Character formulas and descents for the hyperoctahedral group

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Abstract

A general setting to study a certain type of formulas, expressing characters of the symmetric group $\mathfrak{S}_n$ explicitly in terms of descent sets of combinatorial objects, has been developed by two of the authors. This theory is further investigated in this paper and extended to the hyperoctahedral group B_n. Key ingredients are a new formula for the irreducible characters of B_n, the signed quasisymmetric functions introduced by Poirier, and a new family of matrices of Walsh–Hadamard type. Applications include formulas for natural B_n-actions on coinvariant and exterior algebras and on the top homology of a certain poset in terms of the combinatorics of various classes of signed permutations, as well as a B_n-analogue of an equidistribution theorem of Désarménien and Wachs.

Keywords: symmetric group, hyperoctahedral group, character, quasisymmetric function, Schur-positivity, descent set, derangement

1. Introduction

One of the main goals of combinatorial representation theory, as described in the survey article [9], is to provide explicit formulas which express the

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values of characters of interesting representations as weighted enumerations of nice combinatorial objects. Perhaps the best known example of such a formula is the Murnaghan–Nakayama rule [29, page 117] [43, Section 7.17] for the irreducible characters of the symmetric group $\mathfrak{S}_n$.

Several such formulas of a more specific type, expressing characters of the symmetric groups and their Iwahori–Hecke algebras in terms of the distribution of the descent set over classes of permutations, or other combinatorial objects, have been discovered in the past two decades. The prototypical example is Roichman's rule [35] for the irreducible characters of $\mathfrak{S}_n$ (and the corresponding Hecke algebra), where the enumerated objects are either Knuth classes of permutations, or standard Young tableaux. Other notable examples include the character of the Gelfand model (i.e., the multiplicity-free sum of all irreducible characters) [3], characters of homogenous components of the coinvariant algebra [2], Lie characters [24], characters of Specht modules of zigzag shapes [23], characters induced from a lower ranked exterior algebra [17] and k-root enumerators [37], determined by the distribution of the descent set over involutions, elements of fixed Coxeter length, conjugacy classes, inverse descent classes, arc permutations and k-roots of the identity permutation, respectively. The formulas in question evaluate these characters by $\{-1, 0, 1\}$-weighted enumerations of the corresponding classes of permutations, where exactly the same weight function appears in all summations. Some new examples are given in Sections 7–8 of this paper.

An abstract framework for this phenomenon, which captures all aforementioned examples, was proposed in [6]. Characters which are expressed by such formulas (see Definition 3.1) are called *fine characters*, and classes which carry them are called *fine sets*. It was shown in [6] that the equality of two fine characters is equivalent to the equidistribution of the descent set over the corresponding fine sets. This implies, in particular, the equivalence of classical theorems of Lusztig–Stanley in invariant theory [40] and Foata–Schützenberger in permutation statistics [21]. Furthermore, it was shown in [6] that fine sets can be characterized by the symmetry and Schur-positivity of the associated quasisymmetric functions. For the latter, the reader is referred to Gessel–Reutenauer's seminal paper [24].

This paper investigates this setting further and provides a nontrivial extension to the hyperoctahedral group B_n. Section 3.1 gives a more explicit version of the main result of [6]. This version (Theorem 3.2) states that a given $\mathfrak{S}_n$-character is carried by a fine set $\mathcal{B}$ if and only if its Frobenius characteristic is equal to the quasisymmetric generating function of the descent set over $\mathcal{B}$. To extend this result to the group B_n, suitable signed analogues of the concepts of fine characters and fine sets have to be introduced (see Definition 3.5) and suitable signed analogues of the fundamental quasisymmetric functions, namely those defined and studied by Poirier [32], are employed. For the former task, a signed analogue of the concept of descent set is used (see Section 2.2) and an analogue of Roichman's rule for the irreducible characters of B_n (Theorem 4.1) is proven. The weights involved in this rule are used to define type B fine sets as those sets whose weighted enumeration determines a non-virtual B_n-character. The main result (Theorem 3.6) of this paper states that a given B_n-character is carried by a fine set $\mathcal{B}$ if and only if its Frobenius characteristic is equal to the quasisymmetric generating function of the signed descent set over $\mathcal{B}$.

The proof of one direction of Theorem 3.6 amounts to the invertibility of a new family of matrices of Walsh–Hadamard type, introduced in Section 4.2. This is shown by a tricky computation of their determinants. Equidistribution and Schur-positivity phenomena, similar to those in [6], are easily derived from Theorem 3.6 (see Corollary 3.7).

Sections 5–8 show that nearly all examples of fine $\mathfrak{S}_n$-characters mentioned earlier have natural B_n-analogues. These include the irreducible characters of B_n, the Gelfand model, the characters of the natural B_n-action on the homogeneous components of the coinvariant algebra of type B, signed analogues of the Lie characters, characters induced from exterior algebras and k-root enumerators, with corresponding fine sets consisting of elements of Knuth classes of type B_n, involutions, signed permutations of fixed flag-inversion number or flag-major index, conjugacy classes in B_n, signed analogues of arc permutations and k-roots of the identity signed permutation, respectively. Theorem 3.6 implies explicit formulas for these characters in terms of the distribution of the signed descent set over the corresponding fine sets. Section 7.1 is concerned with the character of the $\mathfrak{S}_n$-action on the top homology of the poset of injective words, studied by Reiner and Webb [34], and with its natural B_n-analogue. This allows us to interpret a theorem of Désarménien and Wachs [15, 16] on the equidistribution of the descent set over derangements and desarrangements in $\mathfrak{S}_n$ in the language of $\mathfrak{S}_n$-fine sets, and to derive a B_n-analogue. The latter task provided much of the motivation behind this paper.

The structure of this paper is as follows. Section 2 reviews background material from combinatorial representation theory and sets up the notation on permutations, partitions, Young tableaux, compositions, descent sets and their signed analogues which is necessary to define fine sets. The main definitions and results are stated in Section 3, which also derives the aforementioned result on $\mathfrak{S}_n$-fine sets (Theorem 3.2) from the main result of [6]. Section 4 states and proves the B_n-analogue of Roichman's rule (Theorem 4.1) and uses this result, together with methods from linear algebra, to prove the main result (Theorem 3.6) on B_n-fine sets. As already mentioned, Sections 5–8 discuss applications and examples. Section 9 concludes with remarks and open problems.

2. Background and notation

This section fixes notation and briefly reviews background material regarding the combinatorics of (signed) permutations and Young (bi)tableaux, the representation theory of the symmetric and hyperoctahedral groups and the theory of symmetric and quasisymmetric functions which will be needed in the sequel. More information on these topics and any undefined terminology can be found in [29, Chapter I] [43, Chapter 7] [44] [45].

Throughout this paper, $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ will be sequences of pairwise commuting indeterminates. For positive integers n we set $[n] := \{1, 2, \ldots, n\}$ and $\Omega_n := \{1, 2, \ldots, n\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. The absolute value $|a|$ of $a \in \Omega_n$ is the element of $[n]$ obtained from a by simply forgetting the bar, if present. We will consider the set $[n]$ totally ordered by its natural order $<$ and Ω_n totally ordered by the order

$$\bar{1} <_r \bar{2} <_r \cdots <_r \bar{n} <_r 1 <_r 2 <_r \cdots <_r n, \tag{2.1}$$

corresponding to the right lexicographic order on the set $[n] \times \{-, +\}$, and endowed with the standard involution $^{-}\colon \Omega_n \to \Omega_n$, mapping a to $\bar{a}$ and vice versa, for every $a \in [n]$.

2.1. Compositions and partitions

A *composition* of a positive integer n, written as $\alpha \vDash n$, is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers (parts) summing to n. We will write $S(\alpha) = \{r_1, r_2, \ldots, r_k\}$ for the set of partial sums $r_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i$ $(1 \le i \le k)$ of such α, and will denote by $\mathrm{Comp}(n)$ the set of all compositions of n. Clearly, the map which assigns $S(\alpha)$ to α is a bijection from $\mathrm{Comp}(n)$ to the set of all subsets of $[n]$ which contain n. A *partition* of n, written as $\lambda \vdash n$, is a composition λ of n whose parts appear in a weakly decreasing order. We shall also consider the empty composition (and partition) of $n = 0$.

A *bipartition* of n, written as $(\lambda, \mu) \vdash n$, is a pair (λ, μ) of (possibly empty) partitions such that $\lambda \vdash k$ and $\mu \vdash n-k$ for some $0 \le k \le n$.

A *signed composition* of n is a composition γ of n some of the parts of which may be barred. Such a γ can be encoded by its set of partial sums $S(\gamma) = \{s_1, s_2, \dots, s_k\}$, where $s_1 < s_2 < \cdots < s_k = n$, together with the sign map $\varepsilon : S(\gamma) \to \{-, +\}$ defined by setting $\varepsilon(s_i) = +$ if the ith part of γ is unbarred and $\varepsilon(s_i) = -$ otherwise. Clearly, the map which assigns the pair $(S(\gamma), \varepsilon)$ to γ is a bijection from the set $\mathrm{Comp}^B(n)$ of all signed compositions of n to the set $\Sigma^B(n)$ consisting of all pairs (S, ε), where S is a subset of $[n]$ which contains n and $\varepsilon : S \to \{-, +\}$ is a map. Thus, as observed for instance in [25, Section 3.2], the total number of signed compositions of n is equal to $2 \cdot 3^{n-1}$. We will consider the elements of $\Sigma^B(n)$, which we think of as "signed sets", instead of the corresponding signed compositions, whenever this is notationally convenient.

Given $\sigma = (S, \varepsilon) \in \Sigma^B(n)$, with $S = \{s_1, s_2, \dots, s_k\}$ and $s_1 < s_2 < \cdots < s_k = n$, we may extend ε to a map $\tilde{\varepsilon} : [n] \to \{-, +\}$ by setting $\tilde{\varepsilon}(j) := \varepsilon(s_i)$ for all $s_{i-1} < j \le s_i$ $(1 \le i \le k)$, where $s_0 := 0$. We will refer to $\tilde{\varepsilon}$ as the *sign vector* of σ (and of the corresponding signed composition). The following notation will be useful in Section 2.4.

Definition 2.1. For $\sigma = (S, \varepsilon) \in \Sigma^B(n)$ we denote by $\mathrm{wDes}(\sigma)$ the set of elements $s_i \in S$, other than $s_k = n$, for which either $\varepsilon(s_i) = \varepsilon(s_{i+1})$, or else $\varepsilon(s_i) = +$ and $\varepsilon(s_{i+1}) = -$.

For example, if $n = 9$ and $\gamma = (2, \bar{1}, \bar{2}, 1, 3)$, then for the signed set $\sigma = (S, \varepsilon)$ corresponding to γ we have $S = \{2, 3, 5, 6, 9\}$, $\varepsilon(2) = \varepsilon(6) = \varepsilon(9) = +$, $\varepsilon(3) = \varepsilon(5) = -$, $\mathrm{wDes}(\sigma) = \{2, 3, 6\}$ and sign vector $\tilde{\varepsilon} = (+, +, -, -, -, +, +, +, +)$.

2.2. Permutations and tableaux

We will denote by $\mathfrak{S}_n$ the symmetric group of all permutations of the set $[n]$, namely bijective maps $w : [n] \to [n]$, and by $\mathrm{SYT}(\lambda)$ the set of all standard Young tableaux of shape $\lambda \vdash n$. We recall that the Robinson–Schensted correspondence is a bijection of fundamental importance from $\mathfrak{S}_n$ to the set of pairs (P, Q) of standard Young tableaux of the same shape and size n. The *descent set* of a permutation $w \in \mathfrak{S}_n$ is defined as $\mathrm{Des}(w) := \{i \in [n-1] : w(i) > w(i+1)\}$. The *descent set* $\mathrm{Des}(Q)$ of a standard Young tableau $Q \in \mathrm{SYT}(\lambda)$, where $\lambda \vdash n$, is the set of all $i \in [n-1]$ for which $i+1$ appears in a lower row in Q than i does. A basic property of the Robinson–Schensted correspondence asserts that $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$, where $(P(w), Q(w))$ is the pair of tableaux associated to $w \in \mathfrak{S}_n$.

The hyperoctahedral group B_n consists of all signed permutations of length n, meaning bijective maps $w : \Omega_n \to \Omega_n$ such that $w(a) = b \Rightarrow w(\bar{a}) = \bar{b}$ for every $a \in \Omega_n$. Unless indicated otherwise, we will think of a signed permutation $w \in B_n$ as the sequence of values $(w(1), w(2), \ldots, w(n))$, namely as a permutation in $\mathfrak{S}_n$, written in one line notation, with some of its entries barred. The (unsigned) descent set of $w \in B_n$ is defined as the descent set of the sequence $(w(1), w(2), \ldots, w(n))$ with respect to the total order (2.1), namely $\mathrm{Des}(w) := \{i \in [n-1] : w(i) >_r w(i+1)\}$.

Definition 2.2. The *signed* (or *colored*) *descent set* of $w \in B_n$, denoted $\mathrm{sDes}(w)$, is the signed set $(S, \varepsilon) \in \Sigma^B(n)$ defined as follows:

- The set S consists of n along with all $s \in [n-1]$ for which either $w(s) >_r w(s+1)$ or $w(s)$ is barred and $w(s+1)$ is unbarred.
- For every $s \in S$, $\varepsilon(s) = -$ if $w(s)$ is barred and $\varepsilon(s) = +$ otherwise.

In other words, denoting $|w| = (|w(1)|, \ldots, |w(n)|) \in \mathfrak{S}_n$ and defining $\varepsilon_w : [n] \to \{-, +\}$ by $\varepsilon_w(i) = -$ if $w(i)$ is barred and $\varepsilon_w(i) = +$ otherwise, the set S is the union of $\mathrm{Des}(|w|)$, $\{n\}$ and $\{i \in [n-1] \, : \, \varepsilon_w(i) \neq \varepsilon_w(i+1)\}$, while ε is the restriction of ε_w to S. For example, if $w = (\bar{5}, \bar{2}, \bar{8}, 1, 3, 9, 4, \bar{6}, \bar{7}) \in B_9$ then $\mathrm{sDes}(w) = (S, \varepsilon)$ with $S = \{1, 3, 6, 7, 9\}$ and $\varepsilon(1) = \varepsilon(3) = \varepsilon(9) = -$, $\varepsilon(6) = \varepsilon(7) = +$. We note that if $\sigma = \mathrm{sDes}(w)$ for some $w \in B_n$ then $\mathrm{wDes}(\sigma) = \mathrm{Des}(w) := \{i \in [n-1] : w(i) >_r w(i+1)\}$, where $\mathrm{wDes}(\sigma)$ is as in Definition 2.1.

