colors attached to the first coordinate. We denote by $P_{n,r}$ the set of words over this alphabet whose letters have pairwise distinct first coordinates, partially ordered by the subword order (so that $u \leq v$ if and only if u can be obtained by deleting some of the letters of v), along with an artificial maximum element $\hat{1}$ (the empty word is the minimum element $\hat{0}$). This is the *poset of r-colored injective words*; it specializes to the poset of injective words P_n, discussed in the introduction, when $r = 1$. The Cohen-Macaulayness of $P_{n,r}$ follows from [12, Theorem 1.2].

The symmetric group $\mathfrak{S}_n$ acts on $P = P_{n,r}$ by acting on the first coordinate of each letter of a word and leaving the colors unchanged. We leave it to the reader to verify that the hypotheses of Theorem 3.1 are satisfied for this action. Clearly, $\alpha_P(\{n\})$ is isomorphic to the direct sum of r^n copies of the regular representation ρ^{reg} of $\mathfrak{S}_n$ and thus, Equation (7) gives

$$\beta_P(S) + \beta_P(S \cup \{n\}) = b_n(S) r^n \cdot \rho^{\text{reg}} \tag{10}$$

for every $S \subseteq [n-1]$. Equivalently, denoting by $b_{r,\lambda}(T)$ the multiplicity of the irreducible $\mathfrak{S}_n$-representation corresponding to $\lambda \vdash n$ in $\beta_P(T)$, we have

$$b_{r,\lambda}(S) + b_{r,\lambda}(S \cup \{n\}) = b_n(S) r^n f^\lambda \tag{11}$$

for every $\lambda \vdash n$, where f^λ is the number of standard Young tableaux of shape λ. Equation (11) suggests that there is a combinatorial interpretation to each of the summands on its left-hand

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side in terms of pairs of elements of $\mathfrak{S}_n$ with descent set S and r-colored standard Young tableaux of shape λ. Theorem 4.1, proved in the following section, provides such an interpretation and thus determines the corresponding summands of (10).

We note that Equation (10) substantially refines

$$\sum_{T \subseteq [n]} \beta_P(T) \;=\; r^n \, n! \cdot \rho^{\mathrm{reg}}, \tag{12}$$

which can be shown directly by observing that the permutation representation $\alpha_P([n])$ of $\mathfrak{S}_n$ on the set of maximal chains of P is isomorphic to the direct sum of $r^n \, n!$ copies of the regular representation of $\mathfrak{S}_n$.

Remark 3.3 There are larger groups than $\mathfrak{S}_n$, such as the wreath products $\mathfrak{S}_n[\mathbb{Z}_r]$ and $\mathfrak{S}_n[\mathfrak{S}_r]$, which act naturally on the poset $P_{n,r}$ of r-colored injective words by good actions. We leave it as an open problem to refine Theorem 4.1 in this direction and note that the induced $\mathfrak{S}_n[\mathbb{Z}_2]$-representation on the top homology group of $\bar{P}_{n,2}$ was computed in [1, Theorem 7.3]. $\qquad\square$

Example 3.4 Let Δ be a finite, abstract simplicial complex of dimension $n-1$. We assume that Δ is pure (every maximal face of Δ has dimension $n-1$) and (completely) balanced (the vertices of Δ are colored with n colors, so that vertices of any maximal face of Δ have distinct colors; see [17, Section III.4] for more information about this class of complexes).

Let P_Δ be the set of faces of Δ, ordered by inclusion, with an artificial maximum element $\hat{1}$ attached, and let G be a subgroup of the group of automorphisms of P_Δ whose action preserves the colors of the vertices of Δ. We leave it again to the reader to verify that the hypotheses of Theorem 3.1 are satisfied for the action of G on P_Δ. The special case in which Δ is the order complex of a poset was analysed by Stanley [16, Section 8].

4 The poset of colored injective words

This section applies Theorem 3.1 to decompose the $\mathfrak{S}_n$-representations $\beta_P(S)$ into irreducibles for the poset of r-colored injective words, thus generalizing Theorem 1.2.

