FLAG SUBDIVISIONS AND γ-VECTORS

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ABSTRACT. The γ-vector is an important enumerative invariant of a flag simplicial homology sphere. It has been conjectured by Gal that this vector is nonnegative for every such sphere Δ and by Reiner, Postnikov and Williams that it increases when Δ is replaced by any flag simplicial homology sphere which geometrically subdivides Δ. Using the nonnegativity of the γ-vector in dimension 3, proved by Davis and Okun, as well as Stanley's theory of simplicial subdivisions and local h-vectors, the latter conjecture is confirmed in this paper in dimensions 3 and 4.

1. INTRODUCTION

This paper is concerned with the face enumeration of an important class of simplicial complexes, that of flag homology spheres, and their subdivisions. The face vector of a homology sphere (more generally, of an Eulerian simplicial complex) Δ can be conveniently encoded by its γ-vector [10], denoted $\gamma(\Delta)$. Part of our motivation comes from the following two conjectures (we refer to Section 2 for all relevant definitions). The first, proposed by Gal [10, Conjecture 2.1.7], can be thought of as a Generalized Lower Bound Conjecture for flag homology spheres (it strengthens an earlier conjecture by Charney and Davis [8]). The second, proposed by Postnikov, Reiner and Williams [12, Conjecture 14.2], is a natural extension of the first.

Conjecture 1.1 (Gal [10]). *For every flag homology sphere Δ we have $\gamma(\Delta) \geq 0$.*

Conjecture 1.2 (Postnikov, Reiner and Williams [12]). *For all flag homology spheres Δ and Δ' for which Δ' geometrically subdivides Δ, we have $\gamma(\Delta') \geq \gamma(\Delta)$.*

The previous statements are trivial for spheres of dimension two or less. Conjecture 1.1 was proved for 3-dimensional spheres by Davis and Okun [9, Theorem 11.2.1] and was deduced from that result for 4-dimensional spheres in [10, Corollary 2.2.3]. Conjecture 1.2 can be thought of as a conjectural analogue of the fact [13, Theorem 4.10] that the h-vector (a certain linear transformation of the face vector) of a Cohen-Macaulay simplicial complex

increases under quasi-geometric simplicial subdivision (a class of topological subdivisions which includes all geometric simplicial subdivisions). The main result of this paper proves its validity in three and four dimensions for a new class of simplicial subdivisions, which includes all geometric ones.

Theorem 1.3. *For every flag homology sphere Δ of dimension 3 or 4 and for every flag vertex-induced homology subdivision Δ' of Δ, we have $\gamma(\Delta') \geq \gamma(\Delta)$.*

The previous result naturally suggests the following stronger version of Conjecture 1.2.

Conjecture 1.4. *For every flag homology sphere Δ and every flag vertex-induced homology subdivision Δ' of Δ, we have $\gamma(\Delta') \geq \gamma(\Delta)$.*

The following structural result on flag homology spheres, which may be of independent interest, will also be proved in Section 4. It implies, for instance, that Conjecture 1.4 is stronger than Conjecture 1.1. Throughout this paper, we will denote by Σ_{d-1} the boundary complex of the d-dimensional cross-polytope (equivalently, the simplicial join of d copies of the zero-dimensional sphere).

Theorem 1.5. *Every flag $(d-1)$-dimensional homology sphere is a vertex-induced (hence quasi-geometric and flag) homology subdivision of Σ_{d-1}.*

The proof of Theorem 1.3 relies on the theory of face enumeration for simplicial subdivisions, developed by Stanley [13]. Given a simplicial complex Δ and a simplicial subdivision Δ' of Δ, the h-vector of Δ' can be expressed in terms of local contributions, one for each face of Δ, and the combinatorics of Δ [13, Theorem 3.2]. The local contributions are expressed in terms of the key concept of a local h-vector, introduced and studied in [13]. When Δ is Eulerian, this formula transforms to one involving γ-vectors (Proposition 5.3) and leads to the concept of a local γ-vector, introduced in Section 5. Using the Davis-Okun theorem [9], mentioned earlier, it is shown that the local γ-vector has nonnegative coefficients for every flag vertex-induced homology subdivision of the 3-dimensional simplex. Theorem 1.3 is deduced from these results in Section 5.

The proof of Theorem 1.5 is motivated by that of [1, Theorem 1.2], stating that the graph of any flag simplicial pseudomanifold of dimension $d-1$ contains a subdivision of the graph of Σ_{d-1}.

We now briefly describe the content and structure of this paper. Sections 2 and 3 provide the necessary background on simplicial complexes, subdivisions and their face enumeration. The notion of a homology subdivision, which is convenient for the results of this paper, as well as those of a flag subdivision and vertex-induced (a natural strengthening of quasi-geometric) subdivision, are introduced in Section 2.3. Section 3.3 includes a simple example (see Example 3.4) which shows that there exist quasi-geometric subdivisions of the simplex with non-unimodal local h-vector.

Section 4 proves Theorem 1.5 and another structural result on flag subdivisions (Proposition 4.6), stating that every flag vertex-induced homology subdivision of the $(d-1)$-dimensional simplex naturally occurs as a restriction of a flag vertex-induced homology subdivision of Σ_{d-1}. These results are used in Section 5.

Local γ-vectors are introduced in Section 5, where examples and elementary properties are discussed. It is conjectured there that the local γ-vector has nonnegative coordinates for every flag vertex-induced homology subdivision of the simplex (Conjecture 5.4). This statement can be considered as a local analogue of Conjecture 1.1. It is shown to imply both Conjectures 1.1 and 1.4 and to hold in dimension three. Section 5 concludes with the proof of Theorem 1.3.

Section 6 discusses some special cases of Conjecture 5.4. For instance, the conjecture is shown to hold for iterated edge subdivisions (in the sense of [8, Section 5.3]) of the simplex.

2. FLAG COMPLEXES, SUBDIVISIONS AND γ-VECTORS

This section reviews background material on simplicial complexes, in particular on their homological properties and subdivisions. For more information on these topics, the reader is referred to [15]. Throughout this paper, **k** will be a field which we will assume to be fixed. We will denote by $|S|$ the cardinality, and by 2^S the set of all subsets, of a finite set S.

2.1. **Simplicial complexes.** All simplicial complexes we consider will be abstract and finite. Thus, given a finite set Ω, a *simplicial complex* on the ground set Ω is a collection Δ of subsets of Ω such that $F \subseteq G \in \Delta$ implies $F \in \Delta$. The elements of Δ are called *faces*. The dimension of a face F is defined as one less than the cardinality of F. The dimension of Δ is the maximum dimension of a face and is denoted by $\dim(\Delta)$. Faces of Δ of dimension zero or one are called *vertices* or *edges*, respectively. A *facet* of Δ is a face which is maximal with respect to inclusion. All topological properties of Δ we mention in the sequel will refer to those of the geometric realization $\|\Delta\|$ of Δ [5, Section 9], uniquely defined up to homeomorphism. For example, we say that Δ is a simplicial (topological) ball (respectively, sphere) if $\|\Delta\|$ is homeomorphic to a ball (respectively, sphere).

The *open star* of a face $F \in \Delta$, denoted $\mathrm{st}_\Delta(F)$, is the collection of all faces of Δ which contain F. The *closed star* of $F \in \Delta$, denoted $\overline{\mathrm{st}}_\Delta(F)$, is the subcomplex of Δ consisting of all subsets of the elements of $\mathrm{st}_\Delta(F)$. The *link* of the face $F \in \Delta$ is the subcomplex of Δ defined as $\mathrm{link}_\Delta(F) = \{G \smallsetminus F : G \in \Delta, \ F \subseteq G\}$. The *simplicial join* $\Delta_1 * \Delta_2$ of two collections Δ_1 and Δ_2 of subsets of disjoint ground sets is the collection whose elements are the sets of the form $F_1 \cup F_2$, where $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$. If Δ_1 and Δ_2 are simplicial complexes, then so is $\Delta_1 * \Delta_2$. The simplicial join of Δ with the zero-dimensional complex $\{\varnothing, \{v\}\}$ is denoted by $v * \Delta$ and called the *cone over* Δ on the (new) vertex v.