Given a bipartition (λ, μ) of n, a *standard Young bitableau* of shape (λ, μ) and size n is a pair $Q = (Q^+, Q^-)$ of tableaux which are (strictly) increasing along rows and columns and have the following properties: (a) Q^+ has shape λ; (b) Q^- has shape μ; and (c) every element of $[n]$ appears (exactly once) in either Q^+ or Q^-. We will denote by $\mathrm{SYT}(\lambda, \mu)$ the set of all standard Young bitableaux of shape (λ, μ).

The *descent set*, denoted by $\mathrm{Des}(Q)$, of a bitableau $Q = (Q^+, Q^-) \in \mathrm{SYT}(\lambda, \mu)$ consists of all $i \in [n-1]$, such that i and $i+1$ appear in the same tableau (Q^+ or Q^-) and $i+1$ appears in a lower row than i, or i appears in Q^+ and $i+1$ appears in Q^-.

Definition 2.3. The *signed* (or *colored*) *descent set*, denoted by $\mathrm{sDes}(Q)$, of a bitableau $Q = (Q^+, Q^-) \in \mathrm{SYT}(\lambda, \mu)$ is the element $(S, \varepsilon) \in \Sigma^B(n)$ defined as follows:

- The set S consists of n along with all $s \in [n-1]$ for which either s and $s+1$ appear in the same tableau (Q^+ or Q^-) and $s+1$ appears in a lower row than s, or s and $s+1$ appear in different tableaux.

- For every $s \in S$, $\varepsilon(s) = +$ if s appears in Q^+ and $\varepsilon(s) = -$ otherwise.

Clearly, for the corresponding sign vector $\tilde{\varepsilon} : [n] \to \{-, +\}$ we have, for every $i \in [n]$, $\tilde{\varepsilon}(i) = +$ if i appears in Q^+ and $\tilde{\varepsilon}(i) = -$ if i appears in Q^-.

The Robinson–Schensted correspondence has a natural B_n-analogue; see, for instance, [41, Section 6] [44] and Section 5. This analogue is a bijection from B_n to the set of pairs (P, Q) of standard Young bitableaux of the same shape and size n. It has the property that $\mathrm{sDes}(w) = \mathrm{sDes}(Q^B(w))$, where $(P^B(w), Q^B(w))$ is the pair associated to $w \in B_n$.

2.3. Characters and symmetric functions

Conjugacy classes in $\mathfrak{S}_n$ consist of permutations of given cycle type $\lambda \vdash n$. As a result, the (complex) irreducible characters of $\mathfrak{S}_n$ are indexed by partitions of n. We will denote by χ^λ the irreducible character corresponding to $\lambda \vdash n$. For a class function $\chi : \mathfrak{S}_n \to \mathbb{C}$ and composition $\alpha \models n$, we will write $\chi(\alpha)$ for the value of χ on any element of $\mathfrak{S}_n$ whose cycle type is (the decreasing rearrangement of) α.

We will denote by $\Lambda(x)$ the $\mathbb{C}$-algebra of symmetric functions in x. The algebra $\Lambda(x)$ encodes the combinatorics of characters of the symmetric groups via the Frobenius characteristic map, defined by

$$\mathrm{ch}(\chi) \;=\; \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi(w) p_w(x) \tag{2.2}$$

where $\chi : \mathfrak{S}_n \to \mathbb{C}$ is a class function, $p_w(x) = p_\alpha(x)$ for every permutation $w \in \mathfrak{S}_n$ of cycle type $\alpha \vdash n$, and $p_\alpha(x)$ is a power sum symmetric function. This map is a $\mathbb{C}$-linear isomorphism from the space of such class functions on $\mathfrak{S}_n$ to the degree n homogeneous part $\Lambda^n(x)$ of $\Lambda(x)$, and satisfies $\mathrm{ch}(\chi^\lambda) = s_\lambda(x)$ for every $\lambda \vdash n$; see [43, Section 7.18] for a detailed discussion and further information.

The values $\chi^\lambda(\alpha)$ of the irreducible characters of $\mathfrak{S}_n$ appear in the Frobenius formula

$$p_\alpha(x) \;=\; \sum_{\lambda \vdash n} \chi^\lambda(\alpha) s_\lambda(x) \tag{2.3}$$

for $\mathfrak{S}_n$, expressing $p_\alpha(x)$ as a linear combination of Schur functions.

We now recall an explicit formula for $\chi^\lambda(\alpha)$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp}(n)$ be a composition of n with $S(\alpha) = \{r_1, r_2, \ldots, r_k\}$, as in Section 2.1 (so that $r_k = n$), and set $r_0 = 0$. A subset S of $[n-1]$ is said to be *α-unimodal* (or *unimodal with respect to α*) [6, Definition 3.1] if the intersection of S with each of the sets $\{r_{i-1}+1, \ldots, r_i-1\}$ $(1 \le i \le k)$ is a prefix (possibly empty) of the latter. For instance, if $\alpha = (n)$, then the α-unimodal subsets of $[n-1]$ are those equal to $[p]$ for some $p \in \{0, 1, \ldots, n-1\}$; and if $\alpha = (1, 1, \ldots, 1)$, then every subset of $[n-1]$ is α-unimodal. As another example, if $\alpha = (3, 1, 4, 2)$, then $\{1, 3, 5, 6\}$ is α-unimodal but $\{1, 3, 5, 7\}$ is not. We will denote by U_α the set of α-unimodal subsets of $[n-1]$.

The following theorem is a special case of [35, Theorem 4]; see also [9, Section I.2] and references therein. A direct combinatorial proof appeared in [33].

Theorem 2.4. ([35]) *For all partitions $\lambda \vdash n$ and compositions $\alpha \models n$,*

$$\chi^\lambda(\alpha) \;=\; \sum_{\substack{Q \in \mathrm{SYT}(\lambda) \\ \mathrm{Des}(Q) \in U_\alpha}} (-1)^{|\mathrm{Des}(Q) \smallsetminus S(\alpha)|}. \tag{2.4}$$

The concept of α-unimodality, and Theorem 2.4 in particular, were recently applied to prove conjectures of Regev concerning induced characters [17, Section 9] and of Shareshian and Wachs concerning chromatic quasisymmetric functions [8].

We now briefly describe the analogue of this theory for the group B_n. Conjugacy classes in B_n, and hence (complex) irreducible B_n-characters, are in one-to-one correspondence with bipartitions of n. More precisely, each element $w \in B_n$, viewed as a permutation of the set Ω_n, can be written as a product of disjoint cycles of total length $2n$. Moreover, if $c = (a_1, a_2, \ldots, a_k)$ is such a cycle then so is $\bar{c} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k)$, and either c and $\bar{c}$ are disjoint or else $c = \bar{c}$. In the former case, the product $c\bar{c}$ is said to be a *positive cycle* of w, of length k; otherwise k is necessarily even and $c = \bar{c}$ is said to be a *negative cycle* of w, of length $k/2$. Then the pair of partitions (α, β) for which the parts of α (respectively, β) are the lengths of the positive (respectively, negative) cycles of w is a bipartition of n, called the *signed cycle type* of w. Two elements of B_n are congugate if and only if they have the same signed cycle type. We will denote by $C_{\alpha,\beta}$ the conjugacy class of elements of B_n of signed cycle type (α, β), by $\chi^{\lambda,\mu}$ the irreducible B_n-character corresponding to $(\lambda, \mu) \vdash n$, and by $\chi(\alpha, \beta)$ the value of a class function $\chi : B_n \to \mathbb{C}$ at an arbitrary element of $C_{\alpha,\beta}$.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$, we set

$$p^+_\lambda(x, y) \ = \ p^+_{\lambda_1} p^+_{\lambda_2} \cdots p^+_{\lambda_r} \qquad \text{and} \qquad p^-_\lambda(x, y) \ = \ p^-_{\lambda_1} p^-_{\lambda_2} \cdots p^-_{\lambda_r}$$

where

$$p^+_k \ = \ \sum_{i\geq 1} (x^k_i + y^k_i) \qquad \text{and} \qquad p^-_k \ = \ \sum_{i\geq 1} (x^k_i - y^k_i)$$

for $k \geq 1$. Thus $p^+_\lambda(x, y)$ and $p^-_\lambda(x, y)$ are homogeneous elements of degree n of $\Lambda(x) \otimes \Lambda(y)$. The Frobenius characteristic of a class function $\chi : B_n \to \mathbb{C}$ is defined as

$$\mathrm{ch}(\chi) \ = \ \frac{1}{2^n n!} \sum_{w\in B_n} \chi(w) p_w(x, y) \tag{2.5}$$

where $p_w(x, y) := p^+_\alpha(x, y)p^-_\beta(x, y)$ if $w \in B_n$ has signed cycle type (α, β). The Frobenius characteristic is a $\mathbb{C}$-linear isomorphism from the space of such class functions on B_n to the degree n homogeneous part of $\Lambda(x) \otimes \Lambda(y)$. It satisfies $\mathrm{ch}(\chi^{\lambda,\mu}) = s_\lambda(x)s_\mu(y)$ for every bipartition $(\lambda, \mu) \vdash n$. The following lemma will be used in Section 7.1.

Lemma 2.5. ([32, Lemma 21 (i)]) *For the trivial character of B_n we have $\mathrm{ch}(1_{B_n}) = s_n(x)$.*

The Frobenius formula for B_n states that

$$p^+_\alpha(x, y)p^-_\beta(x, y) \ = \ \sum_{(\lambda,\mu)\vdash n} \chi^{\lambda,\mu}(\alpha, \beta)s_\lambda(x)s_\mu(y) \tag{2.6}$$

for every bipartition $(\alpha, \beta) \vdash n$.

An analogue of Theorem 2.4 for B_n will be proved in Section 4 (see Theorem 4.1).

We write $\langle \cdot\, , \cdot \rangle$ for the inner product on $\Lambda(x)$ (respectively, on $\Lambda(x) \otimes \Lambda(y)$) for which the Schur functions $s_\lambda(x)$ (respectively, the functions $s_\lambda(x)s_\mu(y)$) form an orthonormal basis.

2.4. Quasisymmetric functions

We denote by QSym^n the $\mathbb{C}$-vector space of homogeneous quasisymmetric functions of degree n in x. The *fundamental quasisymmetric function* associated to $S \subseteq [n-1]$ is defined as

$$F_{n,S}(x) \ = \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{2.7}$$

The set $\{F_{n,S}(x) : S \subseteq [n-1]\}$ is known to be a basis of QSym^n. The following proposition expresses the Schur function $s_\lambda(x)$ as a linear combination of the elements of this basis.

Proposition 2.6. ([43, Theorem 7.19.7]) *For every $\lambda \vdash n$,*

$$s_\lambda(x) \ = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(Q)}(x). \tag{2.8}$$

Different type B analogues of quasisymmetric functions have been suggested [14, 30]. The B_n-analogues of the fundamental quasisymmetric functions that we need were introduced (in the more general setting of r-colored permutations) by Poirier [32, Section 3] and were further studied in [10, 25]. For a signed set $\sigma = (S, \varepsilon) \in \Sigma^B(n)$ define

$$F_\sigma(x,y) \ = \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n \\ j \in \mathrm{wDes}(\sigma) \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n} \tag{2.9}$$

where $z_{i_j} = x_{i_j}$ if $\tilde{\varepsilon}_j = +$, and $z_{i_j} = y_{i_j}$ if $\tilde{\varepsilon}_j = -$ (we use the notation F_σ, instead of $F_{n,\sigma}$, since n is determined by $\sigma = (S, \varepsilon)$ as the largest element of S). For example, if $n = 6$ and $S = \{2,3,5,6\}$ with sign vector $\tilde{\varepsilon} = (+,+,-,-,-,+)$, then

$$F_\sigma(x,y) \ = \sum_{1 \le i_1 \le i_2 < i_3 < i_4 \le i_5 \le i_6} x_{i_1} x_{i_2} y_{i_3} y_{i_4} y_{i_5} x_{i_6}.$$

We leave it to the reader to verify that this definition is equivalent to the one in [32]. The functions $F_\sigma(x,y)$, where σ ranges over all signed sets in $\Sigma^B(n)$, are linearly independent over $\mathbb{C}$ [32, Corollary 9] and, in fact, form a $\mathbb{C}$-basis of a natural signed analogue of QSym^n; see [25, Section 3.2].

3. Main results: Character formulas and fine sets

This section introduces the concept of fine set for the hyperoctahedral group and explains the main results of this paper, to be proved in the following section. The corresponding concepts and results from [6] for the symmetric group will first be reviewed.

3.1. Review of results for the symmetric group

The following key definition from [6] uses the notion of unimodality with respect to a composition, defined in Section 2.3.

Definition 3.1. ([6, Definition 1.3]) Let χ be a character of the symmetric group $\mathfrak{S}_n$. A set $\mathcal{B}$, endowed with a map $\mathrm{Des} : \mathcal{B} \to 2^{[n-1]}$, is said to be a *fine set* for χ if

$$\chi(\alpha) \;=\; \sum_{b \in \mathcal{B}_\alpha} (-1)^{|\mathrm{Des}(b) \smallsetminus S(\alpha)|} \tag{3.1}$$

for every composition α of n, where $\mathcal{B}_\alpha$ is the set of all elements $b \in \mathcal{B}$ for which $\mathrm{Des}(b)$ is α-unimodal.

For example, according to Theorem 2.4, the set $\mathrm{SYT}(\lambda)$, endowed with the usual descent map $\mathrm{Des} : \mathrm{SYT}(\lambda) \to 2^{[n-1]}$, is a fine set for the irreducible character χ^λ. Several other examples (including some new ones) appear in Sections 5–8.

The following theorem is a version of the main result of [6]; we include a proof since this version is only implicit in [6].