To state the main result of this section, we need to modify some of the definitions introduced before Theorem 1.2. An *r-colored standard Young tableau* of shape $\lambda \vdash n$ is a standard Young tableau of shape λ, each entry of which has been colored with one of the elements of $\mathbb{Z}_r$. Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq [n-1]$ with $s_1 < s_2 < \cdots < s_k$ and recall the definition of $w_S \in \mathfrak{S}_n$, given in the introduction. Given a permutation $w \in \mathfrak{S}_n$ with descent set S and an r-colored standard Young tableau Q of size n, we denote by $\tau(w, Q)$ the largest $i \in \{0, 1, \ldots, k+1\}$ for which: (a) $w(x) = w_S(x)$ for $x > s_{k-i+1}$; and (b) all numbers $1, 2, \ldots, n - s_{k-i+1}$ appear in the first row of Q and are colored with the zero color, where we have set $s_0 = 0$ and $s_{k+1} = n$. Note that these conditions are vacuously satisfied for $i = 0$ and that the definition of $\tau(w, Q)$ agrees with that given in the introduction in the special case $r = 1$.

Theorem 4.1 *Let P be the poset $P_{n,r}$ of r-colored injective words and for $T \subseteq [n]$, denote by $b_{r,\lambda}(T)$ the multiplicity of the irreducible $\mathfrak{S}_n$-representation corresponding to $\lambda \vdash n$ in $\beta_P(T)$. Then, for $S \subseteq [n-1]$ and $\lambda \vdash n$,*

(a) *$b_{r,\lambda}(S)$ is equal to the number of pairs (w, Q) of permutations $w \in \mathfrak{S}_n$ with descent set S and r-colored standard Young tableaux Q of shape λ for which $\tau(w, Q)$ is odd; and*

(b) $b_{r,\lambda}(S \cup \{n\})$ *is equal to the number of pairs* (w, Q) *of permutations* $w \in \mathfrak{S}_n$ *with descent set* S *and* r-*colored standard Young tableaux* Q *of shape* λ *for which* $\tau(w, Q)$ *is even.*

Proof. To apply Theorem 3.1 and (9), we need to determine $\alpha_P(\{m\})$ for $m \in [n]$. The elements of P of rank m are the r-colored injective words of length m over the alphabet $[n]$. Clearly, the action of $\mathfrak{S}_n$ on these words has r^m orbits, corresponding to the r^m possible coloring patterns, and the stabilizer of the action on each orbit is isomorphic to the Young subgroup $(\mathfrak{S}_1)^m \times \mathfrak{S}_{n-m}$ of $\mathfrak{S}_n$. Using Young's rule [15, Theorem 2.12.2] to decompose the permutation representation of $\mathfrak{S}_n$ on the set of left cosets of this subgroup we find that

$$\alpha_P(\{m\}) = r^m \sum_{\lambda \vdash n} f^{\lambda, n-m} \rho^\lambda, \tag{13}$$

where ρ^λ stands for the irreducible $\mathfrak{S}_n$-representation corresponding to $\lambda \vdash n$ and $f^{\lambda, n-m}$ denotes the number of standard Young tableaux of size n whose first row contains the numbers $1, 2, \ldots, n-m$. Now let $S = \{s_1, s_2, \ldots, s_k\} \subseteq [n-1]$ with $s_1 < s_2 < \cdots < s_k$ and set $s_0 := 0$ and $s_{k+1} := n$. Equation (6) implies the expression

$$b_{r,\lambda}(S) = (-1)^k \delta_{\lambda,(n)} + \sum_{i=1}^{k} (-1)^{k-i} b_{s_i}(\{s_1, s_2, \ldots, s_{i-1}\}) r^{s_i} f^{\lambda, n-s_i} \tag{14}$$

for the multiplicity of ρ^λ in $\beta_P(S)$, where $\delta_{\lambda,(n)}$ is a Kronecker delta. For $1 \le i \le k$, clearly, $r^{s_i} f^{\lambda, n-s_i}$ is equal to the number of r-colored standard Young tableaux of shape λ whose first row contains the numbers $1, 2, \ldots, n-s_i$, all colored with the zero color. We may also interpret $b_{s_i}(\{s_1, s_2, \ldots, s_{i-1}\})$ as the number of permutations $w \in \mathfrak{S}_n$ with descent set S for which $w(x) = w_S(x)$ for $x > s_i$. Therefore, Equation (14) can be rewritten as

$$b_{r,\lambda}(S) = \sum_{i=1}^{k+1} (-1)^{i-1} |\mathcal{T}_i|, \tag{15}$$

where $\mathcal{T}_i$ is the set of all pairs (w, Q) of permutations $w \in \mathfrak{S}_n$ with descent set S and r-colored standard Young tableaux Q of shape λ for which: (a) $w(x) = w_S(x)$ for $x > s_{k-i+1}$; and (b) all numbers $1, 2, \ldots, n-s_{k-i+1}$ appear in the first row of Q and are colored with the zero color. Noting that $\mathcal{T}_0$ is the set of all pairs (w, Q) of permutations $w \in \mathfrak{S}_n$ with descent set S and r-colored standard Young tableaux Q of shape λ and that $\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \cdots$ we conclude that