A simplicial complex Δ is called *flag* if every minimal non-face of Δ has two elements. The closed star and the link of any face of a flag complex, and the simplicial join of two (or more) flag complexes, are also flag complexes. In particular, the simplicial join of d copies of the zero-dimensional complex with two vertices is a flag complex (in fact, a flag triangulation of the $(d-1)$-dimensional sphere), which will be denoted by Σ_{d-1}. Explicitly, Σ_{d-1} can be described as the simplicial complex on the $2d$-element ground set $\Omega_d = \{u_1, u_2, \ldots, u_d\} \cup \{v_1, v_2, \ldots, v_d\}$ whose faces are those subsets of Ω_d which contain at most one element from each of the sets $\{u_i, v_i\}$ for $1 \le i \le d$.

2.2. **Homology balls and spheres.** Let Δ be a simplicial complex of dimension $d - 1$. We call Δ a *homology sphere* (over $\mathbf{k}$) if for every $F \in \Delta$ (including $F = \varnothing$) we have

$$\widetilde{H}_i\left(\mathrm{link}_\Delta(F), \mathbf{k}\right) = \begin{cases} \mathbf{k}, & \text{if } i = \dim \mathrm{link}_\Delta(F) \\ 0, & \text{otherwise,} \end{cases}$$

where $\widetilde{H}_*(\Gamma, \mathbf{k})$ denotes reduced simplicial homology of Γ with coefficients in $\mathbf{k}$. We call Δ a *homology ball* (over $\mathbf{k}$) if there exists a subcomplex $\partial\Delta$ of Δ, called the *boundary* of Δ, so that the following hold:

- $\partial\Delta$ is a $(d-2)$-dimensional homology sphere over $\mathbf{k}$,
- for every $F \in \Delta$ (including $F = \varnothing$) we have

$$\widetilde{H}_i\left(\mathrm{link}_\Delta(F), \mathbf{k}\right) = \begin{cases} \mathbf{k}, & \text{if } F \notin \partial\Delta \text{ and } i = \dim \mathrm{link}_\Delta(F) \\ 0, & \text{otherwise.} \end{cases}$$

The *interior* of Δ is defined as $\mathrm{int}(\Delta) = \Delta$, if Δ is a homology sphere, and $\mathrm{int}(\Delta) = \Delta \smallsetminus \partial\Delta$, if Δ is a homology ball. For example, the simplicial complex $\{\varnothing, \{v\}\}$ with a unique vertex v is a zero-dimensional homology ball (over any field) with boundary $\{\varnothing\}$ and interior $\{\{v\}\}$. If Δ is a homology ball of dimension $d-1$, then $\partial\Delta$ consists exactly of the faces of those $(d-2)$-dimensional faces of Δ which are contained in a unique facet of Δ.

Remark 2.1. It follows from standard facts [5, (9.12)] on the homology of simplicial joins that the simplicial join of a homology sphere and a homology ball, or of two homology balls, is a homology ball and that the simplicial join of two homology spheres is again a homology sphere. Moreover, in each case the interior of the simplicial join is equal to the simplicial join of the interiors of the two complexes in question. $\square$

2.3. **Subdivisions.** We will adopt the following notion of homology subdivision of an abstract simplicial complex. This notion generalizes that of topological subdivision of [13, Section 2]. We should point out that the class of homology subdivisions of simplicial complexes is contained in the much broader class of formal subdivisions of Eulerian posets, introduced and studied in [13, Section 7].
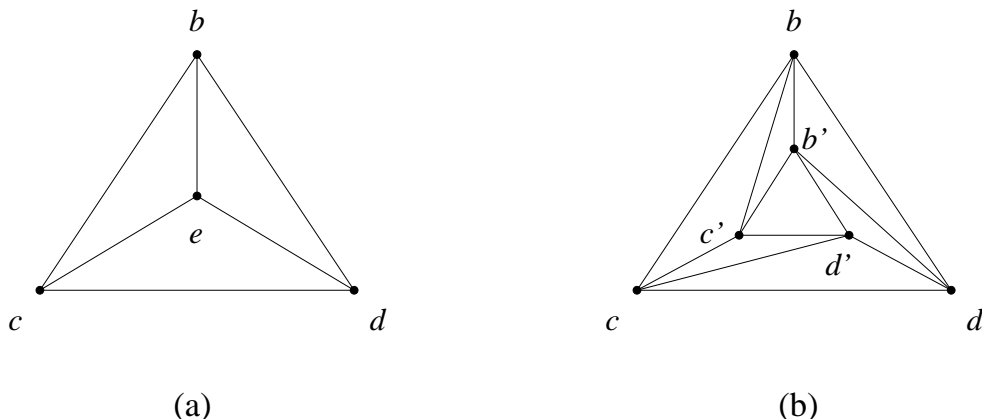
FIGURE 1. Two non-flag subdivisions of a triangle.

Definition 2.2. Let Δ be a simplicial complex. A (finite, simplicial) *homology subdivision* of Δ (over $\mathbf{k}$) is a simplicial complex Δ' together with a map $\sigma : \Delta' \to \Delta$ such that the following hold for every $F \in \Delta$: (a) the set $\Delta'_F := \sigma^{-1}(2^F)$ is a subcomplex of Δ' which is a homology ball (over $\mathbf{k}$) of dimension $\dim(F)$; and (b) $\sigma^{-1}(F)$ consists of the interior faces of Δ'_F.

Such a map σ is said to be a *topological* subdivision if the complex Δ'_F is homeomorphic to a ball of dimension $\dim(F)$ for every $F \in \Delta$.

Let $\sigma : \Delta' \to \Delta$ be a homology subdivision of Δ. From the defining properties, it follows that the map σ is surjective and that $\dim(\sigma(E)) \geq \dim(E)$ for every $E \in \Delta'$. Given faces $E \in \Delta'$ and $F \in \Delta$, the face $\sigma(E)$ of Δ is called the *carrier* of E; the subcomplex Δ'_F is called the *restriction* of Δ' to F. The subdivision σ is called *quasi-geometric* [13, Definition 4.1 (a)] if there do not exist $E \in \Delta'$ and face $F \in \Delta$ of dimension smaller than $\dim(E)$, such that the carrier of every vertex of E is contained in F. Moreover, σ is called *geometric* [13, Definition 4.1 (b)] if there exists a geometric realization of Δ' which geometrically subdivides a geometric realization of Δ, in the way prescribed by σ.

Clearly, if $\sigma : \Delta' \to \Delta$ is a homology (respectively, topological) subdivision, then the restriction of σ to Δ'_F is also a homology (respectively, topological) subdivision of the simplex 2^F for every $F \in \Delta$. Moreover, if σ is quasi-geometric (respectively, geometric), then so are all its restrictions Δ'_F for $F \in \Delta$. As part (c) of the following example shows, the restriction of σ to a face $F \in \Delta$ need not be a flag complex, even when Δ' and Δ are flag complexes and σ is quasi-geometric.

Example 2.3. Consider a 3-dimensional simplex 2^V, with $V = \{a, b, c, d\}$, and set $F = \{b, c, d\}$.

(a) Let Γ be the simplicial complex consisting of the subsets of V and the subsets of $\{b, c, d, e\}$ and let $\sigma : \Gamma \to 2^V$ be the subdivision (considered in

part (h) of [13, Example 2.3]) which pushes Γ into the simplex 2^V, so that the face F of Γ ends up in the interior of 2^V and e ends up in the interior of 2^F. Formally, for $E \in \Gamma$ we let $\sigma(E) = E$, if $E \in 2^V \smallsetminus \{F\}$, we let $\sigma(E) = V$, if E contains F and otherwise we let $\sigma(E) = F$. Then Γ is a flag complex and the restriction Γ_F of σ is the cone over the boundary of 2^F (with new vertex e), shown in Figure 1 (a), which is not flag.

(b) Let Γ' be the simplicial complex consisting of the faces of the simplex 2^V and those of the cone, on a vertex v, over the boundary of the simplex with vertex set $\{b, c, d, e\}$ (note that Γ' is not flag). Consider the subdivision $\sigma' : \Gamma' \to 2^V$ which satisfies $\sigma'(E) = V$ for every face $E \in \Gamma'$ containing v and otherwise agrees with the subdivision σ of part (a). Then σ' is quasi-geometric and its restriction $\Gamma'_F = \Gamma_F$ is again the non-flag complex shown in Figure 1 (a).