Theorem 3.2. *Let χ be a character of the symmetric group $\mathfrak{S}_n$ and $\mathcal{B}$ be a set endowed with a map $\mathrm{Des} : \mathcal{B} \to 2^{[n-1]}$. Then $\mathcal{B}$ is a fine set for χ if and only if*

$$\mathrm{ch}(\chi) \;=\; \sum_{b \in \mathcal{B}} F_{n,\mathrm{Des}(b)}(x). \tag{3.2}$$

In particular, the distribution of the descent set over $\mathcal{B}$ is uniquely determined by χ.

Proof. Let us express $\chi = \sum_{\lambda \vdash n} c_\lambda \chi^\lambda$ as a linear combination of irreducible characters. Using Proposition 2.6 we find that

$$\mathrm{ch}(\chi) \;=\; \sum_{\lambda \vdash n} c_\lambda \mathrm{ch}(\chi^\lambda) \;=\; \sum_{\lambda \vdash n} c_\lambda s_\lambda(x) \;=\; \sum_{\lambda \vdash n} c_\lambda \sum_{Q \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(Q)}(x).$$

From this we conclude that (3.2) holds if and only if

$$|\{b \in \mathcal{B} : \mathrm{Des}(b) = S\}| \;=\; \sum_{\lambda \vdash n} c_\lambda \cdot |\{Q \in \mathrm{SYT}(\lambda) : \mathrm{Des}(Q) = S\}| \tag{3.3}$$

for every $S \subseteq [n-1]$. Assuming (3.2) holds and using (3.3) and Theorem 2.4, we get

$$\begin{aligned}
\chi(\alpha) \;&=\; \sum_{\lambda \vdash n} c_\lambda \cdot \chi^\lambda(\alpha) \;=\; \sum_{\lambda \vdash n} c_\lambda \sum_{\substack{Q \in \mathrm{SYT}(\lambda) \\ \mathrm{Des}(Q) \in U_\alpha}} (-1)^{|\mathrm{Des}(Q) \smallsetminus S(\alpha)|} \\
&=\; \sum_{\substack{S \subseteq [n-1] \\ S \in U_\alpha}} (-1)^{|S \smallsetminus S(\alpha)|} \sum_{\lambda \vdash n} c_\lambda \cdot |\{Q \in \mathrm{SYT}(\lambda) : \mathrm{Des}(Q) = S\}| \\
&=\; \sum_{\substack{S \subseteq [n-1] \\ S \in U_\alpha}} (-1)^{|S \smallsetminus S(\alpha)|} \cdot |\{b \in \mathcal{B} : \mathrm{Des}(b) = S\}| \\
&=\; \sum_{b \in \mathcal{B}_\alpha} (-1)^{|\mathrm{Des}(b) \smallsetminus S(\alpha)|}
\end{aligned}$$

for every $\alpha \models n$, and conclude that $\mathcal{B}$ is a fine set for χ.

Conversely, assume that $\mathcal{B}$ is a fine set for χ and let $\mathcal{B}'$ be the disjoint union, taken over all partitions $\lambda \vdash n$, of c_λ copies of $\mathrm{SYT}(\lambda)$, endowed with the usual descent map $\mathrm{Des} : \mathcal{B}' \to 2^{[n-1]}$. By Theorem 2.4 the set $\mathcal{B}'$ is also fine for χ. Therefore, [6, Theorem 1.4] implies that the descent set is equidistributed over $\mathcal{B}$ and $\mathcal{B}'$. This exactly means that (3.3) holds for every $S \subseteq [n-1]$ and the proof follows. $\square$

Corollary 3.3. (see [6, Theorem 1.5]) *Let $\mathcal{B}$ be a set endowed with a map $\mathrm{Des} : \mathcal{B} \to 2^{[n-1]}$. The following are equivalent:*

(i) *$\mathcal{B}$ is a fine set for some (non-virtual) character of $\mathfrak{S}_n$.*
(ii) *The quasisymmetric function $F_{\mathcal{B}}(x) := \sum_{b \in \mathcal{B}} F_{n,\mathrm{Des}(b)}(x)$ is symmetric and Schur-positive.*
(iii) *There exist nonnegative integers a_λ, for $\lambda \vdash n$, such that*

$$\sum_{b \in \mathcal{B}} \mathbf{z}^{\mathrm{Des}(b)} \;= \sum_{\lambda \vdash n} a_\lambda \sum_{Q \in \mathrm{SYT}(\lambda)} \mathbf{z}^{\mathrm{Des}(Q)} \tag{3.4}$$

where $\mathbf{z} = (z_1, \ldots, z_{n-1})$. Moreover, if these conditions hold, then $a_\lambda = \langle F_{\mathcal{B}}(x), s_\lambda(x) \rangle$ for each $\lambda \vdash n$.

3.2. Results for the hyperoctahedral group

To state the B_n-analogues of these results, we will use the language of signed compositions and sets, explained in Section 2. Let $\sigma = (S, \varepsilon) \in \Sigma^B(n)$ be a signed set, $\gamma \in \mathrm{Comp}^B(n)$ a signed composition of n with set of partial sums $S(\gamma) = \{r_1, r_2, \ldots, r_k\}$ (so $r_k = n$), and set $r_0 = 0$. The signed set σ is called γ-*unimodal* if S is unimodal with respect to the unsigned composition of n corresponding to γ.

Definition 3.4. The *weight* $\mathrm{wt}_\gamma(\sigma)$ of $\sigma = (S, \varepsilon) \in \Sigma^B(n)$ (equivalently, of the signed composition of n corresponding to σ) with respect to γ is defined as follows:

- $\mathrm{wt}_\gamma(\sigma) := 0$ if either σ is not γ-unimodal, or else for some index $1 \le i \le k$ the sign vector $\tilde{\varepsilon} \in \{-, +\}^n$ of σ is not constant on the set $\{r_{i-1} + 1, \ldots, r_i\}$.
- Otherwise we set

$$\mathrm{wt}_\gamma(\sigma) \;:=\; (-1)^{|S \smallsetminus S(\gamma)| \,+\, n_\gamma(\sigma)} \tag{3.5}$$

  where $n_\gamma(\sigma)$ is the number of indices i for which the elements of $\{r_{i-1} + 1, \ldots, r_i\}$ are assigned the negative sign by (the sign vectors of) both σ and γ.

Given a character χ of B_n and a signed composition $\gamma \in \mathrm{Comp}^B(n)$, we will write $\chi(\gamma)$ for the value of χ at the elements of B_n of signed cycle type $(\alpha, \beta) \vdash n$, where α and β are the partitions obtained by reordering the unbarred and barred parts of γ, respectively, in weakly decreasing order.

Definition 3.5. Let χ be a character of the hyperoctahedral group B_n. A set $\mathcal{B}$, endowed with a map $\mathrm{sDes} : \mathcal{B} \to \Sigma^B(n)$, is said to be a *fine set* for χ if

$$\chi(\gamma) \;=\; \sum_{b \in \mathcal{B}} \mathrm{wt}_\gamma(\mathrm{sDes}(b)) \tag{3.6}$$

for every signed composition γ of n.

The main results of this paper are as follows.

Theorem 3.6. *Let χ be a character of the hyperoctahedral group B_n and $\mathcal{B}$ a set endowed with a map $\mathrm{sDes} : \mathcal{B} \to \Sigma^B(n)$. Then $\mathcal{B}$ is a fine set for χ if and only if*

$$\mathrm{ch}(\chi) \;=\; \sum_{b \in \mathcal{B}} F_{\mathrm{sDes}(b)}(x, y). \tag{3.7}$$

In particular, the distribution of the signed descent set over $\mathcal{B}$ is uniquely determined by χ.

To state the third condition of the following corollary, given $\sigma = (S, \varepsilon) \in \Sigma^B(n)$ we write $z^\sigma = \prod_{i \in S} z_i$, where $z_i = x_i$ (respectively, $z_i = y_i$) if $\varepsilon_i = +$ (respectively, $\varepsilon_i = -$).

Corollary 3.7. *Let $\mathcal{B}$ be a set endowed with a map* $\mathrm{sDes} : \mathcal{B} \to \Sigma^B(n)$. *The following are equivalent:*

(i) *$\mathcal{B}$ is a fine set for some (non-virtual) character of B_n.*
(ii) *The quasisymmetric function $F_{\mathcal{B}}(x, y) := \sum_{b \in \mathcal{B}} F_{\mathrm{sDes}(b)}(x, y)$ is a Schur-positive element of $\Lambda(x) \otimes \Lambda(y)$.*
(iii) *There exist nonnegative integers $a_{\lambda,\mu}$, for $(\lambda, \mu) \vdash n$, such that*

$$\sum_{b \in \mathcal{B}} z^{\mathrm{sDes}(b)} = \sum_{(\lambda,\mu) \vdash n} a_{\lambda,\mu} \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} z^{\mathrm{sDes}(Q)}. \tag{3.8}$$

Moreover, if these conditions hold, then $a_{\lambda,\mu} = \langle F_{\mathcal{B}}(x, y), s_\lambda(x) s_\mu(y) \rangle$ for each $(\lambda, \mu) \vdash n$.

4. Proofs

This section proves Theorem 3.6 and deduces Corollary 3.7 from it. The proof of one direction of Theorem 3.6 follows that of Theorem 3.2, as described in Section 3; it is based on a formula for the values of the irreducible characters of B_n (see Theorem 4.1) which is analogous to Roichman's rule for the irreducible characters of $\mathfrak{S}_n$ (Theorem 2.4). The challenge of finding analogues of Roichman's rule for other (complex) reflection groups was implicitly raised in [9, page 45]. The proof of the other direction of Theorem 3.6 requires the invertibility of a suitable weight matrix.

4.1. Irreducible characters and bitableaux

Given a composition α, denote by α^+ (respectively, α^-) the signed composition obtained from α by considering its parts as unbarred (respectively, barred). The following result states that the set of standard Young bitableaux of shape $(\lambda, \mu) \vdash n$, endowed with the standard colored descent map defined in Section 2.2, is a fine set for the irreducible chararcter $\chi^{\lambda,\mu}$ of B_n.

Theorem 4.1. *For all bipartitions $(\lambda, \mu) \vdash n$ and $(\alpha, \beta) \vdash n$ and for every permutation γ of the parts of α^+ and β^-,*

$$\chi^{\lambda,\mu}(\alpha, \beta) \;=\; \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} \mathrm{wt}_\gamma(\mathrm{sDes}(Q)). \tag{4.1}$$

In other words, $\mathrm{SYT}(\lambda, \mu)$ *(with the associated map* sDes*) is a fine set for the irreducible character* $\chi^{\lambda,\mu}$*.*

Proof. To compute $\chi^{\lambda,\mu}(\alpha, \beta)$ we expand the left-hand side of (2.6) as

$$\begin{aligned} p^+_\alpha(x,y)p^-_\beta(x,y) \;&=\; \prod_{i=1}^{\ell(\alpha)} (p_{\alpha_i}(x) + p_{\alpha_i}(y)) \prod_{j=1}^{\ell(\beta)} (p_{\beta_j}(x) - p_{\beta_j}(y)) \\ &=\; \sum_{\varepsilon,\zeta} (-1)^{|\{j:\,\zeta_j=-\}|}\, p_\nu(x)p_\xi(y) \end{aligned}$$

where the sum ranges over all the vectors $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\ell(\alpha)}) \in \{-,+\}^{\ell(\alpha)}$ and $\zeta = (\zeta_1, \dots, \zeta_{\ell(\beta)}) \in \{-,+\}^{\ell(\beta)}$, and where $\nu = \nu^{\varepsilon,\zeta}$ (respectively, $\xi = \xi^{\varepsilon,\zeta}$) is the composition consisting of the parts α_i of α with $\varepsilon_i = +$ (respectively, $\varepsilon_i = -$) followed by the parts β_j of β with $\zeta_j = +$ (respectively, $\zeta_j = -$). Expressing each of $p_\nu(x)$ and $p_\xi(y)$ in the basis of Schur functions, according to (2.3), and comparing to (2.6) we get

$$\chi^{\lambda,\mu}(\alpha, \beta) \;=\; \sum_{\varepsilon,\zeta} (-1)^{|\{j:\,\zeta_j=-\}|}\, \chi^\lambda(\nu^{\varepsilon,\zeta})\chi^\mu(\xi^{\varepsilon,\zeta}) \tag{4.2}$$

where the summation and the compositions $\nu^{\varepsilon,\zeta}$ and $\xi^{\varepsilon,\zeta}$ determined by the sign vectors ε and ζ are as before.