$$b_{r,\lambda}(S) = |\mathcal{T}_1 \smallsetminus \mathcal{T}_2| + |\mathcal{T}_3 \smallsetminus \mathcal{T}_4| + \cdots \tag{16}$$
$$b_{r,\lambda}(S \cup \{n\}) = |\mathcal{T}_0 \smallsetminus \mathcal{T}_1| + |\mathcal{T}_2 \smallsetminus \mathcal{T}_3| + \cdots, \tag{17}$$

where the first equation follows from (15) and the second by a similar argument, or from the first by appealing to the last statement of Theorem 3.1. The two equations are equivalent to parts (a) and (b), respectively, of the theorem. $\qquad\square$

Example 4.2 Keeping the notation of Theorem 4.1, let us consider the multiplicity of the trivial and the sign representation in $\beta_P(T)$ for $T \subseteq [n]$, corresponding to $\lambda = (n)$ and $\lambda = (1^n)$, respectively. Equation (11) holds with $f^\lambda = 1$ in both cases, for $S \subseteq [n-1]$. For $\lambda = (1^n)$, the

tableaux Q in the definition of $\mathcal{T}_i$ have a single column and for $\mathcal{T}_i$ to be nonempty, we must have either $i = 0$, or $i = 1$ and $n - 1 \in S$. From (16) and (17) we infer that

$$b_{r,(1^n)}(S) = \begin{cases} 0, & \text{if } n - 1 \notin S, \\ r^{n-1} b_{n-1}(S \smallsetminus \{n-1\}), & \text{if } n - 1 \in S \end{cases}$$

and

$$b_{r,(1^n)}(S \cup \{n\}) = \begin{cases} r^n b_n(S), & \text{if } n - 1 \notin S, \\ r^n b_n(S) - r^{n-1} b_{n-1}(S \smallsetminus \{n-1\}), & \text{if } n - 1 \in S. \end{cases}$$

Similarly, for $\lambda = (n)$, the pairs (w, Q) of permutations $w \in \mathfrak{S}_n$ and r-colored standard Young tableaux Q of shape λ may be identified with the elements of the wreath product $\mathfrak{S}_n[\mathbb{Z}_r]$, viewed as the r-colored permutations of the set $[n]$. Thus, Theorem 4.1 gives a combinatorial interpretation to the multiplicity $b_{r,(n)}(T)$ in terms of such permutations and their descent and color patterns.

Example 4.3 Suppose $S = [n-1]$. Then, w_S is the only element of $\mathfrak{S}_n$ with descent set equal to S. Therefore, for $\lambda \vdash n$, the multiplicity $b_{r,\lambda}([n-1])$ (respectively, $b_{r,\lambda}([n])$) is equal to the number of r-colored standard Young tableaux Q of shape λ for which the largest $i \in \{0, 1, \ldots, n\}$ such that all numbers $1, 2, \ldots, i$ appear in the first row of Q and are colored with the zero color is odd (respectively, even). This extends Theorem 1.1 to r-colored injective words.

Remark 4.4 As discussed in the beginning of the proof of Theorem 4.1, we have

$$\alpha_P(\{m\}) = r^m \cdot \mathbb{1} {\uparrow}_{(\mathfrak{S}_1)^m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}$$

for $n \geq m$, where $P = P_{n,r}$. By [3, Theorem 2.8] [11, Lemma 2.2], this expression and (9) show that for a fixed set S of positive integers, the representation $\beta_P(S)$ stabilizes for $n \geq 2 \max(S)$, in the sense of [4]. □

As before, we view the elements of the group $\mathfrak{S}_n[\mathbb{Z}_r]$ as the r-colored permutations of the set $[n]$, meaning permutations u of $[n]$ with the entries $u(i)$ in their one-line notation colored, each with one of the elements of $\mathbb{Z}_r$. Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq [n-1]$ be as in the paragraph preceding Theorem 4.1. Given a permutation $w \in \mathfrak{S}_n$ with descent set S and an r-colored permutation $u \in \mathfrak{S}_n[\mathbb{Z}_r]$, we denote by $\tau(w, u)$ the largest $i \in \{0, 1, \ldots, k+1\}$ for which: (a) $w(x) = w_S(x)$ for $x > s_{k-i+1}$; and (b) the numbers $u(1), u(2), \ldots, u(n - s_{k-i+1})$ are all colored with the zero color in u and are in increasing order.

