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Zonotopal Subdivisions of Cyclic Zonotopes

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Abstract. The cyclic zonotope $\mathcal{Z}(n, d)$ is the zonotope in $\mathbb{R}^d$ generated by any n distinct vectors of the form $(1, t, t^2, \ldots, t^{d-1})$. It is proved that the refinement poset of all proper zonotopal subdivisions of $\mathcal{Z}(n, d)$ which are induced by the canonical projection $\pi: \mathcal{Z}(n, d') \to \mathcal{Z}(n, d)$, in the sense of Billera and Sturmfels, is homotopy equivalent to a sphere and that any zonotopal subdivision of $\mathcal{Z}(n, d)$ is shellable. The first statement gives an affirmative answer to the generalized Baues problem in a new special case and refines a theorem of Sturmfels and Ziegler on the extension space of an alternating oriented matroid. An important ingredient in the proofs is the fact that all zonotopal subdivisions of $\mathcal{Z}(n, d)$ are stackable in a suitable direction. It is shown that, in general, a zonotopal subdivision is stackable in a given direction if and only if a certain associated oriented matroid program is Euclidean, in the sense of Edmonds and Mandel.

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1 Introduction

Zonotopes are the centrally symmetric polytopes which are affine projections of a cube or, equivalently, Minkowski sums of line segments [7, §2.2], [25, §7.3]. The cyclic zonotope $\mathcal{Z}(n, d)$ is the Minkowski sum of $n \geqslant d$ segments in $\mathbb{R}^d$ pointing in the directions of any n distinct vectors of the form $(1, t, t^2, \ldots, t^{d-1})$. Its combinatorial structure is described by the alternating oriented matroid $C^{n,d}$ and depends only on n and d. Our main object of study will be the set of zonotopal tilings [25, §7.5], or zonotopal subdivisions, of $\mathcal{Z}(n, d)$. The significance of this set comes from two directions. First, it provides an analogue to the much studied set of polyhedral subdivisions of the classical cyclic polytope [1, 2, 10, 11, 17–19]. Second, by the Bohne–Dress Theorem on zonotopal tilings [8], it bijects to the set of single element extensions of the dual oriented matroid of $C^{n,d}$, a set which is important in the study of the higher Bruhat orders of Manin and Schechtman [16]; see [15, 24].

Our first result is motivated by the generalized Baues problem of Billera *et al.* [5] and provides an analogue to the main result of [2]. The set of zonotopal subdivisions

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of $\mathcal{Z}(n,d)$ can be partially ordered by refinement. The subposet $\omega(n,d)$ of proper zonotopal subdivisions is isomorphic to the extension poset of $C^{n,n-d}$ and, endowed with a standard topology [6], was shown to have the homotopy type of an $(n-d-1)$-sphere by Sturmfels and Ziegler [23]. As in the case of cyclic polytopes [1, 2], given integers $1 \leqslant d < d' \leqslant n$, the canonical projection $\pi: \mathbb{R}^{d'} \to \mathbb{R}^d$ which 'forgets' the last $d'-d$ coordinates induces a linear surjection $\pi: \mathcal{Z}(n,d') \to \mathcal{Z}(n,d)$ of cyclic zonotopes. The proper zonotopal subdivisions of $\mathcal{Z}(n,d)$ which are induced by π [4], [25, §9.1], meaning that their tiles are projections of faces of $\mathcal{Z}(n,d')$, form a subposet of $\omega(n,d)$. This poset is known as the Baues poset associated to the projection π and reduces to $\omega(n,d)$ when $d'=n$.

THEOREM 1.1. *If $\pi: \mathcal{Z}(n,d') \to \mathcal{Z}(n,d)$ is the canonical projection of cyclic zonotopes then the poset of proper zonotopal subdivisions of $\mathcal{Z}(n,d)$ which are induced by π is homotopy equivalent to a $(d'-d-1)$-sphere.*

The previous theorem can be viewed as a refinement of the result of [23], mentioned earlier, on the extension space of an alternating oriented matroid since it reduces to that result when $d'=n$. It gives an affirmative answer to the generalized Baues problem [5] in a new special case. Given any affine surjection of polytopes $\pi: P \to Q$, this problem asks to determine whether the poset of all proper polyhedral subdivisions of Q which are induced by π has the homotopy type of a sphere of dimension $\dim(P) - \dim(Q) - 1$; see [17] and [20] for more information.

Our second result relates to the concept of shellability [6, Section 11] and provides an analogue to a theorem of Rambau [18, Theorem 6.1], which states that all triangulations of a cyclic polytope are shellable.

THEOREM 1.2. *All zonotopal subdivisions of a cyclic zonotope are shellable.*

The proof of both results relies on the fact (see Lemma 3.2) that all zonotopal subdivisions of $\mathcal{Z}(n,d)$ are stackable in the direction of the vector $u = (1, t, t^2, \ldots, t^{d-1})$, where t is sufficiently large. Stackability can be defined for any polyhedral subdivision and was previously exploited for triangulations of cyclic polytopes [17, 18] and later for their subdivisions [2, 19]. In the case of two-dimensional zonotopal subdivisions, stackability corresponds to what is known as 'topologically sweeping' the associated arrangement of pseudolines (see [7, p. 472], [14, Section 3] and the references cited there). We will show in Theorem 6.2 that, for arbitrary dimension, a zonotopal subdivision is stackable in a given direction if and only if a certain associated oriented matroid program is Euclidean, in the sense of Edmonds and Mandel [13], [7, §10.5]. As an application, we give an interpretation to the concept of strong Euclideanness of Sturmfels and Ziegler [23] for realizable oriented matroids and derive an alternative proof (Corollary 6.7) of strong Euclideanness, due to these authors, for the alternating oriented matroids.

Section 2 contains the necessary background except for that related to oriented matroid programs, which is included in Section 6. The definition of a zonotopal subdivision we use is that of a 'strong zonotopal tiling' of [21], which coincides with that of a subdivision induced by a projection from a cube [4, Section 4]. In Section 3 we define stackability for zonotopal subdivisions and use it to prove Theorem 1.2. Theorem 1.1 is proved in Section 5 after we introduce cellular sections for zonotopal subdivisions in Section 4. Our proofs in both cases proceed by 'deletion and contraction'. In Section 6 we discuss the connection between stackability of zonotopal subdivisions and Euclideanness of oriented matroid programs.

2. Background

We begin with basic notation, definitions and facts related to zonotopal subdivisions. For general background on convex polytopes, zonotopes and oriented matroids we rely on the texts [7] and [25].

2.1. SIGN VECTORS

We denote by Λ_n the set $\{-, 0, +\}^n$ of sign vectors of length n and write $X = (X_1, X_2, \ldots, X_n)$ for $X \in \Lambda_n$. The set Λ_n is partially ordered by extending coordinatewise the partial order on $\{-, 0, +\}$ defined by the relations $0 < -$ and $0 < +$. Its unique minimal element is the zero vector, denoted by 0. The sign vector $-X$ is obtained by negating each coordinate of X. The *composition* $X \circ Y$ of two sign vectors is defined by $(X \circ Y)_i = X_i$ or Y_i if $X_i \neq 0$ or $X_i = 0$, respectively, and their *separation set* is $S(X, Y) = \{i: X_i = -Y_i \neq 0\}$.

2.2. ZONOTOPES

Let $V = (v_1, v_2, \ldots, v_n)$ be a configuration of n vectors in $\mathbb{R}^d$. The *zonotope* $\mathcal{Z}$ generated by V is the pair (V, Z), where Z is the Minkowski sum of line segments $Z = \sum_{i=1}^n [-v_i, v_i]$ or, equivalently, the image in $\mathbb{R}^d$ of the cube $[-1, 1]^n \subseteq \mathbb{R}^n$ under the linear map which sends the ith coordinate vector in $\mathbb{R}^n$ to v_i. We assume that the generators v_i linearly span $\mathbb{R}^d$, so that $\mathcal{Z}$ has dimension d. The quantity $n - d$ is the *codimension* of $\mathcal{Z}$. A generator v_i of $\mathcal{Z}$ is called a *loop* if $v_i = 0$ and a *coloop* if the dimension of the linear span of $V - v_i$ is $d - 1$.