We now derive a similar formula for the right-hand side of (4.1), denoted by $\psi^{\lambda,\mu}(\gamma)$. Given sign vectors $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{\ell(\alpha)}) \in \{-,+\}^{\ell(\alpha)}$ and $\zeta = (\zeta_1, \dots, \zeta_{\ell(\beta)}) \in \{-,+\}^{\ell(\beta)}$, denote by $\gamma^{\varepsilon,\zeta}$ the signed composition whose underlying composition is that of γ and whose parts are unbarred or barred, according to whether the corresponding parts of α and β are assigned the $+$ or $-$ sign by ε and ζ, respectively. Denote by $\gamma^{\varepsilon,\zeta}_+$ (respectively, $\gamma^{\varepsilon,\zeta}_-$) the composition obtained from $\gamma^{\varepsilon,\zeta}$ by removing the barred (respectively, unbarred) parts and forgetting the bars. Express the set $[n]$ as the disjoint union of contiguous segments whose cardinalities are the parts of γ, and denote by $R^{\varepsilon,\zeta}_+$ (respectively, $R^{\varepsilon,\zeta}_-$) the union of those segments which correspond to the unbarred (respectively, barred) parts of $\gamma^{\varepsilon,\zeta}$. The definitions of weight and signed descent set show that

$$\psi^{\lambda,\mu}(\gamma)$$
$$= \sum_{\varepsilon,\zeta} (-1)^{|\{j:\, \zeta_j = -\}|} \left(\sum_{Q^+} (-1)^{|\mathrm{Des}(Q^+) \smallsetminus S(\gamma_+^{\varepsilon,\zeta})|} \right) \left(\sum_{Q^-} (-1)^{|\mathrm{Des}(Q^-) \smallsetminus S(\gamma_-^{\varepsilon,\zeta})|} \right)$$

where the outer sum ranges over all sign vectors ε, ζ as above and the inner sums range over all Young tableaux Q^+ of shape λ with content $R_+^{\varepsilon,\zeta}$ and $\gamma_+^{\varepsilon,\zeta}$-unimodal descent set and over all Young tableaux Q^- of shape μ with content $R_-^{\varepsilon,\zeta}$ and $\gamma_-^{\varepsilon,\zeta}$-unimodal descent set. Theorem 2.4 and the previous formula imply that

$$\psi^{\lambda,\mu}(\gamma) \; = \; \sum_{\varepsilon,\zeta} (-1)^{|\{j:\, \zeta_j = -\}|} \chi^{\lambda}(\gamma_+^{\varepsilon,\zeta}) \chi^{\mu}(\gamma_-^{\varepsilon,\zeta}) \tag{4.3}$$

where the sum ranges over all sign vectors $\varepsilon \in \{-,+\}^{\ell(\alpha)}$ and $\zeta \in \{-,+\}^{\ell(\beta)}$. Since the values $\chi(\nu)$ of an irreducible character χ of a symmetric group do not depend on the ordering of the parts of the composition ν, Equations (4.2) and (4.3) imply that $\chi^{\lambda,\mu}(\alpha,\beta) = \psi^{\lambda,\mu}(\gamma)$, as claimed by the theorem. □

Proposition 4.2. *For all partitions* λ, μ

$$s_\lambda(x) s_\mu(y) \; = \; \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} F_{\mathrm{sDes}(Q)}(x,y).$$

Proof. This statement follows from [25, Corollary 8] and Proposition 2.6. For a direct proof, it suffices to describe a bijection from the set of all pairs of semistandard Young tableaux S and T of shape λ and μ, respectively, to the set of all pairs (Q,u) of standard Young bitableau $Q \in \mathrm{SYT}(\lambda,\mu)$ and monomials u which appear in the expansion of $F_{\mathrm{sDes}(Q)}(x,y)$, such that if (Q,u) corresponds to (S,T) then $x^S y^T = u$. Given (S,T), we define Q by numbering the entries of T equal to 1, read from bottom to top and from left to right, with the first few positive integers $1, 2, \ldots$; then the entries of S equal to 1, read in the same fashion, with the next few positive integers; then the entries of T equal to 2 and so on, and set $u = x^S y^T$. We leave to the interested reader to verify that this induces a well defined map which has the claimed properties. □

The following statement furnishes one direction of Theorem 3.6.

Proposition 4.3. *Let χ be a character of the group B_n and let $\mathcal{B}$ be a set endowed with a map* $\mathrm{sDes} : \mathcal{B} \to \Sigma^B(n)$. *If*

$$\mathrm{ch}(\chi) \;=\; \sum_{b \in \mathcal{B}} F_{\mathrm{sDes}(b)}(x, y), \tag{4.4}$$

then $\mathcal{B}$ is a fine set for χ.

Proof. Let us express $\chi = \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu}\chi^{\lambda,\mu}$ as a linear combination of irreducible characters. We then have, by Proposition 4.2,

$$\begin{aligned}
\mathrm{ch}(\chi) \;&=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu}\,\mathrm{ch}(\chi^{\lambda,\mu}) \;=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu}\, s_\lambda(x) s_\mu(y) \\
&=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} F_{\mathrm{sDes}(Q)}(x, y).
\end{aligned}$$

Comparing with (4.4) we get

$$|\{b \in \mathcal{B} : \mathrm{sDes}(b) = \sigma\}| \;=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} \cdot |\{Q \in \mathrm{SYT}(\lambda,\mu) : \mathrm{sDes}(Q) = \sigma\}|$$

for every $\sigma \in \Sigma^B(n)$. Using this equation and Theorem 4.1, we get

$$\begin{aligned}
\chi(\alpha,\beta) \;&=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu}\,\chi^{\lambda,\mu}(\alpha,\beta) \;=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} \sum_{Q \in \mathrm{SYT}(\lambda,\mu)} \mathrm{wt}_\gamma(\mathrm{sDes}(Q)) \\
&=\; \sum_{\sigma \in \Sigma^B(n)} \mathrm{wt}_\gamma(\sigma) \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} \cdot |\{Q \in \mathrm{SYT}(\lambda,\mu) : \mathrm{sDes}(Q) = \sigma\}| \\
&=\; \sum_{\sigma \in \Sigma^B(n)} \mathrm{wt}_\gamma(\sigma) \cdot |\{b \in \mathcal{B} : \mathrm{sDes}(b) = \sigma\}| \\
&=\; \sum_{b \in \mathcal{B}} \mathrm{wt}_\gamma(\mathrm{sDes}(b))
\end{aligned}$$

for every bipartition $(\alpha,\beta) \vdash n$ and every permutation γ of the parts of α^+ and β^- and the proof follows. $\square$

For the proof of the other direction of Theorem 3.6 we need the following B_n-analogue of [6, Theorem 1.4], which will be proved in Section 4.2.

Theorem 4.4. *If $\mathcal{B}$ is a fine set for a B_n-character χ, then the distribution of* sDes *over $\mathcal{B}$ is uniquely determined by χ.*

Proof of Theorem 3.6. We only need to prove the converse of Proposition 4.3. Let $\mathcal{B}$ be a set which, endowed with a map $\mathrm{sDes} : \mathcal{B} \to \Sigma^B(n)$, is fine for a B_n-character χ. We have to show that (4.4) holds. Express $\chi = \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu}\chi^{\lambda,\mu}$ as a linear combination of irreducible characters. Let $\mathcal{B}'$ be the disjoint union, taken over all bipartitions $(\lambda, \mu) \vdash n$, of $c_{\lambda,\mu}$ copies of $\mathrm{SYT}(\lambda, \mu)$, endowed with the corresponding descent map sDes. By Theorem 4.1 the set $\mathcal{B}'$ is also fine for χ. Theorem 4.4 now implies that sDes is equidistributed over $\mathcal{B}$ and $\mathcal{B}'$. As shown at the beginning of the proof of Proposition 4.3,

$$\mathrm{ch}(\chi) \;=\; \sum_{b\in\mathcal{B}'} F_{\mathrm{sDes}(b)}(x, y).$$

Therefore, (4.4) holds as well. □

Finally, we deduce Corollary 3.7 from Theorem 3.6.

Proof of Corollary 3.7. The equivalence (i) $\Leftrightarrow$ (ii) is a direct consequence of Theorem 3.6. The equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition 4.2 and the linear independence of the functions $F_\sigma(x, y)$ for $\sigma \in \Sigma^B(n)$. □

4.2. The weight matrix

We now prove Theorem 4.4, the missing step in the proof of Theorem 3.6, using a B_n-analogue of the Hadamard-type matrix A_n from [6, 7].

By Definition 3.4, a weight $\mathrm{wt}_\gamma(\sigma)$ is defined for any signed composition $\gamma \in \mathrm{Comp}^B(n)$ and signed set $\sigma \in \Sigma^B(n)$. The size of each of the sets $\mathrm{Comp}^B(n)$ and $\Sigma^B(n)$ is $d_n := 2 \cdot 3^{n-1}$. Fixing a linear order (to be specified later) on each of these sets, the weights can be arranged into a *weight matrix*

$$A_n \;=\; (\mathrm{wt}_\gamma(\sigma))_{\gamma\in\mathrm{Comp}^B(n),\ \sigma\in\Sigma^B(n)} \in \{1, 0, -1\}^{d_n\times d_n}.$$

For a set $\mathcal{B}$, endowed with a map $\mathrm{sDes} : \mathcal{B} \to \Sigma^B(n)$, let $v = v(\mathcal{B})$ be the column vector of length d_n with entries

$$v_\sigma \;=\; |\{b \in \mathcal{B} :\ \mathrm{sDes}(b) = \sigma\}| \qquad (\forall \sigma \in \Sigma^B(n)),$$

where $\Sigma^B(n)$ is in the specified linear order. For a character χ of the hyperoctahedral group B_n, let $u = u(\chi)$ be the column vector of length d_n with entries

$$u_\gamma = \chi(\gamma) \qquad (\forall \gamma \in \mathrm{Comp}^B(n)),$$

where $\mathrm{Comp}^B(n)$ is in the specified order. Definition 3.5 can now be restated as follows:

$$\mathcal{B} \text{ is a fine set for } \chi \iff u(\chi) = A_n \cdot v(\mathcal{B}).$$

Theorem 4.4 is thus implied by the following result.

Theorem 4.5. *The matrix A_n is invertible. In fact,*

$$|\det(A_n)| \;=\; \prod_{\gamma \in \mathrm{Comp}^B(n)} m_\gamma$$

where, for a signed composition $\gamma \in \mathrm{Comp}^B(n)$ with part sizes $\gamma_1, \ldots, \gamma_k$,

$$m_\gamma \;:=\; 2^{k/2} \cdot \prod_{i=1}^{k} \gamma_i.$$

Theorem 4.5 will be proved using a recursive formula for A_n and for an auxiliary matrix $\widehat{A}_n$.

Consider now the alphabet $L = \{0, 1, *\}$, the set L^n of all words of length n with letters from L, and the set L^n_* of all words $w \in L^n$ in which the last letter is *not* $*$. Any signed set $\sigma = (S, \varepsilon) \in \Sigma^B(n)$ is uniquely represented by a word $w(\sigma) = a_1 \cdots a_n \in L^n_*$, where

$$a_i \;=\; \begin{cases} 0, & \text{if } i \in S \text{ and } \varepsilon(i) = +; \\ 1, & \text{if } i \in S \text{ and } \varepsilon(i) = -; \\ *, & \text{if } i \notin S. \end{cases}$$

Recalling that we always have $n \in S$, the map $\sigma \mapsto w(\sigma)$ is well defined and is clearly a bijection from $\Sigma^B(n)$ to L^n_*. The natural bijection from $\mathrm{Comp}^B(n)$ to $\Sigma^B(n)$ (see Section 2.1) yields a corresponding representation of signed compositions by words in L^n_*.

By the above, we can consider the rows and columns of the weight matrix A_n to be indexed by words $w \in L^n_*$. Let $\widehat{A}_n$ be the matrix obtained from A_n by setting to zero all the entries indexed by (w, w') for which the initial sequence of $*$-s in w is longer than the initial sequence of $*$-s in w', namely: $(\widehat{A}_n)_{w,w'} = 0$ if $i(w) > i(w')$, where

$$i(w) := \min\{i \mid w_i \neq *\}.$$

This corresponds to a pair (γ, σ) for which there is an element of $S(\sigma)$ before the end of the first part of the signed composition γ.

For a word $w \in L_*^n$, let $c(w)$ be the first letter in w which is not $*$:

$$c(w) := w_{i(w)}.$$

For a letter $a' \in \{0,1\}$, let $A_n^{a'}$ be the matrix obtained from A_n by setting to zero all the entries in columns indexed by words w' with $c(w') \neq a'$. Clearly, $A_n = A_n^0 + A_n^1$. We use similar notation for $\widehat{A}_n$.

Definition 3.4 of the weight $\mathrm{wt}_\gamma(\sigma)$ implies the following recursive properties of A_n and $\widehat{A}_n$:

Lemma 4.6. *For $n \geq 1$, letters $a, a' \in L$ and words $w, w' \in L_*^n$,*

$$(A_{n+1})_{aw,a'w'} \;=\; \begin{cases} (-1)^{a \cdot a'} (A_n)_{w,w'}, & \textit{if } a, a' \in \{0,1\}; \\ (-1)^{a \cdot c(w')} (A_n)_{w,w'}, & \textit{if } a \in \{0,1\},\ a' = *; \\ -(A_n^{a'})_{w,w'}, & \textit{if } a = *,\ a' \in \{0,1\}; \\ (\widehat{A}_n)_{w,w'}, & \textit{if } a = a' = * \end{cases}$$

and

$$(\widehat{A}_{n+1})_{aw,a'w'} \;=\; \begin{cases} (-1)^{a \cdot a'} (A_n)_{w,w'}, & \textit{if } a, a' \in \{0,1\}; \\ (-1)^{a \cdot c(w')} (A_n)_{w,w'}, & \textit{if } a \in \{0,1\},\ a' = *; \\ 0, & \textit{if } a = *,\ a' \in \{0,1\}; \\ (\widehat{A}_n)_{w,w'}, & \textit{if } a = a' = *. \end{cases}$$

Also,

$$(A_1)_{a,a'} \;=\; (\widehat{A}_1)_{a,a'} \;=\; (-1)^{a \cdot a'} \qquad (\forall a, a' \in \{0,1\}).$$

Specifying a linear order on L_*^n, we can write these recursions in matrix form.