Corollary 4.5 *Let P be as in Theorem 4.1. Then, for $S \subseteq [n-1]$,*

(a) *$b_P(S)$ is equal to the number of pairs (w, u) of permutations $w \in \mathfrak{S}_n$ with descent set S and r-colored permutations $u \in \mathfrak{S}_n[\mathbb{Z}_r]$ for which $\tau(w, u)$ is odd; and*

(b) *$b_P(S \cup \{n\})$ is equal to the number of pairs (w, u) of permutations $w \in \mathfrak{S}_n$ with descent set S and r-colored permutations $u \in \mathfrak{S}_n[\mathbb{Z}_r]$ for which $\tau(w, u)$ is even.*

Proof. Since $b_P(T)$ is the dimension of the $\mathfrak{S}_n$-representation $\beta_P(T) = \sum_{\lambda \vdash n} b_{r,\lambda}(T) \rho^\lambda$, we have

$$b_P(T) = \sum_{\lambda \vdash n} b_{r,\lambda}(T) f^\lambda \tag{18}$$

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for $T \subseteq [n]$. This equation, combined with part (a) of Theorem 4.1, implies that $b_P(S)$ counts the triples (w, P_0, Q_0) of permutations $w \in \mathfrak{S}_n$ with descent set S, standard Young tableaux P_0 of size n and r-colored standard Young tableaux Q_0 of the same shape as P_0 for which $\tau(w, Q_0)$ is odd. A trivial extension of the Robinson-Schensted correspondence (see, for instance, [15, Section 3.1]) provides a bijection from the group $\mathfrak{S}_n[\mathbb{Z}_r]$ of r-colored permutations to the set of pairs (P_0, Q_0) of standard Young tableaux P_0 of size n and r-colored standard Young tableaux Q_0 of the same shape as P_0 (under this correspondence, the entry $i \in [n]$ of the tableau Q_0 associated to $u \in \mathfrak{S}_n[\mathbb{Z}_r]$ is colored with the color that $u(i)$ has in u). Standard properties of the Robinson-Schensted correspondence imply that $\tau(w, u) = \tau(w, Q_0)$, if $u \in \mathfrak{S}_n[\mathbb{Z}_r]$ is mapped to (P_0, Q_0) under the extended correspondence. Thus, the set of triples (w, P_0, Q_0) mentioned above, the cardinality of which is equal to $b_P(S)$, bijects to the set of pairs (w, u) mentioned in part (a). A similar argument works for part (b). $\qquad\square$

Remark 4.6 Part (b) of Corollary 4.5 implies that $b_P([n])$ is equal to the number, say $E_{n,r}$, of r-colored permutations $u \in \mathfrak{S}_n[\mathbb{Z}_r]$ for which the largest $i \in \{0, 1, \ldots, n\}$ such that the numbers $u(1), u(2), \ldots, u(i)$ are in increasing order and all colored with the zero color is even. On the other hand, from Equation (6) we get

$$
\begin{aligned}
b_P([n]) &= \dim \beta_P([n]) = (-1)^n + \sum_{i=1}^{n} (-1)^{n-i} a_P(\{i\}) \\
&= \sum_{i=1}^{n} (-1)^{n-i} r^i \frac{n!}{(n-i)!} = D_{n,r},
\end{aligned}
$$

where $D_{n,r}$ is the number of derangements (elements without fixed points of zero color) in $\mathfrak{S}_n[\mathbb{Z}_r]$. The fact that $D_{n,r} = E_{n,r}$ was first discovered for $r = 1$ by Désarménien [5]; for a much stronger statement, see [6, Theorem 1] [7, Corollary 3.3] for $r = 1$ and [1, Theorem 7.3] for $r = 2$. In fact, this stronger statement can be expressed in terms of the Frobenius characteristic of the $\mathfrak{S}_n[\mathbb{Z}_r]$-representation on the homology of $\bar{P}_{n,r}$, induced by the action of $\mathfrak{S}_n[\mathbb{Z}_r]$ on $P_{n,r}$; see [14, Section 2] for $r = 1$ and [1, Theorem 7.3] for $r = 2$.

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