A *subzonotope* $\mathcal{Z}_X$ of $\mathcal{Z}$ is a pair (X, Z_X) of a sign vector $X \in \Lambda_n$ and the *convex hull*

$$Z_X = \sum_{X_i = +} v_i - \sum_{X_i = -} v_i + \sum_{X_i = 0} [-v_i, v_i]$$

of $\mathcal{Z}_X$. The (translated) zonotope $\mathcal{Z}_X$ is generated by the subset $\{v_i: X_i = 0\}$ of V and coincides with $\mathcal{Z}$ if $X = 0$. Its dimension is that of Z_X. If S is a set of subzonotopes of $\mathcal{Z}$ we denote by $\sigma(S)$ the set of sign vectors of the elements of S.

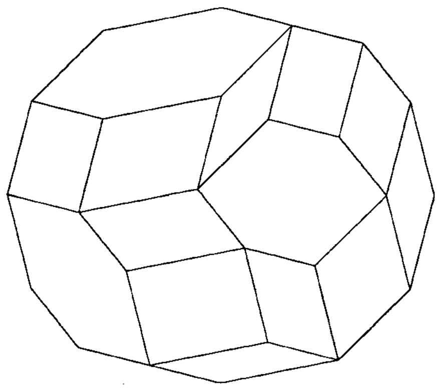

Figure 1. A zonotopal subdivision of a two-dimensional zonotope with seven generators.

We say that $\mathcal{Z}_C$ is a *face* of $\mathcal{Z}_X$ if there exists a face F of the polytope Z_X such that C is minimum among all sign vectors $Y \geqslant X$ with $Z_Y = F$. A face of $\mathcal{Z}$ of dimension $d-1$ is called a *facet*. The set $\mathcal{L} = \mathcal{L}(\mathcal{Z})$ of all $X \in \Lambda_n$ such that $\mathcal{Z}_X$ is a face of $\mathcal{Z}$ is the set of *covectors* of an *oriented matroid* $\mathcal{M}$, in the sense that it satisfies the axioms:

(L0) $0 \in \mathcal{L}$,
(L1) $X \in \mathcal{L}$ implies $-X \in \mathcal{L}$,
(L2) $X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$,
(L3) if $X, Y \in \mathcal{L}$ and $e \in S(X, Y)$, then there exists a $C \in \mathcal{L}$ such that $C_e = 0$ and $C_f = (X \circ Y)_f = (Y \circ X)_f$ for $f \notin S(X, Y)$.

The oriented matroid $\mathcal{M} = \mathcal{M}(\mathcal{Z})$ is the (realizable) oriented matroid associated to $\mathcal{Z}$. It is *uniform* if the generators of $\mathcal{Z}$ are in a general position, i.e. if any d generators or less are linearly independent. The *cocircuits* of an oriented matroid $\mathcal{M}$ are the covectors which cover 0 in the partial order inherited from Λ_n, so that the cocircuits of $\mathcal{M}(\mathcal{Z})$ are the sign vectors of the facets of $\mathcal{Z}$.

If $\mathcal{Z}_X$ is a facet of $\mathcal{Z}$ and u is a vector in $\mathbb{R}^d$ then there exists a unique sign σ such that $(X, \sigma) \in \Lambda_{n+1}$ is the sign vector of a facet of the zonotope generated by $v_1, \ldots, v_n$ and $v_{n+1} = u$. We say that $\mathcal{Z}_X$ is a *lower* or *upper* facet of $\mathcal{Z}$ in the direction of u, or that $\mathcal{Z}_X$ is *parallel* to u if $\sigma = -, +$ or 0, respectively.

2.3. ZONOTOPAL SUBDIVISIONS

Two subzonotopes $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ of $\mathcal{Z}$ are said to *intersect properly* if either $Z_X \cap Z_Y = \emptyset$ or there exists a sign vector C such that

(i) $Z_X \cap Z_Y = Z_C$ and
(ii) $\mathcal{Z}_C$ is a face of both $\mathcal{Z}_X$ and $\mathcal{Z}_Y$.

In the latter case we have $C \geqslant X, Y$ and, in fact, $C = X \circ Y = Y \circ X$.

A *zonotopal subdivision* S of $\mathcal{Z}$ is a set of subzonotopes of $\mathcal{Z}$ with the following properties:

(i) if $\mathcal{Z}_C$ is a face of $\mathcal{Z}_X \in S$ then $\mathcal{Z}_C \in S$,
(ii) any two elements of S intersect properly and
(iii) $\bigcup_{\mathcal{Z}_X \in S} Z_X = Z$.

The d-dimensional elements of S are called *tiles*. Two tiles of S are said to be *adjacent* if they share a facet. The ith *zone* of S is the collection of elements $\mathcal{Z}_X$ of S with $X_i = 0$. The set of zonotopal subdivisions of $\mathcal{Z}$ is partially ordered by refinement: $S \leqslant T$ if and only if for any $\mathcal{Z}_X \in S$ there exists a $\mathcal{Z}_Y \in T$ with $X \geqslant Y$. The poset obtained by removing the maximum element, the *trivial* or *improper* subdivision consisting only of $\mathcal{Z}$ and its faces, is the *Baues poset* $\omega(\mathcal{Z})$. Its minimal elements are the *cubical subdivisions*, each tile of which is affinely equivalent to the d-cube.

Let $\mathcal{Z}' = (V', Z')$ be a d'-dimensional zonotope with n generators. A linear map $\pi: \mathbb{R}^{d'} \to \mathbb{R}^d$ which sends the ith generator of $\mathcal{Z}'$ to the corresponding generator of $\mathcal{Z}$ for each i is called a linear projection of zonotopes, denoted by $\pi: \mathcal{Z}' \to \mathcal{Z}$. Given such a projection, a zonotopal subdivision S of $\mathcal{Z}$ is *induced by* π, or π-*induced*, if for each element (equivalently tile) $\mathcal{Z}_X$ of S, $\mathcal{Z}'_X$ is a face of $\mathcal{Z}'$. The subposet of $\omega(\mathcal{Z})$ consisting of the proper π-induced subdivisions of $\mathcal{Z}$ is the *Baues poset* $\omega(\mathcal{Z}' \xrightarrow{\pi} \mathcal{Z})$. This poset reduces to $\omega(\mathcal{Z})$ if the generators of $\mathcal{Z}'$ are linearly independent. Moreover, $\omega(\mathcal{Z}' \xrightarrow{\pi} \mathcal{Z})$ is isomorphic to the poset of π-induced subdivisions of the projection $\pi: Z' \to Z$, as defined in [4] or [25, §9.1] for an arbitrary affine surjection of polytopes.

2.4. DELETION AND CONTRACTION

Let $\mathcal{Z}$ be generated by V and $v = v_n$ be its last generator. For $X \in \Lambda_n$ we denote by $X - n$ the sign vector obtained from X by omitting its last coordinate.

The *deletion* $\mathcal{Z} - v$ is the zonotope generated by the vector configuration $V - v = (v_1, \ldots, v_{n-1})$. Its oriented matroid is the deletion of n from that of $\mathcal{Z}$, in the sense that

$$\mathcal{L}(\mathcal{Z} - v) = \{X - n: X \in \mathcal{L}(\mathcal{Z})\}.$$

Given a zonotopal subdivision S of $\mathcal{Z}$, the *deletion* $S \backslash v$ is the collection of subzonotopes of $\mathcal{Z} - v$ defined by

$$\sigma(S \backslash v) = \{X - n: \mathcal{Z}_X \in S\}.$$

This collection $S \backslash v$ is a zonotopal subdivision of $\mathcal{Z} - v$ (see [21, Lemma 4.2]), obtained by 'contracting' the nth zone of S.

The following lemma includes some elementary properties of deletion.