(c) Let Γ_0 be the simplicial complex on the ground set $F \cup \{b', c', d'\}$ whose faces are F and those of the simplicial subdivision of 2^F, shown in Figure 1 (b). Let Γ'' consist of the faces of 2^V and those of the cone over Γ_0 on a new vertex v. We leave to the reader to verify that Γ'' is a flag simplicial complex and that it admits a quasi-geometric subdivision $\sigma'' : \Gamma'' \to 2^V$ (satisfying $\sigma''(v) = \sigma''(F) = V$) for which the restriction Γ''_F is the non-flag simplicial complex shown in Figure 1 (b). $\square$

The previous examples suggest the following definitions.

Definition 2.4. Let Δ', Δ be simplicial complexes and let $\sigma : \Delta' \to \Delta$ be a homology subdivision.

(i) We say that σ is *vertex-induced* if for all faces $E \in \Delta'$ and $F \in \Delta$ the following condition holds: if every vertex of E is a vertex of Δ'_F, then $E \in \Delta'_F$.

(ii) We say that σ is a *flag subdivision* if the restriction Δ'_F is a flag complex for every face $F \in \Delta$.

For homology (respectively, topological) subdivisions we have the hierarchy of properties: geometric $\Rightarrow$ vertex-induced $\Rightarrow$ quasi-geometric. The subdivision Γ of Example 2.3 is not quasi-geometric, while Γ' and Γ'' are quasi-geometric but not vertex-induced (none of the three subdivisions is flag). Thus the second implication above is strict. An example discussed on [7, p. 468] shows that the first implication is strict as well. We also point out here that if $\sigma : \Delta' \to \Delta$ is a vertex-induced homology subdivision and the simplicial complex Δ' is flag, then σ is a flag subdivision.

Joins and links. The notion of a (vertex-induced, or flag) homology subdivision behaves well with respect to simplicial joins and links, as we now explain. Let $\sigma_1 : \Delta'_1 \to \Delta_1$ and $\sigma_2 : \Delta'_2 \to \Delta_2$ be homology subdivisions of two simplicial complexes Δ_1 and Δ_2 on disjoint ground sets. The simplicial join $\Delta'_1 * \Delta'_2$ is naturally a homology subdivision of $\Delta_1 * \Delta_2$ with subdivision map $\sigma : \Delta'_1 * \Delta'_2 \to \Delta_1 * \Delta_2$ defined by $\sigma(E_1 \cup E_2) = \sigma_1(E_1) \cup \sigma_2(E_2)$ for $E_1 \in \Delta'_1$ and $E_2 \in \Delta'_2$. Indeed, given faces $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$,

the restriction of $\Delta_1' * \Delta_2'$ to the face $F = F_1 \cup F_2 \in \Delta_1 * \Delta_2$ is equal to $(\Delta_1')_{F_1} * (\Delta_2')_{F_2}$ which, by Remark 2.1, is a homology ball of dimension equal to that of $F_1 \cup F_2$. Moreover, $\sigma^{-1}(F) = \sigma_1^{-1}(F_1) * \sigma_2^{-1}(F_2)$ and hence $\sigma^{-1}(F)$ is the interior of this ball.

Similarly, let $\sigma : \Delta' \to \Delta$ be a homology subdivision and let F be a common face of Δ and Δ' (such as a vertex of Δ) which satisfies $\sigma(F) = F$. An easy application of part (ii) of Lemma 4.1 shows that $\mathrm{link}_{\Delta'}(F)$ is a homology subdivision of $\mathrm{link}_\Delta(F)$ with subdivision map $\sigma_F : \mathrm{link}_{\Delta'}(F) \to \mathrm{link}_\Delta(F)$ defined by $\sigma_F(E) = \sigma(E \cup F) \smallsetminus F$. We will refer to this subdivision as the link of σ at F; its restriction to a face $G \in \mathrm{link}_\Delta(F)$ satisfies $(\mathrm{link}_{\Delta'}(F))_G = \mathrm{link}_{\Delta'_{F \cup G}}(F)$.

The following statement is an easy consequence of the relevant definitions; the proof is left to the reader.

Lemma 2.5. *The simplicial join of two vertex-induced (respectively, flag) homology subdivisions is also vertex-induced (respectively, flag). The link of a vertex-induced (respectively, flag) homology subdivision is also vertex-induced (respectively, flag).*

Stellar subdivisions. We recall the following standard way to subdivide a simplicial complex. Given a simplicial complex Δ on the ground set Ω, a face $F \in \Delta$ and an element v not in Ω, the *stellar subdivision* of Δ on F (with new vertex v) is the simplicial complex

$$\Delta' = (\Delta \smallsetminus \mathrm{st}_\Delta(F)) \cup (\{v\} * \partial(2^F) * \mathrm{link}_\Delta(F))$$

on the ground set $\Omega \cup \{v\}$, where $\partial(2^F) = 2^F \smallsetminus \{F\}$. The map $\sigma : \Delta' \to \Delta$, defined by

$$\sigma(E) = \begin{cases} E, & \text{if } E \in \Delta \\ (E \smallsetminus \{v\}) \cup F, & \text{otherwise} \end{cases}$$

for $E \in \Delta'$, is a topological (and thus a homology) subdivision of Δ. We leave to the reader to check that if Δ is a flag complex and $F \in \Delta$ is an edge, then the stellar subdivision of Δ on F is again a flag complex.

3. FACE ENUMERATION, γ-VECTORS AND LOCAL h-VECTORS

This section reviews the definitions and main properties of the enumerative invariants of simplicial complexes and their subdivisions which will appear in the following sections, namely the h-vector of a simplicial complex, the γ-vector of an Eulerian simplicial complex and the local h-vector of a simplicial subdivision of a simplex. Some new results on local h-vectors are included.

3.1. h-vectors. A fundamental enumerative invariant of a $(d-1)$-dimensional simplicial complex Δ is the h-polynomial, defined by

$$h(\Delta, x) = \sum_{F \in \Delta} x^{|F|}(1-x)^{d-|F|}.$$

The h-vector of Δ is the sequence $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, where $h(\Delta, x) = \sum_{i=0}^{d} h_i(\Delta) x^i$. The number

$$(-1)^{d-1} h_d(\Delta) \ = \ \sum_{F \in \Delta} (-1)^{|F|-1}$$

is the reduced Euler characteristic of Δ and is denoted by $\widetilde{\chi}(\Delta)$. The polynomial $h(\Delta, x)$ satisfies $h_i(\Delta) = h_{d-i}(\Delta)$ [14, Section 3.14] if Δ is an *Eulerian* complex, meaning that

$$\widetilde{\chi}(\mathrm{link}_\Delta(F)) \ = \ (-1)^{\dim \mathrm{link}_\Delta(F)}$$

holds for every $F \in \Delta$. For the simplicial join of two simplicial complexes Δ_1 and Δ_2 we have $h(\Delta_1 * \Delta_2, x) = h(\Delta_1, x) h(\Delta_2, x)$. For a homology ball or sphere Δ of dimension $d - 1$ we set

$$h(\mathrm{int}(\Delta), x) \ = \ \sum_{F \in \mathrm{int}(\Delta)} x^{|F|} (1-x)^{d-|F|}$$

and recall the following well-known statement (see, for instance, Theorem 7.1 in [15, Chapter II] and [2, Section 2.1] for additional references).

Lemma 3.1. *Let Δ be a $(d-1)$-dimensional simplicial complex. If Δ is either a homology ball or a homology sphere over $\mathbf{k}$, then $x^d h(\Delta, 1/x) = h(\mathrm{int}(\Delta), x)$.*

3.2. γ**-vectors.** Let $h = (h_0, h_1, \ldots, h_d)$ be a vector with real coordinates and let $h(x) = \sum_{i=0}^{d} h_i x^i$ be the associated real polynomial of degree at most d. We say that $h(x)$ *has symmetric coefficients*, and that the vector h is *symmetric*, if $h_i = h_{d-i}$ holds for $0 \leq i \leq d$. It is easy to check [10, Proposition 2.1.1] that $h(x)$ has symmetric coefficients if and only if there exists a real polynomial $\gamma(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i$ of degree at most $\lfloor d/2 \rfloor$, satisfying

$$(3.1) \qquad h(x) \ = \ (1+x)^d \, \gamma\left(\frac{x}{(1+x)^2}\right) \ = \ \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1+x)^{d-2i}.$$

In that case, $\gamma(x)$ is uniquely determined by $h(x)$ and called the γ-*polynomial* associated to $h(x)$; the sequence $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ is called the γ-*vector* associated to h. We will refer to the γ-polynomial associated to the h-polynomial of an Eulerian complex Δ as the γ-*polynomial of* Δ and will denote it by $\gamma(\Delta, x)$. Similarly, we will refer to the γ-vector associated to the h-vector of an Eulerian complex Δ as the γ-*vector of* Δ and will denote it by $\gamma(\Delta)$.