Corollary 4.7. *Using the linear order $0 < 1 < *$ on L and the resulting lexicographic order on L_*^n,*

$$A_{n+1} \;=\; \begin{pmatrix} A_n & A_n & A_n \\ A_n & -A_n & A_n^0 - A_n^1 \\ -A_n^0 & -A_n^1 & \widehat{A}_n \end{pmatrix} \qquad (n \geq 1),$$

$$\widehat{A}_{n+1} \;=\; \begin{pmatrix} A_n & A_n & A_n \\ A_n & -A_n & A_n^0 - A_n^1 \\ 0 & 0 & \widehat{A}_n \end{pmatrix} \qquad (n \geq 1)$$

and

$$A_1 \;=\; \widehat{A}_1 \;=\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Corollary 4.8. *For $n \geq 1$,*

$$\det(A_{n+1}) \;=\; \det(A_n) \cdot \det(-2A_n) \cdot \det(A_n + \widehat{A}_n)$$

and

$$\det(\widehat{A}_{n+1}) \;=\; \det(A_n) \cdot \det(-2A_n) \cdot \det(\widehat{A}_n),$$

with

$$\det(A_1) \;=\; \det(\widehat{A}_1) \;=\; -2.$$

More generally, for every real α,

$$\det(\alpha A_{n+1} + (1-\alpha)\widehat{A}_{n+1}) \;=\; \det(A_n) \cdot \det(-2A_n) \cdot \det(\alpha A_n + \widehat{A}_n)$$

for $n \geq 1$, with

$$\det(\alpha A_1 + (1-\alpha)\widehat{A}_1) \;=\; -2.$$

Proof. Using the fact that $A_n = A_n^0 + A_n^1$ (and similarly for $\widehat{A}_n$), we can write

$$A_{n+1} \;=\; \begin{pmatrix} A_n^0 + A_n^1 & A_n^0 + A_n^1 & A_n^0 + A_n^1 \\ A_n^0 + A_n^1 & -A_n^0 - A_n^1 & A_n^0 - A_n^1 \\ -A_n^0 & -A_n^1 & \widehat{A}_n^0 + \widehat{A}_n^1 \end{pmatrix} \qquad (n \geq 1)$$

and, more generally,

$$\alpha A_{n+1} + (1-\alpha)\widehat{A}_{n+1} \;=\; \begin{pmatrix} A_n^0 + A_n^1 & A_n^0 + A_n^1 & A_n^0 + A_n^1 \\ A_n^0 + A_n^1 & -A_n^0 - A_n^1 & A_n^0 - A_n^1 \\ -\alpha A_n^0 & -\alpha A_n^1 & \widehat{A}_n^0 + \widehat{A}_n^1 \end{pmatrix} \qquad (n \geq 1),$$

with

$$\alpha A_1 + (1-\alpha)\widehat{A}_1 \;=\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The set of nonzero columns of A_n^0 is disjoint from the corresponding set for A_n^1. We can thus perform elementary column operations using, separately, the columns of A_n^0 and A_n^1, to get

$$\det(\alpha A_{n+1} + (1-\alpha)\widehat{A}_{n+1}) = \det\begin{pmatrix} A_n^0 + A_n^1 & A_n^0 + A_n^1 & A_n^0 + A_n^1 \\ A_n^0 + A_n^1 & -A_n^0 - A_n^1 & A_n^0 - A_n^1 \\ -\alpha A_n^0 & -\alpha A_n^1 & \widehat{A}_n^0 + \widehat{A}_n^1 \end{pmatrix}$$

$$= \det\begin{pmatrix} A_n^0 + A_n^1 & 0 & 0 \\ A_n^0 + A_n^1 & -2A_n^0 - 2A_n^1 & -2A_n^1 \\ -\alpha A_n^0 & \alpha A_n^0 - \alpha A_n^1 & \alpha A_n^0 + \widehat{A}_n^0 + \widehat{A}_n^1 \end{pmatrix}$$

$$= \det\begin{pmatrix} A_n^0 + A_n^1 & 0 & 0 \\ A_n^0 + A_n^1 & -2A_n^0 - 2A_n^1 & 0 \\ -\alpha A_n^0 & \alpha A_n^0 - \alpha A_n^1 & \alpha A_n^0 + \alpha A_n^1 + \widehat{A}_n^0 + \widehat{A}_n^1 \end{pmatrix}.$$

Note that in the last step we used only half of the columns of the middle $-2A_n^0 - 2A_n^1$ to annihilate the columns of $-2A_n^1$. The block triangular structure of the resulting matrix implies that

$$\det(\alpha A_{n+1} + (1-\alpha)\widehat{A}_{n+1}) \;=\; \det(A_n) \cdot \det(-2A_n) \cdot \det(\alpha A_n + \widehat{A}_n)$$

for $n \geq 1$, with

$$\det(\alpha A_1 + (1-\alpha)\widehat{A}_1) \;=\; -2.$$

The special cases $\alpha = 1$ and $\alpha = 0$ give the results for A_n and $\widehat{A}_n$. $\square$

The following statement implies Theorem 4.5 and completes the proof of Theorem 3.6.

Corollary 4.9. *For all real numbers α and positive integers n,*

$$\det(\alpha A_n + (1-\alpha)\widehat{A}_n) \;=\; -\prod_{\gamma \in \mathrm{Comp}^B(n)} m_\gamma(\alpha) \tag{4.5}$$

where, for a signed composition $\gamma \in \mathrm{Comp}^B(n)$ with part sizes $\gamma_1, \ldots, \gamma_k$,

$$m_\gamma(\alpha) \;=\; 2^{k/2} \cdot (\alpha\gamma_1 + (1-\alpha)) \cdot \prod_{i=2}^{k} \gamma_i.$$

Proof. Setting

$$\Delta_n(\alpha) \;:=\; \det(\alpha A_n + (1-\alpha)\widehat{A}_n),$$

Corollary 4.8 implies that, for $\alpha \neq -1$,

$$\Delta_{n+1}(\alpha) \;=\; (2(\alpha+1))^{d_n} \cdot \Delta_n(1)^2 \cdot \Delta_n\left(\frac{\alpha}{\alpha+1}\right)$$

for $n \geq 1$ and

$$\Delta_1(\alpha) \;=\; -2.$$

We have used here the fact that $(-1)^{d_n} = 1$, since $d_n = 2 \cdot 3^{n-1}$ is even for all $n \geq 1$.

Consider now a signed composition $\gamma \in \mathrm{Comp}^B(n)$ with a corresponding word $w \in L_*^n$. It gives rise to three signed compositions of $n+1$, corresponding to the words $0w, 1w, *w \in L_*^{n+1}$. Using the notation $m_w(\alpha)$ instead of $m_\gamma(\alpha)$, it is clear that

$$m_{0w}(\alpha) \;=\; m_{1w}(\alpha) \;=\; 2^{(k+1)/2} \cdot (\alpha \cdot 1 + (1-\alpha)) \cdot \prod_{i=1}^{k} \gamma_i \;=\; 2^{1/2} \cdot m_w(1),$$

whereas

$$m_{*w}(\alpha) \;=\; 2^{k/2} \cdot (\alpha \cdot (\gamma_1 + 1) + (1-\alpha)) \cdot \prod_{i=2}^{k} \gamma_i \;=\; (\alpha+1) \cdot m_w\left(\frac{\alpha}{\alpha+1}\right).$$

Therefore

$$m_{0w}(\alpha) \cdot m_{1w}(\alpha) \cdot m_{*w}(\alpha) \;=\; 2(\alpha+1) \cdot m_w(1)^2 \cdot m_w\left(\frac{\alpha}{\alpha+1}\right),$$

which implies that, at least for $\alpha > 0$, both sides of equation (4.5) satisfy the same recursion (and, clearly, also the same initial conditions for $n = 1$). The two sides are therefore equal for all $\alpha > 0$ and (being polynomials in α), are actually equal for all α. □

5. Knuth classes and involutions

This section discusses irreducible B_n-characters again, describes a different interpretation of Theorem 4.1 in terms of Knuth classes of type B and confirms that the set of involutions in B_n is fine for the character of the Gelfand model for this group.

We first recall the definition of the natural analogue of the Robinson–Schensted correspondence for the group B_n [41, pages 145–146] [44]. Let $w = (w(1), w(2), \ldots, w(n)) \in B_n$ and let $(w(a_1), w(a_2), \ldots, w(a_k))$ be the subsequence of unbarred elements of w. Applying Schensted's correspondence to the two-line array $\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ w(a_1) & w(a_2) & \cdots & w(a_k) \end{pmatrix}$ results in a pair (P^+, Q^+) of Young tableaux of the same shape $\lambda \vdash k$. Similarly, from the subsequence of barred elements of w and the corresponding two-line array we get a pair (P^-, Q^-) of Young tableaux of the same shape $\mu \vdash n-k$. Then $P^B(w) = (P^+, P^-)$ and $Q^B(w) = (Q^+, Q^-)$ are standard Young bitableaux of shape $(\lambda, \mu) \vdash n$ and the map which assigns the pair $(P^B(w), Q^B(w))$ to w is a bijection from the group B_n to the set of pairs of standard Young bitableaux of the same shape and size n.

The following two properties of this map are explicit in [44, Section 8] and implicit in [41, page 146], respectively.

Proposition 5.1. *For every $w \in B_n$:*

(a) $P^B(w^{-1}) = Q^B(w)$,
(b) $\mathrm{sDes}(w) = \mathrm{sDes}(Q^B(w))$.

A *Knuth class* of (type B and) shape (λ, μ) is a set of the form $\{w \in B_n : P^B(w) = T\}$ for some fixed $T \in \mathrm{SYT}(\lambda, \mu)$ (these Knuth classes should not be confused with the ones considered in the study of Kazdhan-Lusztig cells of type B; see [12, 31]). The first part of the following corollary, already discussed in Section 4, is a restatement of Theorem 4.1. The second part follows from the first and Proposition 5.1 (b).

Corollary 5.2. *For every bipartition $(\lambda, \mu) \vdash n$, the following are fine sets for the irreducible B_n-character $\chi^{\lambda,\mu}$:*

(i) *The set of standard Young bitableaux of shape (λ, μ).*
(ii) *All Knuth classes of shape (λ, μ).*

We recall that a *Gelfand model* for a group G is any representation of G which is equivalent to the multiplicity-free direct sum of its irreducible representations. The following proposition is a B_n-analogue of the corresponding statement [3, Proposition 1.5] (see also [6, Proposition 3.12 (iii)]) for $\mathfrak{S}_n$.

Proposition 5.3. *The set of involutions in B_n, endowed with the standard signed descent map, is fine for the character of the Gelfand model of B_n.*

Proof. Corollary 5.2 implies that the set of standard Young bitableaux of size n is fine for the character of the Gelfand model of B_n. Moreover, as a direct consequence of Proposition 5.1, the signed descent set is equidistributed over the set of standard Young bitableaux of size n and the set of involutions in B_n. The proof follows from these two statements. $\square$

Example 5.4. We confirm this statement for $n = 2$. The involutions in B_2 are the signed permutations $(1, 2)$, $(2, 1)$, $(\bar{1}, 2)$, $(1, \bar{2})$, $(\bar{1}, \bar{2})$ and $(\bar{2}, \bar{1})$. Therefore,

$$\begin{aligned}
&\sum_{w \in B_2 \colon w^{-1} = w} F_{\mathrm{sDes}(w)}(x, y) \\
&= F_{(2)}(x, y) + F_{(1,1)}(x, y) + F_{(\bar{1},1)}(x, y) + F_{(1,\bar{1})}(x, y) + F_{(\bar{2})}(x, y) + F_{(\bar{1},\bar{1})}(x, y) \\
&\qquad = s_{(2)}(x) + s_{(1,1)}(x) + s_{(1)}(x)s_{(1)}(y) + s_{(2)}(y) + s_{(1,1)}(y) \\
&\qquad\qquad = \mathrm{ch}\,(\chi^{((2),\varnothing)} + \chi^{((1,1),\varnothing)} + \chi^{((1),(1))} + \chi^{(\varnothing,(2))} + \chi^{(\varnothing,(1,1))}),
\end{aligned}$$

where we have indexed functions $F_\sigma(x, y)$ by signed compositions, rather than signed sets, and the second equality follows by direct computation or use of Proposition 4.2.

An *inverse signed descent class* in B_n is a set of the form $\{w \in B_n : \mathrm{sDes}(w^{-1}) = \sigma\}$ for some $\sigma \in \Sigma^B(n)$. We postpone the definition of flag-major index to Section 6, where the representation of B_n corresponding to the fine set (ii) in the following proposition will also be described.

Proposition 5.5. *The following subsets of B_n are fine for some B_n-characters:*

(i) *All inverse signed descent classes.*
(ii) *The set of elements of B_n whose inverses have a given flag-major index.*

Proof. By Proposition 5.1, the signed descent set sDes is fixed over an inverse Knuth class. It follows that inverse signed descent classes are unions of Knuth classes, which are fine sets by Corollary 5.2, and hence are fine sets as well. Morever, subsets of B_n with fixed signed descent set have fixed flag-major index. As a result, the set of elements of B_n whose inverses have a given flag-major index is a union of inverse signed descent classes and hence is fine, as a disjoint union of fine sets. $\square$

6. Coinvariant algebra and flag statistics

This section provides a B_n-analogue of a result essentially due to Roichman [36], which describes explicitly fine sets for the characters of the action of $\mathfrak{S}_n$ on the homogeneous components of the coinvariant algebra of type A.

Throughout this section we denote by P_n the polynomial ring $F[x_1, \ldots, x_n]$ in n variables over a field F of characteristic zero. The symmetric group $\mathfrak{S}_n$ acts on P_n by permuting the variables. Let I_n be the ideal of P_n generated by the $\mathfrak{S}_n$-invariant (symmetric) polynomials with zero constant term. The group $\mathfrak{S}_n$ acts on the quotient ring P_n/I_n, known as the *coinvariant algebra* of $\mathfrak{S}_n$, and the resulting representation is isomorphic to the regular representation; see, for instance, [27, Section 3.6]. Since P_n and I_n are naturally graded by degree and the action of $\mathfrak{S}_n$ respects this grading, the coinvariant algebra is also graded by degree and $\mathfrak{S}_n$ acts on each homogeneous component. Denote by $\chi_{n,k}$ the character of the $\mathfrak{S}_n$-action on the k-th homogeneous component of P_n/I_n, for $0 \le k \le \binom{n}{2}$.

Let $w \in \mathfrak{S}_n$ be a permutation. Recall [42, Sections 1.3–1.4] that $\mathrm{inv}(w)$ denotes the number of inversions of w, $\mathrm{Des}(w)$ its set of descents, and $\mathrm{maj}(w)$ its major index (the sum of the elements of $\mathrm{Des}(w)$). Note also that $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$ for $w \in \mathfrak{S}_n$.

Theorem 6.1. ([36] [6, Theorem 3.7]) *For $0 \le k \le \binom{n}{2}$, each of the following subsets of $\mathfrak{S}_n$, endowed with the standard descent map, is a fine set for the $\mathfrak{S}_n$-character $\chi_{n,k}$:*

(i) $\{w \in \mathfrak{S}_n : \mathrm{inv}(w^{-1}) = k\}$;
(ii) $\{w \in \mathfrak{S}_n : \mathrm{maj}(w^{-1}) = k\}$.

Proof. The claim for (i) is a restatement of [36, Theorem 1]. The claim for (ii) follows from the one for (i) by the fact [21, Theorem 1] that the number of $w \in \mathfrak{S}_n$ with $\mathrm{Des}(w^{-1}) = S$ and $\mathrm{inv}(w) = k$ is equal to that of $w \in \mathfrak{S}_n$ with $\mathrm{Des}(w^{-1}) = S$ and $\mathrm{maj}(w) = k$ for all $S \subseteq [n-1]$ and k. $\square$

Consider now the group B_n. It acts on the polynomial ring P_n by permuting the variables $x_1, \ldots, x_n$ and flipping their signs. Let I_n^B be the ideal of P_n generated by the B_n-invariant polynomials (i.e., symmetric functions in the squares $x_1^2, \ldots, x_n^2$) with zero constant term. The *coinvariant algebra* of B_n is the quotient ring P_n/I_n^B. B_n acts on each of its homogeneous components, and the resulting representation is isomorphic to the regular representation of B_n. It is graded by degree; denote by $\chi^B_{n,k}$ the character of the B_n-action on the k-th homogeneous component of P_n/I^B_n, for $0 \le k \le n^2$.