LEMMA 2.1. *Let S, T denote zonotopal subdivisions of $\mathcal{Z}$, v be the last generator of $\mathcal{Z}$ and $\pi: \mathcal{Z}' \to \mathcal{Z}$ be a linear projection of zonotopes.*

(i) *If S refines T then $S \backslash v$ refines $T \backslash v$.*
(ii) *If v is neither a loop nor a coloop of $\mathcal{Z}$ then there are exactly two proper subdivisions S of $\mathcal{Z}$ such that $S \backslash v$ is the trivial subdivision of $\mathcal{Z} - v$.*
(iii) *If S is induced from π and v' is the last generator of $\mathcal{Z}'$ then $S \backslash v$ is induced from $\pi: \mathcal{Z}' - v' \to \mathcal{Z} - v$.*

Proof. (i) is clear, (iii) follows from the fact that if X is a covector of $\mathcal{Z}'$ then $X - n$ is a covector of $\mathcal{Z}' - v'$ and (ii) is a special case of Lemma 4.1 (see Section 4). □

Let H be a subspace in $\mathbb{R}^d$ complementary to the linear span of v, so that H is a linear hyperplane unless v is a loop, and let $\pi_0: \mathbb{R}^d \to H$ be the projection parallel to v. The *contraction* $\mathcal{Z}/v$ of $\mathcal{Z}$ by v is the zonotope generated by the vectors $\pi_0(v_i)$ for $1 \leqslant i < n$. Unless v is a loop of $\mathcal{Z}$, we have $\dim(\mathcal{Z}/v) = d - 1$. The oriented matroid of $\mathcal{Z}/v$ is the contraction of that of $\mathcal{Z}$ by n, in the sense that

$$\mathcal{L}(\mathcal{Z}/v) = \{X - n : X \in \mathcal{L}(\mathcal{Z}), X_n = 0\}.$$

Given any set of subzonotopes S of $\mathcal{Z}$, we define the collection S/v of subzonotopes of $\mathcal{Z}/v$ by

$$\sigma(S/v) = \{X - n : \mathcal{Z}_X \in S, X_n = 0\}.$$

If S is a zonotopal subdivision of $\mathcal{Z}$ then S/v is a zonotopal subdivision of $\mathcal{Z}/v$ (see [21, Lemma 4.2]) called the *contraction of S by v.*

Deletions and contractions of other generators can be defined by relabelling, and of more than one generators by iteration.

2.5. SHELLABILITY

We will use the notion of shellability for polytopal complexes described in [25, §8.1]. More specifically, such a complex $\mathcal{C}$ is *shellable* if it is pure of dimension d, i.e. each inclusion-maximal polytope in $\mathcal{C}$ has dimension d, and there exists a linear ordering $P_1, P_2, \ldots, P_k$ of its d-polytopes, called a *shelling*, such that for $1 < j \leqslant k$,

$$P_j \cap \left(\bigcup_{1 \leqslant i < j} P_i \right) \tag{1}$$

is nonempty and equals the union of the $(d-1)$-polytopes in an initial segment of a shelling of the boundary of P_j.

We call a pure d-dimensional polytopal complex $\mathcal{C}$ *visibly shellable* if there exists a linear ordering $P_1, P_2, \ldots, P_k$ of its d-polytopes such that for all $1 < j \leqslant k$, the set (1) is the union of the elements of a *visible* set of facets of P_j, i.e. the set of all facets of P_j which are visible from a fixed point outside P_j and in the affine hull of P_j but not in the affine hull of any facet of P_j. It follows from the construction of line shellings of Bruggesser and Mani [9], [25, §8.2] that any visibly shellable polytopal complex is shellable.

In the following lemma we assume that the generator v of the zonotope $\mathcal{Z}$ is neither a loop nor a coloop of $\mathcal{Z}$.

LEMMA 2.2. *Let $\mathcal{F}$ be a set of facets of $\mathcal{Z}$ in the zone corresponding to the generator v. If $\mathcal{F}/v$ is a visible set of facets of $\mathcal{Z}/v$ and the length of v is chosen to be sufficiently large then $\mathcal{F}$ is a visible set of facets of $\mathcal{Z}$.*

Proof. Let H be the linear span of $\mathcal{Z}/v$, as in Section 2.4, and let $\mathcal{F}/v$ be the set of facets of $\mathcal{Z}/v$ visible from $x \in H$. If the length of v is sufficiently large then no lower or upper facet of $\mathcal{Z}$ in the direction of v is visible from x and the set of facets of $\mathcal{Z}$ visible from this point is $\mathcal{F}$. □

2.6. POSET TOPOLOGY

When refering to the topology of a finite poset (short for partially ordered set) ω we mean the topology of the geometric realization of its *order complex* $\Delta(\omega)$, i.e. the simplicial complex of chains (totally ordered subsets) of ω [6]. For $x \in \omega$ we write $\omega_{\leqslant x} = \{y \in \omega : y \leqslant x\}$. The following variation of the Quillen Fiber Lemma [6, Theorem 10.5] will be useful in relating the homotopy type of posets. A simple proof (due to A. Björner) can be found in [23, Section 3].

LEMMA 2.3. (Babson [3]). *Let $f : \omega \to \omega'$ be an order preserving map of posets. If*

(i) *$f^{-1}(y)$ is contractible for every $y \in \omega'$ and*
(ii) *$\omega_{\leqslant x} \cap f^{-1}(y)$ is contractible for every $x \in \omega$ and $y \in \omega'$ with $f(x) > y$*

then f induces a homotopy equivalence.

2.7. CYCLIC ZONOTOPES

The *cyclic zonotope* $\mathcal{Z}(n, d)$ is the d-zonotope generated by the n vectors $u_i = (1, t_i, \ldots, t_i^{d-1})$ in $\mathbb{R}^d$, where $t_1 < t_2 < \cdots < t_n$. Although $\mathcal{Z}(n, d)$ depends on the set of parameters $\{t_1, \ldots, t_n\}$, its face poset depends only on n and d. More precisely, given $X = (X_1, X_2, \ldots, X_n) \in \Lambda_n$, let

$$\widetilde{X}_i = \begin{cases} 0, & \text{if } X_i = 0, \\ X_i \cdot (-1)^{m_i}, & \text{otherwise,} \end{cases} \tag{2}$$

where m_i is the number of indices $j > i$ with $X_j = 0$. Let m_X be the sum of the number of zero entries of X and the number of pairs $i < j$ such that $\widetilde{X}_j = -\widetilde{X}_i \neq 0$ and $X_r = 0$ for all r with $i < r < j$, if any. The next proposition is a rephrasing of [1, Proposition 3.2] and is equivalent to Gale's evenness criterion [25, p. 14].

PROPOSITION 2.4. *For $\mathcal{Z} = \mathcal{Z}(n, d)$ and $X \in \Lambda_n$, $\mathcal{Z}_X$ is a proper face of $\mathcal{Z}$ if and only if $m_X \leqslant d - 1$.*

The oriented matroid of $\mathcal{Z}(n, d)$ is known as the *alternating oriented matroid* [7, §9.4], denoted by $C^{n,d}$. It follows from the Bohne–Dress Theorem [8, 21] that the Baues poset $\omega(\mathcal{Z}(n, d))$ of proper zonotopal subdivisions of $\mathcal{Z}(n, d)$ depends only on the oriented matroid $\mathrm{C}^{n,d}$, and hence only on n and d. We denote this poset by $\omega(n, d)$. For $1 \leqslant d < d' \leqslant n$, the map $\pi: \mathbb{R}^{d'} \to \mathbb{R}^d$ which 'forgets' the last $d' - d$ coordinates restricts to a linear projection of zonotopes

$$\pi: \mathcal{Z}(n, d') \to \mathcal{Z}(n, d), \tag{3}$$

where the same set of parameters is used to define the two cyclic zonotopes. We denote by $\omega_{d'}(n, d)$ the Baues poset of the projection (3), which depends only on n, d' and d. Note that $\omega_{d'}(n, d)$ reduces to $\omega(n, d)$ if $d' = n$.

We conclude with the following characterization of the facets of $\mathcal{Z}(n, d)$. Let $u = (1, t, \ldots, t^{d-1})$, where $t > t_n$. We call $X \in \Lambda_n$ *positive* (resp. *negative*) if $X_i = +$ (resp. $X_i = -$) for all i with $X_i \neq 0$.

COROLLARY 2.5. *Let $\mathcal{Z} = \mathcal{Z}(n, d)$ and $X \in \Lambda_n$.*

(i) *$\mathcal{Z}_X$ is a facet of $\mathcal{Z}$ if and only if X has exactly $d - 1$ zero entries and $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ is either positive or negative.*
(ii) *A facet $\mathcal{Z}_X$ of $\mathcal{Z}$ is an upper (resp. lower) facet in the direction of u if and only if $\widetilde{X}$ is positive (resp. negative).*

Proof. Part (i) follows immediately from Proposition 2.4. For part (ii) observe that a facet $\mathcal{Z}_X$ of $\mathcal{Z}(n, d)$ is a lower facet in the direction of u if and only if $(X, -)$ is the sign vector of a (lower) facet of $\mathcal{Z}(n + 1, d)$, generated by $u_1, \ldots, u_n$ and $u_{n+1} = u$, and apply part (i). □

3. Stackability and Shellability

In this section we define the concept of stackability for zonotopal subdivisions and use it to prove Theorem 1.2. Our proof is valid for any zonotope having an alternating oriented matroid, although we will use the cyclic zonotopes as specific realizations.