3.3. **Local h-vectors.** We now recall some of the basics of the theory of face enumeration for subdivisions of simplicial complexes [13] [15, Section III.10]. The following definition is a restatement of [13, Definition 2.1] for homology (rather than topological) subdivisions of the simplex.

Definition 3.2. Let V be a set with d elements and let Γ be a homology subdivision of the simplex 2^V. The polynomial $\ell_V(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_d x^d$ defined by

$$(3.2) \qquad \ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{d-|F|} h(\Gamma_F, x)$$

is the *local h-polynomial* of Γ (with respect to V). The sequence $\ell_V(\Gamma) = (\ell_0, \ell_1, \ldots, \ell_d)$ is the *local h-vector* of Γ (with respect to V).

The following theorem, stated for homology subdivisions, summarizes some of the main properties of local h-vectors (see Theorems 3.2 and 3.3 and Corollary 4.7 in [13]).

Theorem 3.3. (i) *For every homology subdivision Δ' of a simplicial complex Δ we have*

$$(3.3) \qquad h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x) \, h(\mathrm{link}_\Delta(F), x).$$

(ii) *The local h-vector $\ell_V(\Gamma)$ is symmetric for every homology subdivision Γ of the simplex 2^V.*

(iii) *The local h-vector $\ell_V(\Gamma)$ has nonnegative coordinates for every quasi-geometric homology subdivision Γ of the simplex 2^V.*

Proof. Parts (i) and (iii) follow from the proofs of Theorems 3.2 and 4.6, respectively, in [13]. Moreover, Lemma 3.1 implies that every homology subdivision of a simplicial complex is a formal subdivision, in the sense of [13, Definition 7.4]. Thus, parts (i) and (ii) are special cases of Corollary 7.7 and Theorem 7.8, respectively, in [13]. □

Example 3.4. The local h-polynomial of the subdivision in part (a) of Example 2.3 was computed in [13] as $\ell_V(\Gamma, x) = -x^2$. This shows that the assumption in Theorem 3.3 (iii) that Γ is quasi-geometric is essential. For part (b) of Example 2.3 we can easily compute that $\ell_V(\Gamma', x) = x + x^3$. Since Γ' is quasi-geometric, this disproves [13, Conjecture 5.4] (see also [7, Section 6] [15, p. 134]), stating that local h-vectors of quasi-geometric subdivisions are unimodal. □

The previous example suggests the following question.

Question 3.5. Is the local h-vector $\ell_V(\Gamma)$ unimodal for every vertex-induced homology subdivision Γ of the simplex 2^V?

We now show that local h-vectors also enjoy a locality property (this will be useful in the proof of Proposition 6.1).

Proposition 3.6. *Let $\sigma : \Gamma \to 2^V$ be a homology subdivision of the simplex 2^V. For every homology subdivision Γ' of Γ we have*

$$(3.4) \qquad \ell_V(\Gamma', x) = \sum_{E \in \Gamma} \ell_E(\Gamma'_E, x) \, \ell_V(\Gamma, E, x),$$

where

$$(3.5) \qquad \ell_V(\Gamma, E, x) \ = \ \sum_{\sigma(E) \subseteq F \subseteq V} (-1)^{d - |F|} \, h(\mathrm{link}_{\Gamma_F}(E), x)$$

for $E \in \Gamma$.

Proof. By assumption, Γ'_F is a homology subdivision of Γ_F for every $F \subseteq V$. Thus, using the defining equation (3.2) for $\ell_V(\Gamma', x)$ and (3.3) to expand $h(\Gamma'_F, x)$ for $F \subseteq V$, we get

$$\begin{aligned}
\ell_V(\Gamma', x) \ &= \ \sum_{F \subseteq V} (-1)^{d - |F|} \, h(\Gamma'_F, x) \\[2mm]
&= \ \sum_{F \subseteq V} (-1)^{d - |F|} \sum_{E \in \Gamma_F} \ell_E(\Gamma'_E, x) \, h(\mathrm{link}_{\Gamma_F}(E), x) \\[2mm]
&= \ \sum_{E \in \Gamma} \ell_E(\Gamma'_E, x) \sum_{F \subseteq V : \, \sigma(E) \subseteq F} (-1)^{d - |F|} \, h(\mathrm{link}_{\Gamma_F}(E), x)
\end{aligned}$$

and the proof follows. $\qquad\qquad\square$

Remark 3.7. We call the polynomial $\ell_V(\Gamma, E, x)$, defined by (3.5), the *relative local h-polynomial* of Γ (with respect to V) at E. This polynomial reduces to $\ell_V(\Gamma, x)$ for $E = \varnothing$ and shares many of the important properties of $\ell_V(\Gamma, x)$, established in [13]. For instance, using ideas of [13] and their refinements in [2], one can show that $\ell_V(\Gamma, E, x)$ has symmetric coefficients, in the sense that

$$x^{d - |E|} \, \ell_V(\Gamma, E, 1/x) \ = \ \ell_V(\Gamma, E, x),$$

for every homology subdivision Γ of 2^V and $E \in \Gamma$ and that $\ell_V(\Gamma, E, x)$ has nonnegative coefficients for every quasi-geometric homology subdivision Γ of 2^V and $E \in \Gamma$. The following monotonicity property of local h-vectors is a consequence of the latter statement and (3.4): for every quasi-geometric homology subdivision Γ of 2^V and every quasi-geometric homology subdivision Γ' of Γ, we have $\ell_V(\Gamma', x) \geq \ell_V(\Gamma, x)$. Since these results will not be used in this paper, detailed proofs will appear elsewhere. $\qquad\square$

4. Flag subdivisions of Σ_{d-1}

This section proves Theorem 1.5, as well as a result on flag subdivisions of the simplex (Proposition 4.6) which will be used in Section 5.

The following lemma gives several technical properties of homology balls and spheres (over the field $\mathbf{k}$). We will only sketch the proof, which is fairly straightforward and uses standard tools from algebraic topology.

Lemma 4.1.   (i) *If Δ is a homology sphere (respectively, ball) of dimension $d - 1$, then $\mathrm{link}_\Delta(F)$ is a homology sphere of dimension $d - |F| - 1$ for every $F \in \Delta$ (respectively, for every interior face $F \in \Delta$).*

(ii) *If Δ is a homology ball of dimension $d-1$ and $F \in \Delta$ is a boundary face, then $\mathrm{link}_\Delta(F)$ is a homology ball of dimension $d-|F|-1$ with interior equal to $\mathrm{link}_\Delta(F) \cap \mathrm{int}(\Delta)$.*

(iii) *If Δ is a homology sphere (respectively, ball), then the cone over Δ is a homology ball whose boundary is equal to Δ (respectively, to the union of Δ with the cone over the boundary of Δ).*

(iv) *Let Δ_1 and Δ_2 be homology balls of dimension d. If $\Delta_1 \cap \Delta_2$ is a homology ball of dimension $d-1$ which is contained in (respectively, is equal to) the boundary of both Δ_1 and Δ_2, then $\Delta_1 \cup \Delta_2$ is a homology ball (respectively, sphere) of dimension d.*

(v) *Let Δ be a homology sphere of dimension $d-1$. If Γ is a subcomplex of Δ which is a homology ball of dimension $d-1$, then the complement of the interior of Γ in Δ is also a homology ball of dimension $d-1$ whose boundary is equal to that of Γ.*

Proof. We first observe that for all faces $F \in \Delta$ and $E \in \mathrm{link}_\Delta(F)$, the link of E in $\mathrm{link}_\Delta(F)$ is equal to $\mathrm{link}_\Delta(E \cup F)$. Moreover, if Δ is a homology ball and F is an interior face, then so is $E \cup F$. Part (i) follows from these facts and the definition of homology balls and spheres. Part (ii) is an easy consequence of part (i) and the relevant definitions. Part (iii) is an easy consequence of the relevant definitions and the fact that cones have vanishing reduced homology. Part (iv) follows by an easy application of the Mayer-Vietoris long exact sequence [11, §25].