The *flag-major index* [4] of a signed permutation $w \in B_n$ is defined as

$$\mathrm{fmaj}(w) \ = \ 2 \cdot \mathrm{maj}(w) + \mathrm{bar}(w),$$

where $\mathrm{maj}(w)$ is the sum of the elements of $\mathrm{Des}(w)$ (as defined in Section 2.2) and $\mathrm{bar}(w)$ is the number of indices $i \in [n]$ for which $w(i)$ is barred. The *flag-major index* of a standard Young bitableau Q of shape (λ, μ) is defined as twice the sum of the elements of $\mathrm{Des}(Q)$ plus the size of μ, so that $\mathrm{fmaj}(w) = \mathrm{fmaj}(Q^B(w))$ for every $w \in B_n$, in the notation of Section 5, by Proposition 5.1 (b). The *flag-inversion number* of $w \in B_n$ is defined as

$$\mathrm{finv}(w) \ = \ 2 \cdot \mathrm{inv}(w) + \mathrm{bar}(w),$$

where $\mathrm{inv}(w)$ is the number of inversions of the sequence $(w(1), w(2), \dots, w(n))$ with respect to the total order (2.1) (this is a variant of the notion of flag-inversion number introduced in [20]; see also [1, 22] and references therein).

The equidistribution result [21, Theorem 1], mentioned in the proof of Theorem 6.1, follows from a theorem of Foata [19], which implies that inv and maj are equidistributed over the set of permutations of an ordered multiset (see, for instance, the discussion in [20, Section 1]), by an application of the inclusion-exclusion principle. A similar argument yields the following analogous result for B_n.

Proposition 6.2. *For every n, k and $\sigma \in \Sigma^B(n)$, the number of $w \in B_n$ with $\mathrm{sDes}(w^{-1}) = \sigma$ and $\mathrm{finv}(w) = k$ is equal to the number of $w \in B_n$ with $\mathrm{sDes}(w^{-1}) = \sigma$ and $\mathrm{fmaj}(w) = k$.*

Proof. Let us denote by $f_{n,k}(\sigma)$ (respectively, $g_{n,k}(\sigma)$) the number of $w \in B_n$ with $\mathrm{sDes}(w^{-1}) = \sigma$ and $\mathrm{inv}(w) = k$ (respectively, $\mathrm{maj}(w) = k$). Since the function $\mathrm{bar}(w)$ is constant within each class $\{w \in B_n : \mathrm{sDes}(w^{-1}) = \sigma\}$, we have to show that $f_{n,k}(\sigma) = g_{n,k}(\sigma)$ for all n, k and $\sigma \in \Sigma^B(n)$.

Fix n, k and $\sigma \in \Sigma^B(n)$. Let $\gamma \in \mathrm{Comp}^B(n)$, with (absolute) parts $\gamma_1, \dots, \gamma_t$, be the signed composition corresponding to σ. Consider the multiset $\mathcal{M}_\sigma$ obtained by breaking the set $[n]$ into contiguous segments of lengths $\gamma_1, \dots, \gamma_t$, replacing all the entries of a segment by copies of its smallest entry, and barring them if the corresponding part of γ is barred. For example, if $\gamma = (2, 2, \bar{2}, \bar{2})$ then $\mathcal{M}_\sigma = \{1, 1, 3, 3, \bar{5}, \bar{5}, \bar{7}, \bar{7}\}$. We consider the ground set of the multiset $\mathcal{M}_\sigma$ totally ordered by (2.1). For $\tau \in \Sigma^B(n)$ we write $\tau \preceq \sigma$ if the signed composition corresponding to τ can be obtained from the one corresponding to σ by replacing consecutive unbarred parts, or consecutive barred parts, by their sum, while keeping the bar when present. For example, for $\gamma = (2, 2, \bar{2}, \bar{2})$, the signed sets $\tau \preceq \sigma$ are those with signed compositions γ, $(4, \bar{2}, \bar{2})$, $(2, 2, \bar{4})$ and $(4, \bar{4})$. A little thought shows that the aforementioned equidistribution result of Foata [19], applied to the ordered multiset $\mathcal{M}_\sigma$, implies that

$$\sum_{\tau \preceq \sigma} f_{n,k}(\tau) \;=\; \sum_{\tau \preceq \sigma} g_{n,k}(\tau)$$

for every $\sigma \in \Sigma^B(n)$. Using the principle of inclusion-exclusion, we conclude that $f_{n,k}(\sigma) = g_{n,k}(\sigma)$ for every $\sigma \in \Sigma^B(n)$ and the proof follows. □

The following statement is our B_n-analogue of Theorem 6.1.

Theorem 6.3. *For $0 \le k \le n^2$, each of the following subsets of B_n, endowed with the standard signed descent map, is a fine set for the B_n-character $\chi^B_{n,k}$:*

(i) $\{w \in B_n : \mathrm{finv}(w^{-1}) = k\}$;
(ii) $\{w \in B_n : \mathrm{fmaj}(w^{-1}) = k\}$.

Proof. For every bipartition $(\lambda, \mu) \vdash n$, the multiplicity of the irreducible character $\chi^{\lambda,\mu}$ in $\chi^B_{n,k}$ was shown by Stembridge (see [45, Theorem 5.3]) to be equal to the number of standard Young bitableaux of shape (λ, μ) and flag-major index k. This fact, Corollary 5.2 (i) and basic properties of the Robinson–Schensted correspondence for B_n (see Proposition 5.1) imply the claim for (ii). The claim for (i) follows from that for (ii) by Proposition 6.2. □

Example 6.4. For $n = k = 3$ we have

$$\begin{aligned}\{w \in B_3 :\ & \mathrm{finv}(w) = 3\} \\ &= \{(\bar{1}, \bar{2}, \bar{3}), (1, \bar{2}, 3), (1, \bar{3}, 2), (2, \bar{1}, 3), (\bar{1}, 3, 2), (\bar{2}, 3, 1), (\bar{3}, 2, 1)\},\end{aligned}$$

$$\begin{aligned}\{w \in B_3 :\ & \mathrm{fmaj}(w) = 3\} \\ &= \{(\bar{1}, \bar{2}, \bar{3}), (1, \bar{2}, 3), (1, \bar{3}, 2), (2, \bar{1}, 3), (3, \bar{1}, 2), (3, \bar{2}, 1), (2, \bar{3}, 1)\},\end{aligned}$$

$$\{w \in B_3 : \mathrm{finv}(w^{-1}) = 3\}$$
$$= \{(\bar{1},\bar{2},\bar{3}),(1,\bar{2},3),(1,3,\bar{2}),(\bar{2},1,3),(\bar{1},3,2),(3,\bar{1},2),(3,2,\bar{1})\},$$

$$\{w \in B_3 : \mathrm{fmaj}(w^{-1}) = 3\}$$
$$= \{(\bar{1},\bar{2},\bar{3}),(1,\bar{2},3),(1,3,\bar{2}),(\bar{2},1,3),(\bar{2},3,1),(3,\bar{2},1),(3,1,\bar{2})\}.$$

The signed descent compositions of the elements of either of the last two sets are $(\bar{3})$, $(1,\bar{1},1)$, $(2,\bar{1})$, $(\bar{1},2)$, $(\bar{1},1,1)$, $(1,\bar{1},1)$ and $(1,1,\bar{1})$ and hence

$$\begin{aligned}\sum_{\substack{w\in B_3\\ \mathrm{finv}(w^{-1})=3}} F_{\mathrm{sDes}(w)}(x,y) &= \sum_{\substack{w\in B_3\\ \mathrm{fmaj}(w^{-1})=3}} F_{\mathrm{sDes}(w)}(x,y)\\ &= F_{(\bar{3})}(x,y) \;+\\ &\quad\; F_{(1,\bar{1},1)}(x,y) + F_{(2,\bar{1})}(x,y) + F_{(\bar{1},2)}(x,y) \;+\\ &\quad\; F_{(\bar{1},1,1)}(x,y) + F_{(1,\bar{1},1)}(x,y) + F_{(1,1,\bar{1})}(x,y)\\ &= s_{(3)}(y) + s_{(2)}(x)s_{(1)}(y) + s_{(1,1)}(x)s_{(1)}(y).\end{aligned}$$

The last equality follows by direct computation or use of Proposition 4.2. Indeed, there are exactly three bitableaux of size 3 and flag-major index 3, namely

$$\left(\varnothing\,,\ \boxed{1\,|\,2\,|\,3}\right),\qquad \left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,,\ \boxed{2}\right),\qquad \left(\boxed{1\,|\,3}\,,\ \boxed{2}\right).$$

Thus, by [45, Theorem 5.3],

$$\begin{aligned}\mathrm{ch}\left(\chi^B_{3,3}\right) &= \mathrm{ch}\left(\chi^{(\varnothing,(3))} + \chi^{((2),(1))} + \chi^{((1,1),(1))}\right)\\ &= s_{(3)}(y) + s_{(1)}(y)s_{(2)}(x) + s_{(1)}(y)s_{(1,1)}(x).\end{aligned}$$

Remark 6.5. Another natural B_n-analogue of the inversion number statistic $\mathrm{inv} : \mathfrak{S}_n \to \mathbb{N}$ is the length function $\ell_B : B_n \to \mathbb{N}$, in terms of the Coxeter generators of B_n; see [11, Section 8.1] for an explicit combinatorial description of this function. A formula of Adin, Postnikov and Roichman [2, Theorem 4] implies that, for every k, the subset $\{w \in B_n : \ell_B(w) = k\}$ of B_n is a $\mathfrak{S}_n$-fine set for the restriction of $\chi^B_{n,k}$ to $\mathfrak{S}_n$. However, this set is not fine for any B_n-character. Indeed, let us write $\mathrm{Neg}(\sigma)$ for the set of negative coordinates of the sign vector of $\sigma \in \Sigma^B(n)$. Given a fine set $\mathcal{B}$ for some B_n-character, condition (iii) of Corollary 3.7 implies that for any $J \subseteq [n]$, the number of elements $b \in \mathcal{B}$ with $\mathrm{Neg}(\mathrm{sDes}(b)) = J$ depends only on the cardinality J and is divisible by $\binom{n}{|J|}$. We leave it to the interested reader to verify that this restriction is violated in the case considered here.

7. Conjugacy classes

This section reviews a result of Poirier [32], which implies that every conjugacy class in B_n is a fine set, and discusses in detail two interesting examples of fine sets which are unions of conjugacy classes, namely the set of derangements in B_n and the set of k-roots of the identity element.

Consider the alphabet $\mathcal{A} = \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$. A *necklace* of length m over $\mathcal{A}$ is an equivalence class of words of length m over $\mathcal{A}$, where two such words are equivalent if one is a cyclic shift of the other. A necklace is called *primitive* if the corresponding word (the choice of representative being irrelevant) is not equal to a power of a word of smaller length. Given a finite multiset $\mathcal{M}$ of primitive necklaces over $\mathcal{A}$, the product of all variables appearing in the necklaces of $\mathcal{M}$ is called the *evaluation* of $\mathcal{M}$. The *signed cycle type* of $\mathcal{M}$ is the bipartition (α, β) for which the parts of α (respectively, β) are the lengths of the necklaces of $\mathcal{M}$ which contain an even (respectively, odd) number of y variables. We will denote by $L^B_{\alpha,\beta}(x, y)$ the formal sum of the evaluations of all multisets of primitive necklaces of signed cycle type (α, β) over $\mathcal{A}$.

The following result extends to B_n analogous results for $\mathfrak{S}_n$ (see, for instance, the discussion in [24, Section 2]). Combined with Theorem 3.6, it shows that each conjugacy class $C_{\alpha,\beta}$, endowed with the standard colored descent map, is a fine set for B_n.

Theorem 7.1. ([32, Theorem 16]) *For every bipartition* $(\alpha, \beta) \vdash n$

$$\sum_{w \in C_{\alpha,\beta}} F_{\mathrm{sDes}(w)}(x, y) \;=\; L^B_{\alpha,\beta}(x, y). \tag{7.1}$$

Moreover, $L^B_{\alpha,\beta}(x, y)$ *is equal to the Frobenius characteristic of a representation of* B_n*.*

The representation of B_n which appears in the theorem is a B_n-analogue of the Lie representation of given type of $\mathfrak{S}_n$. We omit the definition, since this representation does not play any major role in this paper, and refer to [32, Section 4] and references therein for more information.

7.1. Derangements

The set of derangements in B_n, being a union of conjugacy classes, is naturally a fine set. This section describes the corresponding representation of B_n and its decomposition as a direct sum of irreducible representations, and expresses its Frobenius characteristic in terms of another set of elements of B_n, which may be called desarrangements of type B. These results are essentially B_n-analogues of results of Désarménien and Wachs [15, 16] and of Reiner and Webb [34, Section 2] for the symmetric group, which will first be reviewed; they provided much of the motivation behind the present paper. A B_n-analogue of the equidistribution [15, 16] of the descent set among derangements and desarrangements in $\mathfrak{S}_n$ will be deduced.

The set of derangements (elements without fixed points) in $\mathfrak{S}_n$, denoted here by $\mathcal{D}_n$, is a fine set for $\mathfrak{S}_n$ by Theorem 3.2 and [24, Theorem 3.6]. The corresponding $\mathfrak{S}_n$-representation, first studied in [34, Section 2], can be described as follows. We denote by $\mathcal{I}_n$ the set of words from the alphabet $[n]$ with no repeated letters (known as injective words), partially ordered by setting $u \le v$ for $u, v \in \mathcal{I}_n$ if u is a subword of v. Thus, the empty word is the minimum element and the $n!$ permutations in $\mathfrak{S}_n$ are the maximal elements of $\mathcal{I}_n$. The poset $\mathcal{I}_n$ is the face poset of a regular cell complex $\mathcal{K}_n$ whose faces are combinatorially isomorphic to simplices. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{I}_n$ and hence on the augmented cellular chain complex of $\mathcal{K}_n$ over $\mathbb{C}$; see [34] and references therein for more explanation and for background on regular cell complexes. Following [34], we denote by χ_n the character of the resulting representation on the top reduced homology $\tilde{H}_{n-1}(\mathcal{K}_n, \mathbb{C})$.