Let $u \in \mathbb{R}^d$ be a nonzero vector, $\mathcal{Z}$ be a d-zonotope in $\mathbb{R}^d$ and S be a zonotopal subdivision of $\mathcal{Z}$. If τ_1 and τ_2 are two adjacent tiles of S, we say that τ_1 and τ_2 are *parallel*, or that τ_1 lies *below* or *above* τ_2, in the direction of u if their common facet is parallel to u, or is a lower or upper facet of τ_2 in the direction of u, respectively.

DEFINITION 3.1. The subdivision S is stackable in the direction of u if there exists a function κ on the set of tiles of S such that for adjacent tiles τ_1, τ_2 of S we have:

(i) $\kappa(\tau_1) < \kappa(\tau_2)$ if τ_1 lies below τ_2 in the direction of u and
(ii) $\kappa(\tau_1) = \kappa(\tau_2)$ if τ_1, τ_2 are parallel in the direction of u.

If no two adjacent tiles of S are parallel in the direction of u, Definition 3.1 simply requires that there exists a linear ordering $\tau_1, \tau_2, \dots, \tau_k$ of the tiles of S such that whenever τ_i and τ_j are adjacent and τ_i lies below τ_j in the direction of u we have $i < j$. Stackability of zonotopal subdivisions of $\mathcal{Z}$ is valid in any direction if $\mathcal{Z}$ has dimension $d \leqslant 2$ or codimension $n - d \leqslant 2$ but may fail otherwise, see Remark 6.4.

The next lemma follows from the results of [23, Section 4] (see also [24, Proposition 3.6] and Section 6). It will be crucial in the proofs of Theorems 1.1 and 1.2. Here we give an alternative, direct proof.

LEMMA 3.2. *Let $t_1 < t_2 < \cdots < t_n$ be the parameters defining the cyclic zonotope $\mathcal{Z}(n, d)$. If $u = (1, t, \dots, t^{d-1})$ and $t > t_n$ then every zonotopal subdivision of $\mathcal{Z}(n, d)$ is stackable in the direction of u.*

Proof. Let S be a zonotopal subdivision of $\mathcal{Z} = \mathcal{Z}(n, d)$ and $\mathcal{X}$ be the set of sign vectors $X \in \Lambda_n$ such that $\mathcal{Z}_X$ is a tile of S. For $X = (X_1, X_2, \dots, X_n) \in \mathcal{X}$ let $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_n)$, where $\widetilde{X}_i$ is defined in (2). For $X, Y \in \mathcal{X}$ we let $X \prec Y$ if $\widetilde{X}$ precedes $\widetilde{Y}$ in the reverse lexicographic order on Λ_n induced by $- < 0 < +$, so that '$\prec$' is a linear order on $\mathcal{X}$. To prove the proposition we show that if $X, Y \in \mathcal{X}$ and $\mathcal{Z}_X$ is adjacent to and lies below $\mathcal{Z}_Y$ in the direction of u, then $X \prec Y$.

Indeed, let X, Y be as before, with $X = (X_1, X_2, \dots, X_n)$ and $Y = (Y_1, Y_2, \dots, Y_n)$. Since $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ are adjacent we have $X_j = Y_j$ whenever both entries are nonzero. Hence, if i is the largest index with $\widetilde{X}_i \neq \widetilde{Y}_i$ then either $X_i = 0$ or $Y_i = 0$; say the former holds, so that $\widetilde{X}_i = 0$. We need to show that $\widetilde{Y}_i = +$. Let $\mathcal{Z}_C$ be the common facet of $\mathcal{Z}_X$ and $\mathcal{Z}_Y$ with $C = (C_1, C_2, \dots, C_n)$, so that $C = X \circ Y$. Since $\mathcal{Z}_C$ is an upper facet of $\mathcal{Z}_X$, by Lemma 2.5(ii) applied to $\mathcal{Z}_X$ we have $C_i \cdot (-1)^m = +$, where m is the number of $j > i$ with $C_j = 0$. Since $C_i = Y_i$ and for $j > i$, $C_j = 0$ if and only if $Y_j = 0$, we get $\widetilde{Y}_i = +$, as desired. A similar argument applies in the case that $Y_i = 0$. □

The following proposition relates the concepts of shellability and stackability. Let $\mathcal{Z}$ and $v = v_n$ be as in Section 2.4.

PROPOSITION 3.3. *Suppose that the generators of $\mathcal{Z}$ are in general position and that v is not a coloop of $\mathcal{Z}$. Let S be a zonotopal subdivision of $\mathcal{Z}$ such that:*

(i) *S/v is visibly shellable and*
(ii) *$S \backslash v$ is stackable in the direction of v.*

Then S is visibly shellable, if the length of v is sufficiently large.

Proof. Let $\mathcal{X}$ be the set of sign vectors of the tiles of S and for $\sigma \in \{-, 0, +\}$ let $\mathcal{X}^\sigma$ be the subset of those sign vectors X with $X_n = \sigma$. Let $\prec_0$ be the linear order of

$\mathcal{X}^0$ induced by a visible shelling of S/v and $\prec$ be a linear order of the set of tiles of $S\backslash v$ such that if τ_1 is adjacent to and lies below τ_2 in the direction of v for such tiles, then $\tau_1 \prec \tau_2$. The order $\prec$ induces a linear order on $\mathcal{X}^- \cup \mathcal{X}^+$. Let $\prec_-$ and $\prec_+$ be the restriction of this order on $\mathcal{X}^-$ and $\mathcal{X}^+$, respectively. We linearly order $\mathcal{X}$ by first ordering $\mathcal{X}^0$ according to $\prec_0$, then ordering $\mathcal{X}^+$ by $\prec_+$ and finally $\mathcal{X}^-$ by the reverse of $\prec_-$. We will show that the resulting linear order of $\mathcal{X}$ is a visible shelling of S.

Let $\tau = \mathcal{Z}_X$ be any tile of S other than the first and let F_τ be the intersection of Z_X with the union of the convex hulls of the previous tiles in the order. We need to verify that F_τ is the union of the elements of a visible set of facets of τ. If $X \in \mathcal{X}^0$ then τ and all previous tiles in the order are in the nth zone of S and can intersect only along common faces in this zone. The claim for F_τ then follows from Lemma 2.2, our choice of $\prec_0$ and our assumption on the length of v. If $X \in \mathcal{X}^+$ then a lower facet of τ in the direction of v is shared by a tile with sign vector either in $\mathcal{X}^0$ or $\mathcal{X}^+$ and an upper facet of τ is either an upper facet of $\mathcal{Z}$ or a facet of another tile with sign vector in $\mathcal{X}^+$. Similar remarks apply if $X \in \mathcal{X}^-$. It follows by our choice of $\prec_+$ and $\prec_-$ that F_τ is the union of all lower or upper facets of τ, respectively, in the direction of v. These two sets of facets are clearly visible. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let S be a zonotopal subdivision of $\mathcal{Z}(n, d)$. Since shellability of S is not affected by rescaling the generators of $\mathcal{Z}(n, d)$, we will assume that the length of the ith generator is sufficiently large with respect to the previous generators and prove that S is visibly shellable.

We proceed by induction on n. The statement is clear if $d = 1$ or $n = d$, so suppose that $2 \leqslant d < n$ and let v be the nth generator of $\mathcal{Z}(n, d)$. By the induction hypothesis, and since the contraction of $\mathcal{Z}(n, d)$ by v has the oriented matroid of $\mathcal{Z}(n-1, d-1)$, we can assume that S/v is visibly shellable. Since $S\backslash v$ is stackable in the direction of v by Lemma 3.2, Proposition 3.3 implies that S is visibly shellable. □

In view of the fact that nonshellable triangulations of balls exist (see, for instance, [25, §8.1]), Theorem 1.2 suggests the following question.

QUESTION 3.4. Is every zonotopal subdivision shellable?

4. Cellular Sections

In what follows $\mathcal{Z}$ is a d-zonotope generated by $V = (v_1, v_2, \ldots, v_n)$, $v = v_n$ is neither a loop nor a coloop of $\mathcal{Z}$ and T is a zonotopal subdivision of $\mathcal{Z} - v$. Let H and the projection π_0 parallel to v be as in Section 2.4 and let U_T be the union of the convex hulls of the facets of the tiles in T. A *geometric* section of T in the direction of

v is a continuous (necessarily piecewise linear) map $h\colon \pi_0(Z_X) \to U_T$ such that

$$h(x) \in \pi_0^{-1}(x) \text{ for } x \in \pi_0(Z_X).$$

A geometric section h divides the tiles of T into those lying below (the image of) h and those lying above it.