For the last part, we let K denote the complement of the interior of Γ in Δ and note that the pairs $(\|\Gamma\|, \|\partial\Gamma\|)$ and $(\|\Delta\|, \|K\|)$ are compact triangulated relative homology manifolds which are orientable over $\mathbf{k}$. Applying the Lefschetz duality theorem [11, §70] and the long exact homology sequence [11, §23] to these pairs shows that K has trivial reduced homology over $\mathbf{k}$. Similar arguments work for the links of faces of K. The details are omitted. □

Remark 4.2. Although not all parts of Lemma 4.1 remain valid if homology balls and spheres are replaced by topological balls and spheres, they do hold for the subclasses of PL balls and PL spheres (we refer the reader to [6, Section 4.7 (d)] for this claim, for a short introduction to PL topology and for additional references). Thus, the results of this paper remain valid when homology balls and spheres are replaced by PL balls and spheres and the notion of homology subdivision is replaced by its natural PL analogue. □

The following lemma will also be essential in the proof of Theorem 1.5. A similar result has appeared in [4, Lemma 3.2].

Lemma 4.3. *Let Δ be a flag $(d-1)$-dimensional homology sphere. For every nonempty face F of Δ, the subcomplex $\bigcup_{v \in F} \overline{\mathrm{st}}_\Delta(v)$ is a homology $(d-1)$-dimensional ball whose interior is equal to $\bigcup_{v \in F} \mathrm{st}_\Delta(v)$.*

Proof. We set $F = \{v_1, v_2, \ldots, v_k\}$, $\bigcup_{i=1}^k \overline{\mathrm{st}}_\Delta(v_i) = \mathrm{K}$ and $\bigcup_{i=1}^k \mathrm{st}_\Delta(v_i) = \mathrm{L}$ and proceed by induction on the cardinality k of F. For $k = 1$, the complex

K is the cone over $\text{link}_\Delta(v_1)$ on the vertex v_1. Since Δ is a homology sphere, the result follows from parts (i) and (iii) of Lemma 4.1. Suppose that $k \geq 2$. We will also assume that $d \geq 3$, since the result is trivial otherwise (we note that the assumption that Δ is flag is essential in the case $d = 2$). Since, by Lemma 4.1 (i), links of flag homology spheres are also flag homology spheres, the complex $\Gamma = \text{link}_\Delta(v_k)$ is a flag homology sphere of dimension $d - 2$ and $\{v_1, \ldots, v_{k-1}\}$ is a nonempty face of Γ. Thus, by the induction hypothesis, the union $\Gamma_1 = \bigcup_{i=1}^{k-1} \overline{\text{st}}_\Gamma(v_i)$ is a homology ball of dimension $d - 2$. Let Γ_0 denote the boundary of Γ_1 and let Γ_2 denote the complement of the interior of Γ_1 in Γ. Thus Γ_0 is a homology sphere of dimension $d - 3$ and, by part (v) of Lemma 4.1, Γ_2 is a homology ball of dimension $d-2$ whose boundary is equal to Γ_0.

Consider the union $K_1 = \bigcup_{i=1}^{k-1} \overline{\text{st}}_\Delta(v_i)$ and the cones $K_2 = v_k * \Gamma_2$ and $K_0 = v_k * \Gamma_0$. It is straightforward to verify that $K = K_1 \cup K_2$ and that $K_1 \cap K_2 = K_0$. We note that K_1 is a homology ball of dimension $d - 1$, by the induction hypothesis, and that K_2 and K_0 are homology balls of dimension $d - 1$ and $d - 2$, respectively, by part (iii) of Lemma 4.1. By the induction hypothesis, the interior of Γ_1 is equal to $\bigcup_{i=1}^{k-1} \text{st}_\Gamma(v_i)$. Therefore, none of the faces of Γ_0 contains any of $v_1, \ldots, v_{k-1}$ and hence the same holds for K_0. Since, by the induction hypothesis, the interior of K_1 is equal to $\bigcup_{i=1}^{k-1} \text{st}_\Delta(v_i)$, we conclude that K_0 is contained in the boundary of K_1. Moreover, K_0 is also contained in the boundary of K_2, since Γ_0 is contained in the boundary of Γ_2. It follows from the previous discussion and Lemma 4.1 (iv) that K is a homology $(d - 1)$-dimensional ball.

We now verify that the interior of K is equal to L. This statement may be derived from the previous inductive argument, since the interior of K is equal to the union of the interiors of K_1, K_2 and K_0. We give the following alternative argument. Since K is a homology ball, its boundary consists of all faces of the $(d-2)$-dimensional faces of K which are contained in exactly one facet of K. The validity of the statement for $k = 1$ implies that these $(d - 2)$-dimensional faces of K are precisely those which do not contain any of the v_i and which are not contained in more than one of the subcomplexes $\text{link}_\Delta(v_i)$. However, since Δ is $(d - 1)$-dimensional and flag, no $(d - 2)$-dimensional face of Δ may be contained in more than one of the $\text{link}_\Delta(v_i)$. Thus, the boundary of K consists precisely of its faces which do not contain any of the v_i and the proof follows. $\square$

Proof of Theorem 1.5. Let Δ be a flag simplicial complex of dimension $d-1$ and Σ_{d-1} be the simplicial join of the zero-dimensional spheres $\{\varnothing, \{u_i\}, \{v_i\}\}$ for $1 \leq i \leq d$. We fix a facet $\{x_1, x_2, \ldots, x_d\}$ of Δ and for $E \in \Delta$ we define

$$(4.1) \qquad \sigma(E) = \{u_i : x_i \in E\} \cup \{v_i : E \notin \overline{\text{st}}_\Delta(x_i)\}.$$

Clearly, $\sigma(E)$ cannot contain any of the sets $\{u_i, v_i\}$. Thus we have $\sigma(E) \in \Sigma_{d-1}$ for every $E \in \Delta$ and hence we get a map $\sigma : \Delta \to \Sigma_{d-1}$. We will prove

that this map is a homology subdivision of Σ_{d-1}, if Δ is a homology sphere. Given a face $F \in \Sigma_{d-1}$, we need to show that $\sigma^{-1}(2^F)$ is a subcomplex of Δ of dimension $\dim(F)$ which is a homology ball with interior $\sigma^{-1}(F)$. We denote by S the subset of $\{x_1, x_2, \ldots, x_d\}$ consisting of all vertices x_i for which $F \cap \{u_i, v_i\} = \varnothing$ and distinguish two cases:

Case 1: $S = \varnothing$. We may assume that $F = \{u_1, \ldots, u_k\} \cup \{v_{k+1}, \ldots, v_d\}$ for some $k \leq d$. Setting $E_0 = \{x_1, \ldots, x_k\}$, the defining equation (4.1) shows that $\sigma^{-1}(2^F)$ is equal to the intersection of $\bigcap_{i=1}^{k} \overline{\text{st}}_\Delta(x_i)$ with the complement of $\bigcup_{i=k+1}^{d} \text{st}_\Delta(x_i)$ in Δ and that $\sigma^{-1}(F)$ consists of those faces of $\sigma^{-1}(2^F)$ which contain E_0 and do not belong to any of the $\text{link}_\Delta(x_i)$ for $k + 1 \leq i \leq d$. Consider the complex $\Gamma = \text{link}_\Delta(E_0)$ and let K denote the complement of the union $\bigcup_{i=k+1}^{d} \text{st}_\Gamma(x_i)$ in Γ. Since links of homology spheres are also homology spheres (see part (i) of Lemma 4.1), the complex Γ is a homology sphere of dimension $d - |F| - 1$. By Lemma 4.3 and part (v) of Lemma 4.1, K is a homology ball of dimension $d - |F| - 1$ whose interior is equal to the set of those faces of K which do not belong to any of the $\text{link}_\Gamma(x_i)$ for $k + 1 \leq i \leq d$. From the above we conclude that $\sigma^{-1}(2^F)$ is equal to the simplicial join of the simplex 2^{E_0} and K and that $\sigma^{-1}(F)$ is equal to the simplicial join of $\{E_0\}$ and the interior of K. The result now follows from part (iii) of Lemma 4.1 and the previous discussion.

Case 2: $S \neq \varnothing$. Then $\sigma^{-1}(2^F)$ is contained in $\text{link}_\Delta(S)$. As a result, replacing Δ by $\text{link}_\Delta(S)$ reduces this case to the previous one.