Using the Hopf trace formula and the shellability of $\mathcal{K}_n$, it was shown in [34, Proposition 2.1] that

$$\chi_n \;=\; \sum_{k=0}^{n} (-1)^{n-k} \; 1\!\uparrow_{(\mathfrak{S}_1)^k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \tag{7.2}$$

for every n (the reader who is unfamiliar with cellular homology may wish to consider this formula as the definition of χ_n).

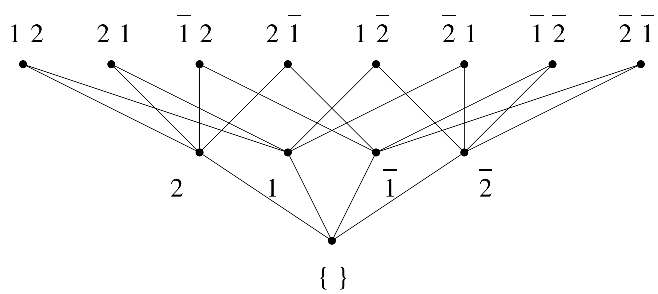
We denote by $\mathcal{E}_n$ the set of permutations (called desarrangements in [16]) $w \in \mathfrak{S}_n$ for which the smallest element of $[n] \smallsetminus \mathrm{Des}(w)$ is even. The following theorem, which combines results of [15, 16] and [34, Section 2], shows that $\mathcal{D}_n$ and $\{w^{-1} : w \in \mathcal{E}_n\}$ are fine sets for the sign twist $\varepsilon_n \otimes \chi_n$ of χ_n and determines its decomposition as a direct sum of irreducible characters.

Theorem 7.2. ([15, 16] [34, Section 2]) *For every positive integer n,*

$$
\begin{aligned}
\mathrm{ch}(\varepsilon_n \otimes \chi_n) &= \sum_{w \in \mathcal{D}_n} F_{n,\mathrm{Des}(w)}(x) = \sum_{w \in \mathcal{E}_n} F_{n,\mathrm{Des}(w^{-1})}(x) \\
&= \sum_{\lambda \vdash n} c_\lambda s_\lambda(x)
\end{aligned}
$$

where c_λ is the number of standard Young tableaux P of shape λ for which the smallest element of $[n] \smallsetminus \mathrm{Des}(P)$ is even. In particular, the number of derangements $w \in \mathcal{D}_n$ with $\mathrm{Des}(w) = S$ is equal to the number of permutations $w \in \mathcal{E}_n$ with $\mathrm{Des}(w^{-1}) = S$ for every $S \subseteq [n-1]$.

We now turn attention to the hyperoctahedral group B_n. We denote by $\mathcal{I}_n^B$ the set of words from the alphabet Ω_n which contain no two letters with equal absolute values and partially order this set by the subword order. The poset $\mathcal{I}_n^B$ is the face poset of a regular cell complex $\mathcal{K}_n^B$ whose faces are combinatorially isomorphic to simplices; see Figure 1 for the case $n = 2$. The group B_n acts naturally on $\mathcal{I}_n^B$ and hence on the augmented cellular chain complex of $\mathcal{K}_n^B$ over $\mathbb{C}$. We denote by ψ_n the character of the resulting representation on the top reduced homology $\tilde{H}_{n-1}(\mathcal{K}_n^B, \mathbb{C})$.



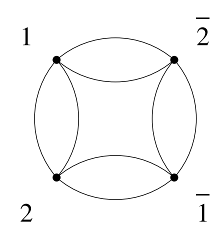


Figure 1: The face poset of the cell complex $\mathcal{K}_2^B$.

We denote by $\mathcal{D}_n^B$ the set of derangements (elements without fixed points, when thought of as permutations of Ω_n) in B_n and by $\mathcal{E}_n^B$ the set of $w \in B_n$ for which the maximum number k such that $w(1) > w(2) > \cdots > w(k) > 0$ is even, possibly equal to zero (where for $a \in \Omega_n$ we write $a > 0$ if a is unbarred). Finally, we denote by ω_x the standard involution on $\Lambda(x) \otimes \Lambda(y)$ acting on the x variables. The following statement is our B_n-analogue of Theorem 7.2.

Theorem 7.3. *For every positive integer n,*

$$\begin{aligned}\omega_x \operatorname{ch}(\psi_n) &= \sum_{w \in \mathcal{D}_n^B} F_{\mathrm{sDes}(w)}(x,y) = \sum_{w \in \mathcal{E}_n^B} F_{\mathrm{sDes}(w^{-1})}(x,y) \\ &= \sum_{(\lambda,\mu) \vdash n} c_{\lambda,\mu} s_\lambda(x) s_\mu(y)\end{aligned}$$

where $c_{\lambda,\mu}$ is the number of standard Young bitableaux (P^+, P^-) of shape (λ, μ) such that the largest number k for which $1, 2, \ldots, k$ appear in the first column of P^+ is even (possibly equal to zero).

In particular, the number of derangements $w \in \mathcal{D}_n^B$ with $\mathrm{sDes}(w) = \sigma$ is equal to the number of signed permutations $w \in \mathcal{E}_n^B$ with $\mathrm{sDes}(w^{-1}) = \sigma$ for every $\sigma \in \Sigma^B(n)$.

Example 7.4. To illustrate the theorem, let us compute explicitly these expressions for $\omega_x \operatorname{ch}(\psi_n)$ when $n = 2$. The bitableaux which satisfy the condition in the statement of the theorem are

$$\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\, , \varnothing \right), \quad \left(\begin{array}{|c|}\hline 2 \\ \hline\end{array}\, , \begin{array}{|c|}\hline 1 \\ \hline\end{array} \right), \quad \left(\varnothing\, , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \right), \quad \left(\varnothing\, , \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \right).$$

As a result, the third expression claimed for $\omega_x \operatorname{ch}(\psi_n)$ by the theorem gives

$$\omega_x \operatorname{ch}(\psi_2) = s_{(1,1)}(x) + s_{(1)}(x) s_{(1)}(y) + s_{(2)}(y) + s_{(1,1)}(y).$$

Moreover, the signed descent sets of the elements of

$$\mathcal{D}_2^B = \{(2,1), (2,\bar{1}), (\bar{2},1), (\bar{2},\bar{1}), (\bar{1},\bar{2})\},$$

written as signed compositions, are $(1,1)$, $(1,\bar{1})$, $(\bar{1},1)$, $(\bar{1},\bar{1})$ and $(\bar{2})$ and the same holds for those of $\{w^{-1} : w \in \mathcal{E}_2^B\} = \{(2,1), (\bar{1},2), (\bar{1},\bar{2}), (2,\bar{1}), (\bar{2},\bar{1})\}$. Hence, either of the first two expressions claimed for $\omega_x \operatorname{ch}(\psi_n)$ gives

$$\omega_x \operatorname{ch}(\psi_2) = F_{(1,1)}(x,y) + F_{(1,\bar{1})}(x,y) + F_{(\bar{1},1)}(x,y) + F_{(\bar{2})}(x,y) + F_{(\bar{1},\bar{1})}(x,y).$$

By direct computation or use of Proposition 4.2, one can verify that the two formulas for $\omega_x \operatorname{ch}(\psi_2)$ are equivalent.

The proof of part of Theorem 7.3 will be based on the following proposition. The proof of the proposition is a direct analogue of the proofs of Propositions 2.1 and 2.2 in [34, Section 2].

Proposition 7.5. *For every positive integer n,*

$$\psi_n \ = \ \sum_{k=0}^{n} (-1)^{n-k} \ 1\!\uparrow_{(\mathfrak{S}_1)^k \times B_{n-k}}^{B_n} . \tag{7.3}$$

For every $n \geq 2$,

$$\psi_n \ = \ (1 \otimes \psi_{n-1})\!\uparrow_{\mathfrak{S}_1 \times B_{n-1}}^{B_n} + (-1)^n \, 1_{B_n}. \tag{7.4}$$

Proof. Following arguments in [34, Section 2], we observe that the words of given rank k in $\mathcal{I}_n^B$ index the elements of a basis of the augmented cellular chain group $C_{k-1}(\mathcal{K}_n^B, \mathbb{C})$. The group B_n acts transitively on these words with stabilizer isomorphic to the Young subgroup $(\mathfrak{S}_1)^k \times B_{n-k}$. As a result, the character of B_n acting on $C_{k-1}(\mathcal{K}_n^B, \mathbb{C})$ is equal to the induced character $1\!\uparrow_{(\mathfrak{S}_1)^k \times B_{n-k}}^{B_n}$. Equation (7.3) then follows from the Hopf trace formula and the shellability (see [28, Theorem 1.2]) of $\mathcal{K}_n^B$. Finally, using (7.3) we compute that

$$\begin{aligned}
\psi_n \ &= \ \sum_{k=0}^{n} (-1)^{n-k} \ 1\!\uparrow_{(\mathfrak{S}_1)^k \times B_{n-k}}^{B_n} \\
&= \ (-1)^n \, 1_{B_n} \ + \ \sum_{k=1}^{n} (-1)^{n-k} \ 1\!\uparrow_{(\mathfrak{S}_1)^k \times B_{n-k}}^{B_n} \\
&= \ (-1)^n \, 1_{B_n} \ + \ \left(1 \otimes \sum_{k=1}^{n} (-1)^{n-k} \ 1\!\uparrow_{(\mathfrak{S}_1)^{k-1} \times B_{n-k}}^{B_{n-1}} \right) \uparrow_{\mathfrak{S}_1 \times B_{n-1}}^{B_n} \\
&= \ (-1)^n \, 1_{B_n} \ + \ (1 \otimes \psi_{n-1})\!\uparrow_{\mathfrak{S}_1 \times B_{n-1}}^{B_n} .
\end{aligned}$$

This verifies Equation (7.4). □

Proof of Theorem 7.3. Applying the Frobenius characteristic to Equation (7.4) and using Lemma 2.5 gives

$$\mathrm{ch}(\psi_n) \ = \ s_1(x,y) \, \mathrm{ch}(\psi_{n-1}) \ + \ (-1)^n s_n(x)$$

and hence

$$\omega_x \, \mathrm{ch}(\psi_n) \ = \ s_1(x,y) \cdot \omega_x \, \mathrm{ch}(\psi_{n-1}) \ + \ (-1)^n e_n(x) \tag{7.5}$$

for $n \geq 2$, where $s_1(x,y) = s_1(x) + s_1(y)$. One can check that ψ_1 is the sign character of B_1, so that $\mathrm{ch}(\psi_1) = \omega_x \, \mathrm{ch}(\psi_1) = s_1(y)$. By this observation and an easy induction argument, the recurrence (7.5) and Pieri's rule imply that

$$\omega_x \, \mathrm{ch}(\psi_n) \;=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} s_\lambda(x) s_\mu(y), \tag{7.6}$$

where $c_{\lambda,\mu}$ is as in the statement of Theorem 7.3. Furthermore, using Proposition 4.2 and basic properties of the Robinson–Schensted correspondence of type B (see Proposition 5.1) we get

$$\begin{aligned}
\sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} s_\lambda(x) s_\mu(y) \;&=\; \sum_{(\lambda,\mu)\vdash n} c_{\lambda,\mu} \sum_{Q\in \mathrm{SYT}(\lambda,\mu)} F_{\mathrm{sDes}(Q)}(x,y) \\
&=\; \sum_{P\in \mathrm{SYT}'(\lambda,\mu)} \; \sum_{Q\in \mathrm{SYT}(\lambda,\mu)} F_{\mathrm{sDes}(Q)}(x,y) \\
&=\; \sum_{w^{-1}\in \mathcal{E}_n^B} F_{\mathrm{sDes}(w)}(x,y), \qquad\qquad (7.7)
\end{aligned}$$

where $\mathrm{SYT}'(\lambda,\mu)$ stands for the set of standard Young bitableaux $P = (P^+, P^-)$ such that the largest number k for which $1, 2, \ldots, k$ appear in the first column of P^+ is even (possibly equal to zero).

Finally, we set

$$D_n^B(x,y) \;:=\; \sum_{w\in \mathcal{D}_n^B} F_{\mathrm{sDes}(w)}(x,y)$$

for $n \geq 1$. Since $\mathcal{D}_n^B$ is a union of conjugacy classes, Theorem 7.1 implies that

$$D_n^B(x,y) \;=\; \sum L^B_{\alpha,\beta}(x,y)$$

where the sum ranges over all bipartitions $(\alpha,\beta) \vdash n$ such that α has no part equal to one. We now recall (see [24, Section 3] and references therein) that there is an evaluation and length preserving bijection from the set of words on the alphabet $\mathcal{A}$ to the set of multisets of primitive necklaces on $\mathcal{A}$. Thus, just as in the symmetric group case (see the proof of [24, Theorem 8.1]), summing $L^B_{\alpha,\beta}(x,y)$ over all $(\alpha,\beta) \vdash n$ and considering the number of parts of α equal to one we get

$$s_1(x,y)^n \;=\; \sum_{k=0}^{n} s_k(x)\, D^B_{n-k}(x,y)$$

for $n \geq 0$, where $s_0(x) = D_0^B(x,y) = 1$. Equivalently, we have

$$D_n^B(x,y) \;=\; \sum_{k=0}^{n} (-1)^k e_k(x)\, s_1(x,y)^{n-k} \tag{7.8}$$

for $n \geq 1$. This formula and (7.5) show that

$$D_n^B(x,y) \;=\; \omega_x \, \mathrm{ch}(\psi_n) \tag{7.9}$$

for every $n \geq 1$. The proof follows by combining Equations (7.6), (7.7) and (7.9). $\square$

Remark 7.6. The last statement of Theorem 7.2 was originally proven in [15]. A bijective proof was later provided in [16]. An analogous proof of the last statement of Theorem 7.3 should be possible.