A *cellular section*, or simply *section*, of T in the direction of v is a (weak) partition s of the set of tiles of T into the three (possibly empty) sets of tiles lying below h_1, between h_1, h_2 and above h_2 for some pair of geometric sections h_1, h_2 for which $h_1(x) \leqslant h_2(x)$ for all x and such that the line $\pi_0^{-1}(x)$ intersects the interior of at most one tile between h_1 and h_2. We denote by below(s), on(s) and above(s) these parts, respectively, consisting of the tiles lying 'below', 'on' or 'above' s.

We denote by $\mathrm{Sec}_v(T)$ the set of cellular sections of T in the direction of v and consider two partial orders on $\mathrm{Sec}_v(T)$. The *refinement order* $\omega_v(T)$ is defined by letting $s_1 \leqslant s_2$ if below$(s_2) \subseteq$ below(s_1) and above$(s_2) \subseteq$ above(s_1). We say that a section s_1 lies *below* a section s_2 if below$(s_1) \subseteq$ below(s_2) and on$(s_1) \subseteq$ below$(s_2) \cup$ on(s_2). We call the resulting partial order on $\mathrm{Sec}_v(T)$ the *stackability* order in the direction of v defined by T and denote it by $\mathrm{St}_v(T)$; see also [2, Definition 4.8].

Let S be a zonotopal subdivision of $\mathcal{Z}$ with $S \backslash v = T$. It follows from [21, Lemma 3.2] that, after deleting v, the lower and upper facets in the direction of v of the tiles in the nth zone of S define two geometric sections of T in the same direction which give rise to a cellular section $s_v(S)$ in $\mathrm{Sec}_v(T)$. The sets of tiles lying below, on or above $s_v(S)$ are precisely the sets of tiles of T whose sign vector Y is such that $(Y, -), (Y, 0)$ or $(Y, +)$, respectively, is the sign vector of a tile of S.

LEMMA 4.1. *Given $T \in \omega(\mathcal{Z} - v)$, the map s_v is an isomorphism between the refinement poset of zonotopal subdivisions S of $\mathcal{Z}$ with $S \backslash v = T$ and $\omega_v(T)$.*

Proof. Given $s \in \mathrm{Sec}_v(T)$, let $\tau(s)$ be the set consisting of the subzonotopes of $\mathcal{Z}$ whose sign vectors (Y, σ) are one of the following two kinds: first, the ones for which Y is the sign vector of a tile of T which lies below, on or above s if $\sigma = -$, 0 or +, respectively and, second, the ones for which $\sigma = 0$ and Y is the sign vector either

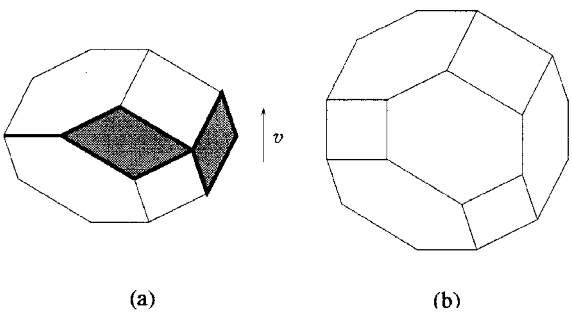

Figure 2. (a) A cellular section s of a zonotopal subdivision in the direction of v. (b) The subdivision $s_v^{-1}(s)$.

of a common facet of two tiles of T, one lying below and the other above s, or of a lower facet of $\mathcal{Z} - v$ in the direction of v which is also a facet of a tile of T lying above s, or of an upper facet of $\mathcal{Z} - v$ in the direction of v which is also a facet of a tile of T lying below s. Let $s_v^{-1}(s)$ be the collection of faces of the elements of $\tau(s)$. It is straightforward to verify that the tiles in $\tau(s)$ intersect properly and cover Z. The maps s_v and s_v^{-1} are inverses of each other and order preserving by construction. □

We now assume that T is stackable in the direction of v and that the generators of $\mathcal{Z}$ are in general position. We denote by $\mathrm{st}_v(T)$ the partial order on the set of tiles of T in which τ_1 is covered by τ_2 if τ_1 is adjacent to and lies below τ_2 in the direction of v. Given any order ideal J in $\mathrm{st}_v(T)$, it is clear (by induction on the cardinality of J) that the upper envelope of J defines a geometric section of T in the direction of v. Hence, in this case, a partition s of the set of tiles of T into the sets below(s), on(s) and above(s) defines a cellular section of T in the direction of v if and only if below(s) and below(s) $\cup$ on(s) are order ideals and on(s) is an antichain in $\mathrm{st}_v(T)$. The proof of the next lemma follows that of [2, Proposition 4.11] in the case of subdivisions of cyclic polytopes (see also [23, Corollary 3.10] and [19, Theorem 4.5]).

LEMMA 4.2. *Suppose that the generators of* $\mathcal{Z}$ *are in general position and that* $T \in \omega(\mathcal{Z} - v)$ *is stackable in the direction of* v. *If* $I = [s_1, s_2]$ *is a closed interval in* $\mathrm{St}_v(T)$, *with* on(s_1) = on(s_2) = $\emptyset$, *then* I *is contractible in* $\omega_v(T)$.

Proof. The sets below(s_1) $\subseteq$ below(s_2) are order ideals in the partial order $\mathrm{st}_v(T)$. We can order the set of tiles in below(s_2) − below(s_1) as $\tau_1, \tau_2, \ldots, \tau_k$ so that if $\tau_i < \tau_j$ in $\mathrm{st}_v(T)$ then $i < j$. Let I_i be the set of sections in I for which all tiles τ_j with $j > i$ lie above s, so that $I_k = I$ and $I_0 = \{s_1\}$ is contractible.

One can show that I_i and I_{i-1} are homotopy equivalent as subposets of $\omega_v(T)$ for $1 \leqslant i \leqslant k$ by defining maps $g_i: I_{i-1} \to I_i$ and $f_i: I_i \to I_{i-1}$ as follows. Given $s \in I_{i-1}$, let $g_i(s) = s$ if τ_i is not a minimal element of on(s) $\cup$ above(s) in $\mathrm{st}_v(T)$ and $g_i(s)$ have below(s) and on(s) $\cup \{\tau_i\}$ as the sets of tiles lying below and on it, respectively, if otherwise. Given $s \in I_i$, let $f_i(s) = s$ if $s \in I_{i-1}$ and $f_i(s)$ have below(s) − $\{\tau_i\}$ and on(s) − $\{\tau_i\}$ as the sets of tiles lying below and on it, respectively, if otherwise. Then f_i and g_i are well defined, order preserving and satisfy $f_i \circ g_i = \mathrm{id}_{I_{i-1}}$, $g_i \circ f_i \geqslant \mathrm{id}_{I_i}$. It follows from Quillen's order homotopy theorem [6, Theorem 10.11] that f_i is a homotopy inverse of g_i. □

5. The Projections Between Cyclic Zonotopes

In this section we prove Theorem 1.1. Our argument is a combination of those used in the proofs of the main results of [23] and [2].

Throughout this section, $\pi: \mathcal{Z}(n, d') \to \mathcal{Z}(n, d)$ is the natural projection (3) between cyclic zonotopes, where $1 \leqslant d < d' \leqslant n$, and v', v are the nth generators of $\mathcal{Z}(n, d')$, $\mathcal{Z}(n, d)$, respectively, so that $\pi(v') = v$. We denote by $\widehat{\omega}_{d'}(n-1, d)$ the Baues poset $\omega_{d'}(n-1, d)$, if $n > d'$, and the poset $\omega(n-1, d)$ augmented by two incomparable elements, say ρ^- and ρ^+, which lie above every element of $\omega(n-1, d)$, if $n = d'$. Let S_v^- and S_v^+ be the two subdivisions of $\mathcal{Z}(n, d)$ defined by Lemma 2.1(ii).