Finally, we note that (4.1) may be rewritten as $\sigma(E) = \bigcup_{x \in E} f(x)$, where

$$f(x) = \begin{cases} \{u_i\}, & \text{if } x = x_i \text{ for some } 1 \leq i \leq d \\ \{v_i : x \notin \text{link}_\Delta(x_i)\}, & \text{otherwise} \end{cases}$$

for every vertex x of Δ. This implies that for every $E \in \Delta$, the carrier of E is equal to the union of the carriers of the vertices of E. As a result, σ is vertex-induced and the proof follows. $\square$

Corollary 4.4. *Given any flag homology sphere Δ of dimension $d - 1$, there exist simplicial complexes Γ_F, one for each face $F \in \Sigma_{d-1}$, with the following properties: (a) Γ_F is a flag vertex-induced homology subdivision of the simplex 2^F for every $F \in \Sigma_{d-1}$; and (b) we have*

$$(4.2) \qquad h(\Delta, x) = \sum_{F \in \Sigma_{d-1}} \ell_F(\Gamma_F, x)(1 + x)^{d - |F|}.$$

Proof. This statement follows by applying (3.3) to the subdivision of Δ guaranteed by Theorem 1.5 and noting that for every $F \in \Sigma_{d-1}$, the restriction Γ_F of this subdivision to F has the required properties and that $h(\text{link}_{\Sigma_{d-1}}(F), x) = (1 + x)^{d - |F|}$. $\square$

Remark 4.5. It follows from (4.2) and Theorem 3.3 (iii) that $h(\Delta, x) \geq (1 + x)^d$ for every flag $(d - 1)$-dimensional homology sphere Δ. This inequality

was proved, more generally, for every flag $(d-1)$-dimensional doubly Cohen-Macaulay simplicial complex Δ in [1, Theorem 1.3]. $\qquad\square$

We now fix a d-element set $V = \{v_1, v_2, \ldots, v_d\}$ and a homology subdivision Γ of 2^V, with subdivision map $\sigma : \Gamma \to 2^V$. We let $U = \{u_1, u_2, \ldots, u_d\}$ be a d-element set which is disjoint from V and consider the union Δ of all collections of the form $2^E * \Gamma_G$, where $E = \{u_i : i \in I\}$ and $G = \{v_j : j \in J\}$ are subsets of U and V, respectively, and (I, J) ranges over all ordered pairs of disjoint subsets of $\{1, 2, \ldots, d\}$. Clearly, Δ is a simplicial complex which contains as a subcomplex Γ (set $I = \varnothing$) and the simplex 2^U (set $J = \varnothing$).

We let Σ_{d-1} be as in the proof of Theorem 1.5 and define the map $\sigma_0 : \Delta \to \Sigma_{d-1}$ by $\sigma_0(E \cup F) = E \cup \sigma(F)$ for all $E \subseteq U$ and $F \in \Gamma$ such that $E \cup F \in \Delta$. The second result of this section is as follows.

Proposition 4.6. *Under the established assumptions and notation, the following hold:*

(i) *The complex Δ is a $(d-1)$-dimensional homology sphere.*
(ii) *Endowed with the map σ_0, the complex Δ is a homology subdivision of Σ_{d-1}.*
(iii) *If Γ is flag and vertex-induced, then Δ is a flag simplicial complex and a flag, vertex-induced homology subdivision of Σ_{d-1}.*

Proof. We first verify (ii). We consider any face $W \in \Sigma_{d-1}$, so that $W = E \cup G$ for some $E \subseteq U$ and $G \subseteq V$, and recall that Γ_G is a homology ball of dimension $\dim(G)$. By definition of σ_0 we have $\sigma_0^{-1}(2^W) = 2^E * \Gamma_G$ and $\sigma_0^{-1}(W) = \{E\} * \sigma^{-1}(G) = \{E\} * \mathrm{int}(\Gamma_G)$. Thus, it follows from part (iii) of Lemma 4.1 that $\sigma_0^{-1}(2^W)$ is a homology ball of dimension $\dim(W)$ and that its interior is equal to $\sigma_0^{-1}(W)$.

Part (i) may be deduced from part (ii) as follows. Let $F_0, F_1, \ldots, F_m$ be a linear ordering of the facets of Σ_{d-1} such that $F_i \cap U \subset F_j \cap U$ implies $i < j$. Thus we have $m = 2^d$, $F_0 = V$ and $F_m = U$. By assumption, $\Delta_{F_0} = \Gamma_V$ is a $(d-1)$-dimensional homology ball. Moreover, Δ_{F_j} is equal to the simplicial join of a face of 2^U with the restriction of Γ to a face of 2^V, for $1 \le j \le m$, and hence a $(d-1)$-dimensional homology ball by part (iii) of Lemma 4.1, and $\Delta_{F_j} \cap \bigcup_{i=0}^{j-1} \Delta_{F_i}$ is equal to the simplicial join of the boundary of this face with the same restriction of Γ. It follows from part (iv) of Lemma 4.1 by induction on j that $\bigcup_{i=0}^{j} \Delta_{F_i}$ is a $(d-1)$-dimensional homology ball, for $0 \le j \le m-1$ and a $(d-1)$-dimensional homology sphere, for $j = m$. This proves (i), since $\Delta = \bigcup_{i=0}^{m} \Delta_{F_i}$.

To verify (iii), assume that Γ is flag and vertex-induced. It is clear from the definition of σ_0 that the subdivision Δ is also vertex-induced. Since the restriction of Δ to any face of Σ_{d-1} is the join of a simplex with the restriction of Γ to a face of 2^V, the subdivision Δ is flag as well. To verify that Δ is a flag complex, let $E \cup F$ be a set of vertices of Δ which are pairwise joined by edges, where $E = \{u_i : i \in I\}$ for some $I \subseteq \{1, 2, \ldots, d\}$ and F consists of vertices of Γ. We need to show that $E \cup F \in \Delta$. We set

$J = \{1, 2, \ldots, d\} \smallsetminus I$ and $G = \{v_j : j \in J\}$ and note that the elements of F are vertices of Γ_G, by definition of Δ. Since the elements of F are pairwise joined by edges in Γ, our assumptions that Γ is vertex-induced and flag imply that $F \in \Gamma_G$. Therefore $E \cup F$ belongs to $2^E * \Gamma_G$, which is contained in Δ, and the result follows. $\square$

Remark 4.7. The conclusion in Proposition 4.6 that Δ is a flag complex does not hold under the weaker hypothesis that Γ is quasi-geometric, rather than vertex-induced. For instance, let Γ be the simplicial complex consisting of the subsets of $V = \{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ and let $\sigma : \Gamma \to 2^V$ be the subdivision which pushes Γ into 2^V, so that the face $F = \{v_2, v_3\}$ of Γ ends up in the interior of 2^V and v_4 ends up in the interior of 2^F. Then Γ is quasi-geometric and flag but the simplicial complex Δ is not flag, since it has $\{u_1, v_2, v_3\}$ as a minimal non-face. $\square$

5. Local γ-vectors

This section defines the local γ-vector of a homology subdivision of the simplex, lists examples and elementary properties, discusses its nonnegativity in the special case of flag subdivisions and concludes with the proof of Theorem 1.3. This proof comes as an application of the considerations and results of the present and the previous section.

Definition 5.1. Let V be a set with d elements and let Γ be a homology subdivision of the simplex 2^V. The polynomial $\xi_V(\Gamma, x) = \xi_0 + \xi_1 x + \cdots + \xi_{\lfloor d/2 \rfloor} x^{\lfloor d/2 \rfloor}$ defined by

$$(5.1) \qquad \ell_V(\Gamma, x) \ = \ (1 + x)^d \, \xi_V \left(\Gamma, \frac{x}{(1 + x)^2} \right) \ = \ \sum_{i=0}^{\lfloor d/2 \rfloor} \xi_i x^i (1 + x)^{d - 2i}$$

is the *local γ-polynomial* of Γ (with respect to V). The sequence $\xi_V(\Gamma) = (\xi_0, \xi_1, \ldots, \xi_{\lfloor d/2 \rfloor})$ is the *local γ-vector* of Γ (with respect to V).

Thus $\xi_V(\Gamma, x)$ is the γ-polynomial associated to $\ell_V(\Gamma, x)$ and $\xi_V(\Gamma)$ is the γ-vector associated to $\ell_V(\Gamma)$, in the sense of Section 3.2. All formulas in the next example follow from corresponding formulas in [13, Example 2.3], or directly from the relevant definitions.

Example 5.2. (a) For the trivial subdivision $\Gamma = 2^V$ of the $(d-1)$-dimensional simplex 2^V we have

$$(5.2) \qquad\qquad\qquad \xi_V(\Gamma, x) \ = \ \begin{cases} 1, & \text{if } d = 0 \\ 0, & \text{if } d \geq 1. \end{cases}$$

(b) Let $\xi_V(\Gamma) = (\xi_0, \xi_1, \ldots, \xi_{\lfloor d/2 \rfloor})$, where Γ and V are as in Definition 5.1. Assuming that $d \geq 1$, we have $\xi_0 = 0$ and $\xi_1 = f_0^\circ$, where f_0° is the number of interior vertices of Γ. Assuming that $d \geq 4$, we also have $\xi_2 = -(2d - 3)f_0^\circ + f_1^\circ - \tilde{f}_0$, where f_1° is the number of interior edges of Γ

and $\tilde{f}_0$ is the number of vertices of Γ which lie in the relative interior of a $(d-2)$-dimensional face of 2^V.

(c) Suppose that $d \in \{2,3\}$. As a consequence of (b) we have $\xi_V(\Gamma, x) = tx$ for every homology subdivision Γ of 2^V, where t is the number of interior vertices of Γ.

(d) Let Γ be the cone over the boundary $2^V \smallsetminus \{V\}$ of the simplex 2^V (so Γ is the stellar subdivision of 2^V on the face V). Then $\ell_V(\Gamma, x) = x + x^2 + \cdots + x^{d-1}$ and hence ξ_2 is negative for $d \geq 4$. For instance, we have $\xi_V(\Gamma, x) = x - x^2$ for $d = 4$.

(e) For the subdivisions of parts (b) and (c) of Example 2.3 we can compute that $\ell_V(\Gamma', x) = \ell_V(\Gamma'', x) = x + x^3$ and hence that $\xi_V(\Gamma', x) = \xi_V(\Gamma'', x) = x - 2x^2$. $\qquad \square$

The following proposition shows the relevance of local γ-vectors in the study of γ-vectors of subdivisions of Eulerian complexes.

Proposition 5.3. *Let Δ be an Eulerian simplicial complex. For every homology subdivision Δ' of Δ we have*

$$(5.3) \qquad \gamma(\Delta', x) = \sum_{F \in \Delta} \xi_F(\Delta'_F, x)\, \gamma(\mathrm{link}_\Delta(F), x).$$

Proof. Since Δ is Eulerian, so is $\mathrm{link}_\Delta(F)$ for every $F \in \Delta$. Thus, applying (3.1) to the h-polynomial of $\mathrm{link}_\Delta(F)$ we get

$$h(\mathrm{link}_\Delta(F), x) = (1+x)^{d-|F|}\, \gamma\left(\mathrm{link}_\Delta(F), \frac{x}{(1+x)^2}\right),$$

where $d-1 = \dim(\Delta)$. Using this and (5.1), Equation (3.3) may be rewritten as

$$h(\Delta', x) = (1+x)^d \sum_{F \in \Delta} \xi_F\left(\Delta'_F, \frac{x}{(1+x)^2}\right) \gamma\left(\mathrm{link}_\Delta(F), \frac{x}{(1+x)^2}\right).$$

The proposed equality now follows from the uniqueness of the γ-polynomial associated to $h(\Delta', x)$. $\qquad \square$

The following statement is the main conjecture of this paper.

Conjecture 5.4. *For every flag vertex-induced homology subdivision Γ of the simplex 2^V we have $\xi_V(\Gamma) \geq 0$.*

Parts (d) and (e) of Example 5.2 show that the conclusion of Conjecture 5.4 fails under various weakenings of the hypotheses. We do not know of an example of a flag quasi-geometric homology subdivision of the simplex for which the local γ-vector fails to be nonnegative.

We now discuss some consequences of Theorem 1.5 and Proposition 5.3, related to Conjecture 5.4.

Corollary 5.5. *For every flag homology sphere Δ of dimension $d - 1$ we have*

$$(5.4) \qquad \gamma(\Delta, x) \;=\; \sum_{F \in \Sigma_{d-1}} \xi_F(\Gamma_F, x),$$

where Γ_F is as in Corollary 4.4 for each $F \in \Sigma_{d-1}$. In particular, the validity of Conjecture 5.4 for homology subdivisions Γ of dimension at most $d - 1$ implies the validity of Conjecture 1.1 for homology spheres Δ of dimension at most $d - 1$.

Proof. Setting $\ell_F(\Gamma_F, x) = \sum_i \xi_{F,i}\, x^i (1 + x)^{|F| - 2i}$ in (4.2) and changing the order of summation, results in (5.4). Alternatively, one can apply (5.3) to the subdivision guaranteed by Theorem 1.5 and note that $\gamma(\mathrm{link}_{\Sigma_{d-1}}(F), x) = 1$ for every $F \in \Sigma_{d-1}$. The last sentence in the statement of the corollary follows from (5.4). $\qquad\square$

Corollary 5.6. *The validity of Conjecture 5.4 for homology subdivisions Γ of dimension at most $d - 1$ implies the validity of Conjecture 1.4 for homology spheres Δ and subdivisions Δ' of dimension at most $d - 1$.*

Proof. We observe that the term corresponding to $F = \varnothing$ in the sum of the right-hand side of (5.3) is equal to $\gamma(\Delta, x)$. Thus, the result follows from (5.3), Corollary 5.5 and the fact that the link of every nonempty face of a flag homology sphere is also a flag homology sphere of smaller dimension. $\qquad\square$

Proposition 5.7. *Conjecture 5.4 holds for subdivisions of the 3-dimensional simplex.*

Proof. Let Γ be a flag vertex-induced homology subdivision of the $(d - 1)$-dimensional simplex 2^V and let Δ be the homology subdivision of Σ_{d-1} considered in Proposition 4.6. Applying (5.3) to this subdivision and noting that $\gamma(\mathrm{link}_{\Sigma_{d-1}}(F), x) = 1$ for every $F \in \Sigma_{d-1}$, we get

$$\gamma(\Delta, x) \;=\; \sum_{F \in \Sigma_{d-1}} \xi_F(\Delta_F, x).$$

By definition of Δ, the restriction Δ_F is a cone over the restriction of Δ to a proper face of F for every $F \in \Sigma_{d-1}$ which is not contained in V. Since every such subdivision has a zero local h-vector [13, p. 821], the previous formula can be rewritten as

$$(5.5) \qquad \gamma(\Delta, x) \;=\; \sum_{F \subseteq V} \xi_F(\Gamma_F, x).$$

Assume now that $d = 4$, so that $\xi(\Gamma, x) = \xi_0 + \xi_1 x + \xi_2 x^2$ for some integers ξ_0, ξ_1, ξ_2. Since $\xi_0 = 0$ and $\xi_1 \geq 0$ by part (b) of Example 5.2, it suffices to show that $\xi_2 \geq 0$. For that, we observe that the only contribution to the coefficient of x^2 in the right-hand side of (5.5) comes from the term with $F = V$. As a result, ξ_2 is equal to the coefficient of x^2 in $\gamma(\Delta, x)$. Since, Δ is a 3-dimensional flag homology sphere (by Proposition 4.6), this coefficient is

nonnegative by the Davis-Okun theorem [9, Theorem 11.2.1] and the result follows.   □

Proof of Theorem 1.3. For 3-dimensional spheres the result follows from Proposition 5.7 and Corollary 5.6. Assume now that Δ and Δ' have dimension 4. Then we can write $\gamma(\Delta, x) = 1 + \gamma_1(\Delta)x + \gamma_2(\Delta)x^2$ and $\gamma(\Delta', x) = 1 + \gamma_1(\Delta')x + \gamma_2(\Delta')x^2$. Since $\gamma_1(\Delta) = f_0(\Delta) - 8$ and $\gamma_1(\Delta') = f_0(\Delta') - 8$, where $f_0(\Delta)$ and $f_0(\Delta')$ is the number of vertices of Δ and Δ', respectively, it is clear that $\gamma_1(\Delta') \geq \gamma_1(\Delta)$. As the computation in the proof of [10, Corollary 2.2.2] shows, we also have

$$2\gamma_2(\Delta) \;=\; \sum_{v \in \mathrm{vert}(\Delta)} \gamma_2(\mathrm{link}_\Delta(v)),$$

where $\mathrm{vert}(\Delta)$ is the set of vertices of Δ. Similarly, we have

$$2\gamma_2(\Delta') \;=\; \sum_{v' \in \mathrm{vert}(\Delta')} \gamma_2(\mathrm{link}_{\Delta'}(v')),$$

where we may assume that $\mathrm{vert}(\Delta) \subseteq \mathrm{vert}(\Delta')$. Since $\mathrm{link}_{\Delta'}(v)$ is a flag vertex-induced homology subdivision of $\mathrm{link}_\Delta(v)$ for every $v \in \mathrm{vert}(\Delta)$, by Lemma 2.5, we have $\gamma_2(\mathrm{link}_{\Delta'}(v)) \geq \gamma_2(\mathrm{link}_\Delta(v))$ by the 3-dimensional case, treated earlier, for every such vertex v. Since $\mathrm{link}_{\Delta'}(v')$ is a 3-dimensional flag homology sphere, we also have $\gamma_2(\mathrm{link}_{\Delta'}(v')) \geq 0$ by the Davis-Okun theorem for every $v' \in \mathrm{vert}(\Delta') \smallsetminus \mathrm{vert}(\Delta)$. Hence $\gamma_2(\Delta') \geq \gamma_2(\Delta)$ and the result follows.   □

Question 5.8. Does $\gamma(\Delta') \geq \gamma(\Delta)$ hold for every flag homology sphere Δ and every flag homology subdivision Δ' of Δ?