PROPOSITION 5.1. *The deletion map*

$$\Pi_n\colon\ \omega_{d'}(n, d) \to \widehat{\omega}_{d'}(n-1, d),$$

defined by

$$\Pi_n(S) = S \backslash v,$$

if $n > d'$, and by

$$\Pi_n(S) = \begin{cases} \rho^- & \text{if } S = S_v^-, \\ \rho^+ & \text{if } S = S_v^+, \\ S\backslash v & \text{otherwise}, \end{cases}$$

if $n = d'$, is well-defined and order preserving.

Proof. If $n = d'$ then $\mathcal{Z}(n, d')$ is affinely equivalent to an n-cube and S_v^- and S_v^+ are clearly π-induced. By Lemma 2.1(ii) and (iii), to show that the maps are well-defined it suffices to show that in the case $n > d'$, if S is proper then so is $S\backslash v$. Indeed, if $S\backslash v$ is the trivial subdivision then S contains a tile τ which has all generators of $\mathcal{Z}(n, d)$ other than v. Then the face τ' of $\mathcal{Z}(n, d')$ with $\pi(\tau') = \tau$ has all generators of $\mathcal{Z}(n, d')$ other than v'. Since these are $n - 1 \geqslant d'$ vectors in general position in $\mathbb{R}^{d'}$, this implies that $\dim(\tau') = d'$, so $\tau' = \mathcal{Z}(n, d')$, $\tau = \mathcal{Z}(n, d)$ and S is the trivial subdivision of $\mathcal{Z}(n, d)$.

The map Π_n is order preserving by Lemma 2.1(i). □

The proof of the following proposition will occupy most of the present section.

PROPOSITION 5.2. *The map Π_n induces a homotopy equivalence between the posets $\omega_{d'}(n, d)$ and $\widehat{\omega}_{d'}(n-1, d)$.*

Theorem 1.1 is a straightforward consequence of Proposition 5.2, as we show next.

Proof of Theorem 1.1. We fix $d \geqslant 1$ and proceed by induction on n. The result is trivial for $n = d + 1$, since this implies that $d' - d = 1$, so assume $n - d \geqslant 2$. By induction, the poset $\widehat{\omega}_{d'}(n-1, d)$ is homotopy equivalent to a sphere of dimension $d' - d - 1$, if $n > d'$, and to the suspension of a sphere of dimension $n - d - 2 = d' - d - 2$, if $n = d'$. In either case, Proposition 5.2 implies that $\omega_{d'}(n, d)$ is homotopy equivalent to a $(d' - d - 1)$-sphere. □

To prove Proposition 5.2 we will make use of Lemma 2.3, so we need to study the fibers of the map Π_n. We fix an element T of $\widehat{\omega}_{d'}(n-1, d)$ other than ρ^-, ρ^+ and an element S of $\omega_{d'}(n, d)$. Let $\mathcal{L}'$ denote the set of covectors of the oriented matroid of $\mathcal{Z}(n, d')$, described in Proposition 2.4, and $Y \in \Lambda_{n-1}$. We will use repeatedly the following statement (valid for any oriented matroid by axiom (L3)):

(L3$'$) if $(Y, -)$ and $(Y, +)$ are in $\mathcal{L}'$ then so does $(Y, 0)$.

In fact, if any two of $(Y, -), (Y, 0)$ and $(Y, +)$ are in $\mathcal{L}'$ then so does the third.

Let $T_0 = S \backslash v$ and $s_0 = s_v(S)$, so that s_0 is a section in $\mathrm{Sec}_v(T_0)$. If $T \leqslant T_0$ we say that a tile τ of T lies below, on or above s_0 if the unique tile of T_0 which contains τ has the corresponding property. We say that a section $s \in \mathrm{Sec}_v(T)$ is *contained* in s_0 if any tile of T which lies below or above s_0 also lies below or above s, respectively.

For $\sigma \in \{-, 0, +\}$ we denote by J_σ the set of tiles τ of T such that if Y is the sign vector of T then $(Y, \sigma) \in \mathcal{L}'$. The following lemma does not depend on the specific projection π of zonotopes under consideration.

LEMMA 5.3. (i) *The map s_v of Lemma 4.1 restricts to an isomorphism between $\Pi_n^{-1}(T)$ and the subposet of $\omega_v(T)$ consisting of all sections $s \in \mathrm{Sec}_v(T)$ such that for each tile τ of T,*

$$\tau \in \begin{cases} J_- & \text{if } \tau \text{ lies below } s, \\ J_0 & \text{if } \tau \text{ lies on } s, \\ J_+ & \text{if } \tau \text{ lies above } s. \end{cases} \tag{4}$$

(ii) *Let $T_0 = S \backslash v$, $s_0 = s_v(S)$ and assume that $T \leqslant T_0$. The map s_v restricts to an isomorphism between $\Pi_n^{-1}(T) \cap \omega_{d'}(n, d)_{\leqslant S}$ and the subposet of $\omega_v(T)$ consisting of all sections $s \in \mathrm{Sec}_v(T)$ contained in s_0 such that* (4) *holds for each tile τ of T which lies on s_0.*

Proof. (i) By Lemma 4.1, it suffices to check that a section $s \in \mathrm{Sec}_v(T)$ satisfies (4) if and only if $s_v^{-1}(s)$ is π-induced or, equivalently, if $X \in \mathcal{L}'$ for all X such that $\mathcal{Z}_X$ is a tile of $s_v^{-1}(s)$. Let ρ denote a subzonotope of $\mathcal{Z} - v$ with sign vector C. By definition of $s_v^{-1}(s)$, we only need to show that if (4) holds for each tile of T and ρ is a common facet of two tiles $\tau_1 \in \mathrm{below}(s)$ and $\tau_2 \in \mathrm{above}(s)$ of T, or ρ is a lower facet of $\mathcal{Z} - v$ in the direction of v and a facet of a tile $\tau_2 \in \mathrm{above}(s)$ of T, or ρ is an upper facet of $\mathcal{Z} - v$ and a facet of a tile $\tau_1 \in \mathrm{below}(s)$ of T then $(C, 0) \in \mathcal{L}'$. Indeed, let Y and W be the sign vectors of τ_1 and τ_2, respectively. In the first case we have $C = Y \circ W = W \circ Y$ and, by (4), $(Y, -)$ and $(W, +)$ are in $\mathcal{L}'$. It follows that $(C, -) = (Y, -) \circ (W, +)$ and $(C, +) = (W, +) \circ (Y, -)$ are also in $\mathcal{L}'$ and by (L3$'$), so does $(C, 0)$. In the second case we have $W < C$ and $(W, +) \in \mathcal{L}'$, as before. Since $\mathcal{Z}_C$ is a lower facet of $\mathcal{Z} - v$, $(C, -)$ is a covector of $\mathcal{Z}$. Hence $(C, -)$ and $(C, +) = (W, +) \circ (C, -)$ are in $\mathcal{L}'$ and so does $(C, 0)$, again by (L3$'$). The third case is similar to the second.

(ii) It is straightforward to verify that for $s \in \mathrm{Sec}_v(T)$, $s_v^{-1}(s)$ refines S if and only if s is contained in s_0. Indeed, to check one direction, suppose that $s_v^{-1}(s)$ refines S and let τ be a tile of T with sign vector Y which lies below s_0. Let $\sigma \in \{-, 0, +\}$ be such that (Y, σ) is the sign vector of a tile of $s_v^{-1}(s)$. To show that τ lies below s we need to show that $\sigma = -$. By assumption, we have $(Y, \sigma) \geqslant X$ for some $X \in \Lambda_n$ such that $\mathcal{Z}_X$ is a tile of S. Then $Y \geqslant X - n$ and $X - n$ is the sign vector of the unique tile τ_0 of T_0 containing τ. Since τ_0 lies below s_0 we must have $X_n = -$, which together with $(Y, \sigma) \geqslant X$ implies that $\sigma = -$. A similar argument applies if τ lies above s_0.

By part (i), it remains to show that if $s \in \mathrm{Sec}_v(T)$ is contained in s_0 then (4) holds for all tiles τ of T below or above s_0. Suppose that τ lies, say, below s_0 (and hence s) and let τ_0 be the unique tile of T_0 containing τ. Let Y and W be the sign vectors of τ and τ_0, respectively, so that $Y \geqslant W$. Since τ_0 lies below s_0 and S is induced from $\mathcal{Z}'$, we have $(W, -) \in \mathcal{L}'$. Since T is induced from $\mathcal{Z}(n-1, d')$, the deletion of $\mathcal{Z}(n, d')$ by v', at least one of $(Y, -)$, $(Y, 0)$ and $(Y, +)$ is in $\mathcal{L}'$. Since $(W, -) \circ (Y, +) = (Y, 0) \circ (W, -) = (Y, -)$, we conclude that $(Y, -) \in \mathcal{L}'$, in other words $\tau \in J_-$, as desired. □

The next lemma lists a few crucial properties of the sets J_-, J_0 and J_+ for the special projection $\pi: \mathcal{Z}(n, d') \to \mathcal{Z}(n, d)$.