6. Special cases

This section provides some evidence in favor of the validity of Conjecture 5.4 other than that provided by Proposition 5.7.

Simplicial joins. Let Γ be a homology subdivision of the simplex 2^V and Γ' be a homology subdivision of the simplex $2^{V'}$, where V and V' are disjoint finite sets. Then $\Gamma * \Gamma'$ is a homology subdivision of the simplex $2^V * 2^{V'} = 2^{V \cup V'}$ and given subsets $F \subseteq V$ and $F' \subseteq V'$, the restriction of $\Gamma * \Gamma'$ to the face $F \cup F'$ of this simplex satisfies $(\Gamma * \Gamma')_{F \cup F'} = \Gamma_F * \Gamma'_{F'}$. Since $h(\Gamma_F * \Gamma'_{F'}, x) = h(\Gamma_F, x) \, h(\Gamma'_{F'}, x)$, the defining equation (3.2) and a straightforward computation show that

$$\ell_{V \cup V'}(\Gamma * \Gamma', x) \;=\; \ell_V(\Gamma, x) \, \ell_{V'}(\Gamma', x).$$

This equation and (5.1) imply that

(6.1) $$\xi_{V \cup V'}(\Gamma * \Gamma', x) \;=\; \xi_V(\Gamma, x) \, \xi_{V'}(\Gamma', x).$$

From the previous formula and Lemma 2.5 we conclude that if Γ and Γ' satisfy the assumptions and the conclusion of Conjecture 5.4, then so does $\Gamma * \Gamma'$.

Edge subdivisions. Following [8, Section 5.3], we refer to the stellar subdivision on an edge of a simplicial complex Γ as an *edge subdivision*. As mentioned in Section 3.2, flagness of a simplicial complex is preserved by edge subdivisions. The following statement describes a class of flag (geometric) subdivisions of the simplex with nonnegative local γ-vectors.

Proposition 6.1. *For every subdivision Γ of the simplex 2^V which can be obtained from the trivial subdivision by successive edge subdivisions, we have $\xi_V(\Gamma) \geq 0$.*

Proof. Let Γ be a subdivision of 2^V and Γ' be the edge subdivision of Γ on $e = \{a, b\} \in \Gamma$. Thus we have $\Gamma' = (\Gamma \smallsetminus \mathrm{st}_\Gamma(e)) \cup (\{v\} * \partial(e) * \mathrm{link}_\Gamma(e))$, where v is the new vertex added and $\partial(e) = \{\varnothing, \{a\}, \{b\}\}$.

By appealing to (3.4) and noticing that the right-hand side of this formula vanishes except when $E \in \{\varnothing, e\}$ (or by direct computation), we find that

$$\ell_V(\Gamma', x) \;=\; \ell_V(\Gamma, x) + x\,\ell_V(\Gamma, e, x).$$

Thus it suffices to prove the following claim: if the γ-polynomial corresponding to $\ell_V(\Gamma, E, x)$ has nonnegative coefficients for every face $E \in \Gamma$ of positive dimension (meaning that $\ell_V(\Gamma, E, x)$ can be written as a linear combination of the polynomials $x^i(1 + x)^{d-|E|-2i}$ with nonnegative coefficients for every $|E| \geq 2$), then the same holds for Γ'. We consider a face $E \in \Gamma'$ of positive dimension and distinguish the following cases (we note that E cannot contain e and that if $E \in \Gamma$, then the carrier $\sigma(E) \subseteq V$ of E is the same, whether considered with respect to Γ or Γ'):

Case 1: $E \in \Gamma \smallsetminus \mathrm{link}_\Gamma(e)$. The links $\mathrm{link}_{\Gamma'_F}(E)$ and $\mathrm{link}_{\Gamma_F}(E)$ are then combinatorially isomorphic for every $F \subseteq V$ which contains the carrier of E (these two links are equal if $E \cup e \notin \Gamma$) and the defining equation (3.5) implies that $\ell_V(\Gamma', E, x) = \ell_V(\Gamma, E, x)$.

Case 2: $E \in \mathrm{link}_\Gamma(e)$. For $F \subseteq V$ which contains the carrier of E, the link $\mathrm{link}_{\Gamma'_F}(E)$ is equal to either $\mathrm{link}_{\Gamma_F}(E)$ or to the edge subdivision of $\mathrm{link}_{\Gamma_F}(E)$ on e, in case F does not or does contain the carrier of e, respectively. It follows from this and (3.3) that (see also [10, Proposition 2.4.3])

$$h(\mathrm{link}_{\Gamma'_F}(E), x) \;=\; \begin{cases} h(\mathrm{link}_{\Gamma_F}(E), x), & \text{if } \sigma(e) \not\subseteq F \\ h(\mathrm{link}_{\Gamma_F}(E), x) + x\,h(\mathrm{link}_{\Gamma_F}(E \cup e), x), & \text{if } \sigma(e) \subseteq F \end{cases}$$

and then from (3.5) that $\ell_V(\Gamma', E, x) = \ell_V(\Gamma, E, x) + x\ell_V(\Gamma, E \cup e, x)$.

Case 3: $E \notin \Gamma$. Then we must have $E \in \{v\} * \partial(e) * \mathrm{link}_\Gamma(e)$ and, in particular, $v \in E$. We distinguish two subcases:

Suppose first that E intersects e and set $E' = (E \smallsetminus \{v\}) \cup e$. Then $\mathrm{link}_{\Gamma'_F}(E) = \mathrm{link}_{\Gamma_F}(E')$ for every $F \subseteq V$ which contains the carrier of E in Γ' (the latter coincides with the carrier of E' in Γ) and hence $\ell_V(\Gamma', E, x) = \ell_V(\Gamma, E', x)$.

Suppose finally that $E \cap e = \varnothing$ and set $E' = (E \smallsetminus \{v\}) \cup e$. Then $\mathrm{link}_{\Gamma'_F}(E) = \mathrm{link}_{\Gamma_F}(E') * \partial(e)$ for every $F \subseteq V$ which contains the carrier of E in Γ'. Therefore we have $h(\mathrm{link}_{\Gamma'_F}(E), x) = (1+x)\, h(\mathrm{link}_{\Gamma_F}(E'), x)$ for every such F and hence $\ell_V(\Gamma', E, x) = (1 + x)\, \ell_V(\Gamma, E', x)$.

The expressions obtained for $\ell_V(\Gamma', E, x)$ and our assumption on Γ show that indeed, the corresponding γ-polynomial has nonnegative coefficients in all cases. $\qquad\square$

Barycentric and cluster subdivisions. As a special case of Proposition 6.1, the (first) barycentric subdivision of the simplex 2^V has nonnegative local γ-vector. Several combinatorial interpretations for its entries are given in [3]. Similar results appear there for the simplicial subdivision of a simplex defined by the positive part of the cluster complex, associated to a finite root system.

The following special case of Conjecture 5.4 might also be interesting to explore. The notion of a CW-regular subdivision can be defined by replacing the simplicial complex Δ' in the definition of a topological subdivision (Definition 2.2) by a regular CW-complex; see [13, p. 839].

Question 6.2. Does Conjecture 5.4 hold for the barycentric subdivision of any CW-regular subdivision of the simplex?

Acknowledgments

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