LEMMA 5.4. *Let T be a subdivision in $\widehat{\omega}_{d'}(n-1, d)$ and let τ_1 and τ_2 denote tiles of T such that $\tau_1 < \tau_2$ in the partial order* $\mathrm{st}_v(T)$.

(i) *$J_- \cup J_+$ is the set of all tiles of T.*
(ii) $J_0 = J_- \cap J_+$.
(iii) *If $\tau_2 \in J_- - J_0$ then $\tau_1 \in J_-$.*
(iv) *If $\tau_1 \in J_+ - J_0$ then $\tau_2 \in J_+$.*

Proof. Note that all parts of the lemma are trivial if $n = d'$, since then $\mathcal{L}' = \Lambda_n$ and all three sets J_-, J_0 and J_+ reduce to the set of all tiles of T. Suppose that $n > d'$, so that $T \in \omega_{d'}(n-1, d)$ and let τ be a tile of T with sign vector Y. Since T is π-induced, Y is a covector of $\mathcal{Z}(n-1, d')$, so $m_Y \leqslant d' - 1$ by the characterization of Proposition 2.4. Parts (i) and (ii) now follow from this characterization and the observations that for each $Y \in \Lambda_{n-1}$ we have $m_{(Y,0)} = m_Y + 1$ and if $Y \neq 0$ then either $m_{(Y,+)} = m_Y$ and $m_{(Y,-)} = m_Y + 1$ or $m_{(Y,+)} = m_Y + 1$ and $m_{(Y,-)} = m_Y$.

We now prove (iv). Let $X, Y \in \Lambda_{n-1}$ be the sign vectors of τ_1 and τ_2, respectively, so that $m_X, m_Y \leqslant d' - 1$. Let $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_{n-1})$ and $\widetilde{Y} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_{n-1})$ be defined by (2) and note that $\widetilde{X} \prec \widetilde{Y}$ in the reverse lexicographic order on Λ_{n-1} induced by $- < 0 < +$, by the proof of Lemma 3.2. Since $\tau_1 \in J_+ - J_0$, we have $m_X = m_{(X,+)} = d' - 1$. The equality $m_X = m_{(X,+)}$ means precisely that $\widetilde{X}_i = +$ for the largest $i \leqslant n-1$ with $\widetilde{X}_i \neq 0$. Since $\widetilde{X} \prec \widetilde{Y}$ the same must be true for $\widetilde{Y}$, so $m_{(Y,+)} = m_Y \leqslant d' - 1$ and $\tau_2 \in J_+$, as desired. The case of (iii) is similar. □

Recall from Lemma 3.2 that T is stackable in the direction of v. We define two sections $s_{\min}$ and $s_{\max}$ in $\mathrm{Sec}_v(T)$ by letting below($s_{\min}$) be the order ideal in $\mathrm{st}_v(T)$ generated by $J_- - J_0$, above($s_{\max}$) be the dual order ideal generated by $J_+ - J_0$ and above($s_{\min}$) and below($s_{\max}$) be the complements of these ideals, respectively, so that on($s_{\min}$) and on($s_{\max}$) are empty. It follows from Lemma (iv) (iii) and (iv) that the intersection of below($s_{\min}$) and above($s_{\max}$) is empty and hence that $s_{\min} \leqslant s_{\max}$ in the stackability order $\mathrm{St}_v(T)$.

LEMMA 5.5. *If T is a subdivision in $\widehat{\omega}_{d'}(n-1, d)$ then $\Pi_n^{-1}(T)$ is isomorphic to the restriction of the refinement order $\omega_v(T)$ to the interval $[s_{\min}, s_{\max}]$ in $\mathrm{St}_v(T)$.*

Proof. From the definition of $s_{\min}$ and $s_{\max}$ we have above($s_{\min}$) $\subseteq J_+$ and below($s_{\max}$) $\subseteq J_-$. By Lemma 5.3(i) it suffices to check that a section $s \in \mathrm{Sec}_v(T)$ satisfies conditions (4) if and only if $s_{\min} \leqslant s \leqslant s_{\max}$ in the stackability order $\mathrm{St}_v(T)$. Indeed, if $s \in \mathrm{Sec}_v(T)$ satisfies (4) then on(s) $\cup$ above(s) $\subseteq J_0 \cup J_+ = J_+$ and, hence, $J_- - J_0 \subseteq$ below(s). Since below(s) is an order ideal in $\mathrm{st}_v(T)$ this implies that below($s_{\min}$) $\subseteq$ below(s), so $s_{\min} \leqslant s$ in $\mathrm{St}_v(T)$ and similarly $s \leqslant s_{\max}$. Conversely, if $s_{\min} \leqslant s \leqslant s_{\max}$ in $\mathrm{St}_v(T)$ then below(s) $\subseteq$ below($s_{\max}$) $\subseteq J_-$, above(s) $\subseteq$ above($s_{\min}$) $\subseteq J_+$ and on(s) $\subseteq$ above($s_{\min}$) $\cap$ below($s_{\max}$) $\subseteq J_+ \cap J_- = J_0$, so s satisfies (4). □

We now prove Proposition 5.2 and complete the proof of Theorem 1.1.

Proof of Proposition 5.2. Let $T \in \widehat{\omega}_{d'}(n-1, d)$ and $S \in \omega_{d'}(n, d)$. By Lemma 2.3 applied to the map Π_n, it suffices to show that the posets

(i) $\Pi_n^{-1}(T)$ and
(ii) $\Pi_n^{-1}(T) \cap \omega_{d'}(n, d)_{\leqslant S}$, if $T \leqslant \Pi_n(S)$

are contractible. If $n = d'$ and $T = \rho^-$ or ρ^+ then both posets are singletons, hence contractible, so we can assume that $T \neq \rho^-, \rho^+$. In view of Lemma 4.2, contractibility of (i) follows directly from Lemma 5.5. Finally, suppose that T refines $\Pi_n(S) = S\backslash v$ and let $T_0 = S\backslash v$ and $s_0 = s_v(S)$ be as in Lemma 5.3(ii). Let also $T|_{\tau_0}$ be the subdivision of a tile τ_0 of T_0 induced by T. Lemma 5.3(ii) implies that $\Pi_n^{-1}(T) \cap \omega_{d'}(n, d)_{\leqslant S}$ is isomorphic to the direct product of posets of the form $\Pi_n^{-1}(T|_{\tau_0})$, one for each tile τ_0 of T_0 lying on s_0 (where Π_n is applied to subdivisions of the appropriate subzonotope of $\mathcal{Z}(n, d)$). Each of these posets is contractible by part (i) and hence so is their product. □

6. Stackability and Euclideanness

In this section we relate the concept of stackability of zonotopal subdivisions to that of Euclideanness for oriented matroid programs. We will go over some basic definitions concerning oriented matroid programs and refer to [7, Chapter 10], the sources [12, 13] as well as [23, Section 3] for more information and background.

6.1. ORIENTED MATROID PROGRAMS

Here we assume familiarity with basic constructions and terminology about oriented matroids from [7].

An oriented matroid program [12] is a triple $(\widetilde{\mathcal{M}}, g, f)$ of an oriented matroid $\widetilde{\mathcal{M}}$ and elements g, f of its ground set such that g is not a loop and f is not a coloop. The program $(\widetilde{\mathcal{M}}, g, f)$ is called Euclidean [13] if for every cocircuit Y of $\mathcal{M} = \widetilde{\mathcal{M}} - f$ with $Y_g = +$ there exists an extension $\mathcal{M} \cup \hat{f}$ of $\mathcal{M}$ such that $(\mathcal{M} \cup \hat{f})\,/\,g = \widetilde{\mathcal{M}}/g$ and the sign vector Y^0 which extends Y with zero $\hat{f}$-coordinate is a cocircuit of $\mathcal{M} \cup \hat{f}$.

Following the conventions of [23], we define the graph G of the program $(\widetilde{\mathcal{M}}, g, f)$ on the vertex set of cocircuits Y of $\mathcal{M}$ with $Y_g = +$ as follows. The edges of G are the rank two covectors of $\mathcal{M}$ joining these cocircuits, some of which have an orientation. Specifically, given an edge joining Y and W, let X be the unique cocircuit of $\mathcal{M}$ obtained from W and $-Y$ by elimination of g, i.e. by applying axiom (L3) with $g \in S(W, -Y)$, and let X^σ be the unique cocircuit of $\widetilde{\mathcal{M}}$ extending X, where $\sigma = X^\sigma_f$. Then the edge is oriented towards W, towards Y or is unoriented if $\sigma = +, -$ or 0, respectively. Note that the definition of Euclideanness of the program $(\widetilde{\mathcal{M}}, g, f)$ and that of the associated graph G depend only on $\mathcal{M}$ and the extension of $\mathcal{M}/g$ on f induced by $\widetilde{\mathcal{M}}$.

A *directed cycle* in G is a cycle of the underlying unoriented graph together with one of two possible orientations o, such that all edges of the cycle are either unoriented in G or oriented as in o and the latter case occurs. We will use the following criterion for Euclideanness; see also [7, Theorem 10.5.5].

PROPOSITION 6.1 (Edmonds–Mandel [13]). *The oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ is euclidean if and only if the associated graph G has no directed cycles.*

6.2. STACKABILITY AND EUCLIDEANNESS

Let $\mathcal{Z}$ be a zonotope with oriented matroid $\mathcal{M}$. In what follows we will identify the set of generators of $\mathcal{Z}$ with the ground set of $\mathcal{M}$. Let g be a generator of $\mathcal{Z}$ which is neither a loop nor a coloop and T be a zonotopal subdivision of $\mathcal{Z} - g$. By the Bohne–Dress Theorem [8], T corresponds to a lifting of $\mathcal{M} - g$ on a (nonloop) element f and hence, by oriented matroid duality, to an extension of the dual oriented matroid $(\mathcal{M} - g)^* = \mathcal{M}^*/g$ by f. For notational convenience, we consider any extension $\widetilde{\mathcal{M}}^* = \mathcal{M}^* \cup f$ of $\mathcal{M}^*$ which contracts to the extension of $\mathcal{M}^*/g$ on f defined by T. The following theorem is the main result of this section.

THEOREM 6.2. *Let $\mathcal{M}$ be the oriented matroid of $\mathcal{Z}$ and g be a generator which is neither a loop nor a coloop. Let T be a zonotopal subdivision of $\mathcal{Z} - g$ and $\widetilde{\mathcal{M}}^*$ be a corresponding extension of $\mathcal{M}^*$ on an element f.*

The subdivision T is stackable in the direction of g if and only if the oriented matroid program $(\widetilde{\mathcal{M}}^*, g, f)$ is Euclidean.

The following definitions and observations will facilitate the proof. We define the *stackability graph* G_T of T on the vertex set of tiles of T as follows. Two tiles τ_1 and τ_2 of T are joined by an edge in T which is either directed, say from τ_1 to τ_2, or undirected if the two tiles are adjacent and either τ_1 lies below τ_2, or τ_1 and τ_2 are parallel in the direction of g, respectively. Clearly, T is stackable in the direction of g if and only if G_T has no directed cycles (defined as in the case of G). Let $\widehat{\mathcal{M}}$ be the lifting of $\mathcal{M}$ on f which corresponds to the extension $\widetilde{\mathcal{M}}^*$, refered to in Theorem 6.2, and note that the triple $(\widehat{\mathcal{M}}, f, g)$ is an oriented matroid program.

LEMMA 6.3. *The stackability graph G_T is isomorphic to the graph of the oriented matroid program $(\widehat{\mathcal{M}}, f, g)$.*

Proof. Let G be the graph of the program $(\widehat{\mathcal{M}}, f, g)$, so that G is a graph on the set of cocircuits X of the induced lifting of $\mathcal{M} - g$ with $X_f = +$. It is clear from the Bohne–Dress Theorem [8, 21] that G and G_T are isomorphic as undirected graphs. That the orientations of the edges correspond is trivial if $\dim(\mathcal{Z}) = 1$. For the general case, it suffices to observe that the definition of the orientation of the edges (τ_1, τ_2) or (Y, W) is preserved by reduction to rank two, i.e. by contracting $\mathcal{Z}$, T or $\widehat{\mathcal{M}}$, respectively, on the set of generators of the common facet of τ_1, τ_2 or the intersection of the zero sets of Y, W. □

We now give the proof of the theorem.

Proof of Theorem 6.2. By oriented matroid duality [12] [7, Corollary 10.5.9], $(\widetilde{\mathcal{M}}^*, g, f)$ is Euclidean if and only if the dual program $(\widehat{\mathcal{M}}, f, g)$ is euclidean. By Proposition 6.1, this happens if and only if the graph of $(\widehat{\mathcal{M}}, f, g)$ has no directed cycles and hence, by Lemma 6.3, if and only if T is stackable in the direction of g. □

Remark 6.4. It is known that any oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ with $\mathcal{M}$ of rank at most three or (by duality) corank at most two is Euclidean [7, Proposition 10.5.7]. It follows directly from Theorem 6.2 that any zonotopal subdivision of a zonotope of dimension or codimension at most two is stackable in any direction. However, the non-Euclidean program discussed in [7, §10.4] implies the existence of a zonotopal subdivision of a three-dimensional zonotope with 6 generators which is not stackable in a specific direction.

6.3. STRONG STACKABILITY AND EUCLIDEANNESS

As an application of Theorem 6.2, we give an interpretation to the concept of strong euclideanness of Sturmfels and Ziegler [23] in the case of realizable oriented matroids in terms of the condition of stackability for zonotopal subdivisions. Recall [23, Definition 3.8 (ii)] that a loopless oriented matroid $\mathcal{M}$ is *strongly Euclidean* if it has rank one or it possesses an element g such that $\mathcal{M}/g$ is strongly Euclidean and the oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ is Euclidean for every extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ by a nonloop element f. We define strong stackability for zonotopes by analogy.

DEFINITION 6.5. The zonotope $\mathcal{Z}$ is strongly stackable if it has corank one or it possesses a generator g such that:

(i) $\mathcal{Z} - g$ is strongly stackable and
(ii) if g is neither a loop nor a coloop then every zonotopal subdivision of $\mathcal{Z} - g$ is stackable in the direction of g.

The next proposition is a consequence of Theorem 6.2.

PROPOSITION 6.6. *A zonotope $\mathcal{Z}$ without coloops is strongly stackable if and only if the dual $\mathcal{M}^*$ of the oriented matroid $\mathcal{M}$ of $\mathcal{Z}$ is strongly Euclidean.*

Proof. By oriented matroid duality, $\mathcal{M}^*$ is strongly Euclidean if $\mathcal{M}$ has corank one or it possesses an element g such that $(\mathcal{M} - g)^*$ is strongly Euclidean and the oriented matroid program $(\widetilde{\mathcal{M}^*}, g, f)$ is euclidean for every extension $\widetilde{\mathcal{M}^*}$ of $\mathcal{M}^*$ by a (nonloop) element f. The result follows directly from Theorem 6.2 and the fact that zonotopes of codimension one are strongly stackable. □

Our proof of Lemma 3.2 gives another proof to the fact [23, Theorem 4.12] that the alternating oriented matroids are strongly Euclidean.

COROLLARY 6.7. *The cyclic zonotope $\mathcal{Z}(n, d)$ is strongly stackable. Equivalently, the alternating oriented matroid $C^{n,n-d}$ is strongly Euclidean.*

Proof. Let g be the last generator of $\mathcal{Z} = \mathcal{Z}(n, d)$. By Lemma 3.2 every zonotopal subdivision of $\mathcal{Z} - g = \mathcal{Z}(n-1, d)$ is stackable in the direction of g, so induction on n proves the first statement. The second statement follows from Proposition 6.6 and the observation [25, Problem 6.13] that the dual of $C^{n,d}$ is isomorphic to a reorientation of $C^{n,n-d}$. □

Note that the proof of Theorem 1.2 in Section 3 remains valid if $\mathcal{Z}(n, d)$ is replaced by any zonotope $\mathcal{Z}$ having a uniform oriented matroid and such that $\mathcal{Z}$ and all of its minors are strongly stackable. As opposed to strong stackability, the uniformity assumption is not essential for the argument but we will not give the relevant details here.

Realizable oriented matroids which are not strongly Euclidean and, hence, zonotopes which are not strongly stackable, were recently constructed by Santos [22].

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