7.2. k-roots

Given a positive integer k, we will denote by $r_{n,k}(w)$ the number of k-roots of $w \in \mathfrak{S}_n$ (meaning, elements $u \in \mathfrak{S}_n$ with $u^k = w$) and by $r^B_{n,k}(w)$ the number of k-roots of $w \in B_n$. Clearly, $r_{n,k}$ and $r^B_{n,k}$ are class functions on $\mathfrak{S}_n$ and B_n, respectively. It was shown by Scharf [38, 39] that these functions are actually (non-virtual) characters of $\mathfrak{S}_n$ and B_n. The first statement in the next theorem follows from [37, Theorem 1.1] and its proof; the second statement follows from the first and Theorem 3.2.

Theorem 7.7. ([37]) *The set $\{w \in \mathfrak{S}_n : w^k = e\}$ of all k-roots of the identity element in $\mathfrak{S}_n$ is a fine set for the character $r_{n,k}$. Equivalently,*

$$\mathrm{ch}(r_{n,k}) \;=\; \sum_{w \in \mathfrak{S}_n :\, w^k = e} F_{n,\mathrm{Des}(w)}(x) \tag{7.10}$$

for all positive integers n, k.

The following theorem gives a partial answer to [37, Question 3.4]. It is expected that this theorem can be extended to the (possibly more complicated) case of even positive integers k, which is left open in the present writing (the case $k = 2$ follows from Proposition 5.3, since $r^B_{n,2}$ is equal to the character of the Gelfand model of B_n; see [26, p. 58]).

Theorem 7.8. *The set $\{w \in B_n : w^k = e\}$ of all k-roots of the identity element in B_n is a fine set for the character $r^B_{n,k}$ for every odd positive integer k and all $n \geq 1$. Equivalently,*

$$\mathrm{ch}(r^B_{n,k}) \;=\; \sum_{w \in B_n \colon w^k = e} F_{\mathrm{sDes}(w)}(x,y) \tag{7.11}$$

for all positive integers n, k with k odd.

Example 7.9. For $n = k = 3$, the k-roots of the identity element in B_n are $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(\bar{2},\bar{3},1)$, $(\bar{2},3,\bar{1})$, $(2,\bar{3},\bar{1})$, $(\bar{3},\bar{1},2)$, $(\bar{3},1,\bar{2})$ and $(3,\bar{1},\bar{2})$. Computing the signed descent compositions of these signed permutations and applying Theorem 7.8, we find that

$$\begin{aligned}\mathrm{ch}(r^B_{3,3}) \;&=\; F_{(3)}(x,y) \,+\, F_{(2,1)}(x,y) \,+\, F_{(1,2)}(x,y) \,+\, F_{(\bar{2},1)}(x,y) \,+ \\ &\quad\; F_{(1,\bar{2})}(x,y) \,+\, F_{(1,\bar{1},1)}(x,y) \,+\, 2F_{(\bar{1},1,\bar{1})}(x,y) \,+\, F_{(1,\bar{1},\bar{1})}.\end{aligned}$$

The multiset of these signed compositions coincides with the multiset of signed descent compositions of the standard Young bitableaux of shapes $((3),\varnothing)$, $((2,1),\varnothing)$, $((1),(2))$ and $((1),(1,1))$. Thus, we may deduce from Proposition 4.2 that

$$\mathrm{ch}(r^B_{3,3}) \;=\; s_{(3)}(x) \,+\, s_{(2,1)}(x) \,+\, s_{(1)}(x)s_{(2)}(y) \,+\, s_{(1)}(x)s_{(1,1)}(y).$$

This yields the decomposition $r^B_{3,3} = \chi^{((3),\varnothing)} + \chi^{((2,1),\varnothing)} + \chi^{((1),(2)} + \chi^{((1),(1,1))}$.

The proof of Theorem 7.8 follows the computation of $\mathrm{ch}(r_{n,k})$ in [46], as presented in the solution to [43, Exercise 7.69 (c)]. We will write $L^B_n(x,y)$ for the formal sum of the evaluations of all primitive necklaces of length n over the alphabet $\mathcal{A}$ having an even number of y variables (i.e. for the function $L^B_{\alpha,\beta}(x,y)$ when $\alpha = (n)$ and $\beta = \varnothing$). We will also write $L_n(x,y)$ for the corresponding formal sum with no restriction on the number of y variables. We will first establish the following lemma.

Lemma 7.10. *For every odd positive integer k*

$$\sum_{d\,|k} \sum_{n \geq 1} \frac{1}{n} L^B_d(x^n, y^n)\, t^{nd} = \sum_{n \geq 1} \frac{1}{2n} \left((p^+_{n/(n,k)}(x,y))^{(n,k)} + (p^-_{n/(n,k)}(x,y))^{(n,k)} \right) t^n,$$

where $x^n = (x_1^n, x_2^n, \dots)$, $y^n = (y_1^n, y_2^n, \dots)$ and (n,k) denotes the greatest common divisor of n and k.

Proof. The definition of the functions $L_d(x, y)$ and $L_d^B(x, y)$ as generating functions of necklaces implies that $L_d^B(x, y) = (L_d(x, y) + L_d(x, -y))/2$. Since

$$\sum_{d\,|\,k} \sum_{n\geq 1} \frac{1}{n} L_d(x^n, y^n)\, t^{nd} \;=\; \sum_{n\geq 1} \frac{1}{n} (p^+_{n/(n,k)}(x, y))^{(n,k)}\, t^n \tag{7.12}$$

follows from [43, Equation (7.216)], it suffices to verify that

$$\sum_{d\,|\,k} \sum_{n\geq 1} \frac{1}{n} L_d(x^n, -y^n)\, t^{nd} \;=\; \sum_{n\geq 1} \frac{1}{n} (p^-_{n/(n,k)}(x, y))^{(n,k)}\, t^n. \tag{7.13}$$

We sketch the proof of this equation, which is similar to that of (7.12). Let us denote by $\mu : \{1, 2, \dots\} \to \mathbb{Z}$ the number theoretic Möbius function. The following well known (see, for instance, [24, Equation 2.2] or [43, Exercise 7.89 (a)]) formula

$$L_m(x) \;=\; \frac{1}{m} \sum_{d\,|\,m} \mu(d) (p_d(x))^{m/d}$$

can be proved by a simple Möbius inversion argument. Replacing x by $(x^{n/m}, -y^{n/m})$ in this equation and assuming that m is odd gives

$$mL_m(x^{n/m}, -y^{n/m}) \;=\; \sum_{d\,|\,m} \mu(d) (p^-_{nd/m}(x, y))^{m/d} \;=\; \sum_{d\,|\,m} \mu(m/d) (p^-_{n/d}(x, y))^d$$

whenever n is divisible by m. Applying Möbius inversion, we get

$$(p^-_{n/m}(x, y))^m \;=\; \sum_{d\,|\,m} dL_d(x^{n/d}, -y^{n/d})$$

whenever m is odd and n is divisible by m. Since k is assumed to be odd, we may replace m by (n, k) in the last equation to get

$$(p^-_{n/(n,k)}(x, y))^{(n,k)} \;=\; \sum_{d\,|\,k} \sum_{d\,|\,n} dL_d(x^{n/d}, -y^{n/d})$$

for every positive integer n. Multiplying by t^n/n and summing over for all $n \geq 1$ we get (7.13) and the proof follows. $\square$

Proof of Theorem 7.8. From the defining equation (2.5) of the Frobenius characteristic map we have

$$\mathrm{ch}(r^B_{n,k}) \;=\; \frac{1}{2^n n!} \sum_{w \in B_n} r^B_{n,k}(w) p_w(x,y) \;=\; \frac{1}{2^n n!} \sum_{u \in B_n} p_{u^k}(x,y).$$

As in the solution to [43, Exercise 7.69 (c)] we observe that the kth power of a positive m-cycle in B_n is a product of (m,k) disjoint positive cycles of length $m/(m,k)$ and, since k is odd, the analogous statement holds for negative cycles. An application of the exponential formula [43, Corollary 5.1.9] then yields

$$\sum_{n \ge 0} \mathrm{ch}(r^B_{n,k}) t^n \;=\; \exp \sum_{n \ge 1} \left((p^+_{n/(n,k)}(x,y))^{(n,k)} \;+\; (p^-_{n/(n,k)}(x,y))^{(n,k)} \right) \frac{t^n}{2n}. \tag{7.14}$$

We now denote by $G_{n,k}(x,y)$ the right-hand side of (7.11) and compute its generating function as follows. Since k is odd, a signed permutation $w \in B_n$ is a k-root of the identity element if and only if the cycle decomposition of w involves only positive cycles whose lengths divide k. Thus, it follows from Theorem 7.1 that

$$\sum_{n \ge 0} G_{n,k}(x,y) t^n \;=\; \sum_{\alpha} L^B_{\alpha,\varnothing}(x,y) t^{|\alpha|}$$

where the sum ranges over all integer partitions α such that every part of α divides k. The definition of $L^B_{\alpha,\beta}(x,y)$ in terms of multisets of necklaces and that of plethysm of symmetric functions in turn imply that

$$\sum_{n \ge 0} G_{n,k}(x,y) t^n \;=\; \prod_{d \,|\, k} h[L^B_d(tx,ty)] \tag{7.15}$$

where $h(x) = \sum_{n \ge 0} h_n(x)$ is the formal sum of all complete homogeneous symmetric functions $h_n(x)$. The basic formula $\log h(x) = \log \prod_{i \ge 1} 1/(1 - x_i) = \sum_{n \ge 1} p_n(x)/n$ and (7.15) then imply that

$$\sum_{n \ge 0} G_{n,k}(x,y) t^n \;=\; \exp \sum_{d \,|\, k} \sum_{n \ge 1} \frac{1}{n} L^B_d(x^n, y^n)\, t^{nd}. \tag{7.16}$$

Comparing (7.14) to (7.16) and using Lemma 7.10 gives $\mathrm{ch}(r^B_{n,k}) = G_{n,k}(x,y)$ for all $n \ge 0$ and the proof follows. $\square$

8. Arc permutations

Arc permutations, originally introduced in the study of triangulations [5], have interesting combinatorial properties. For instance, they can be characterized by pattern avoidance, they carry interesting graph and poset structures, as well as an affine Weyl group action, and afford well factorized unsigned and signed enumeration formulas [17]. Two type B extensions were introduced in [18]. This section reviews the relevant definitions and shows that one of these extensions is a fine set for an $\mathfrak{S}_n$-character, while the other is a fine set for a B_n-character.

A permutation $w \in \mathfrak{S}_n$ is said to be an *arc permutation* if, for every $i \in [n]$, the set $\{w(1), w(2), \ldots, w(i)\}$ is an interval in $\mathbb{Z}_n$ (where the letter n is identified with zero). For example, $(2, 1, 5, 3, 4)$ is an arc permutation in $\mathfrak{S}_5$ but $(2, 1, 5, 6, 3, 4)$ is not an arc permutation in $\mathfrak{S}_6$. The set of arc permutations in $\mathfrak{S}_n$ is denoted by $\mathcal{A}_n$.

As will be explained in the sequel, the following theorem can be deduced from results of Elizalde and Roichman [17]. This theorem will be extended to type B in this section. Let V_{n-1} be an $(n-1)$-dimensional vector space over $\mathbb{C}$, on which $\mathfrak{S}_{n-1}$ acts by permuting coordinates.

Theorem 8.1. *For $n \geq 2$, the set $\mathcal{A}_n$ is fine for the character of the induced representation $\bigwedge V_{n-1} \uparrow^{\mathfrak{S}_n}$ of the exterior algebra of V_{n-1} from $\mathfrak{S}_{n-1}$ to $\mathfrak{S}_n$.*

Equivalently, $\sum_{w \in \mathcal{A}_n} F_{n,\mathrm{Des}(w)}(x) = \mathrm{ch}(\chi)$ where

$$\chi = \sum_{k=1}^{n-1} \chi^{(k,1^{n-1-k})} \uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = \chi^{(n)} + \chi^{(1^n)} + 2\sum_{k=2}^{n-1} \chi^{(k,1^{n-k})} + \sum_{k=2}^{n-2} \chi^{(k,2,1^{n-k-2})}.$$

Two type B extensions of the concept of an arc permutation, suggested in [18], can be described as follows. Let us identify Ω_n with $\mathbb{Z}_{2n}$ by associating $\bar{i} \in \Omega_n$ with $n + i \in \mathbb{Z}_{2n}$ (and thus $\bar{n} \in \Omega_n$ with $0 \in \mathbb{Z}_{2n}$) for every $i \in [n]$. A signed permutation $w \in B_n$ is said to be a *B-arc permutation* if $\{w(i), w(i+1), \ldots, w(n)\}$ is an interval in $\mathbb{Z}_{2n}$ for every $i \in [n]$. For instance, $(\bar{2}, 3, \bar{1}, 5, 4)$ is a B-arc permutation in B_5. The set of B-arc permutations in B_n will be denoted by $\mathcal{A}_n^B$. Note that $|\mathcal{A}_n^B| = n2^n$.

We also say that $w \in B_n$ is a *signed arc permutation* if, for each $i \in \{2, \ldots, n-1\}$,

- the set $\{|w(1)|, \ldots, |w(i-1)|\}$ is an interval in $\mathbb{Z}_n$; and

- $w(i)$ is unbarred if $|w(i)|-1 \in \{|w(1)|,\ldots,|w(i-1)|\}$ and barred if $|w(i)|+1 \in \{|w(1)|,\ldots,|w(i-1)|\}$ (with addition in $\mathbb{Z}_n$).

For instance, $(\bar{3},\bar{2},4,1)$ is a signed arc permutation in B_4. The set of signed arc permutations in B_n will be denoted by $\mathcal{A}_n^s$. Since there is no restriction on the signs of $w(1)$ and $w(n)$, the number of signed arc permutations in B_n satisfies $|\mathcal{A}_n^s| = 4|\mathcal{A}_n| = n2^n$ for $n \geq 2$.

The main results of this section state that $\mathcal{A}_n^B$ and $\mathcal{A}_n^s$ are fine sets for characters of $\mathfrak{S}_n$ and B_n, respectively, which are described explicitly. We recall from Section 2.2 that the descent set of $w \in B_n$ is defined as $\mathrm{Des}(w) = \{i \in [n-1] : w(i) >_r w(i+1)\}$, and that its signed descent set $\mathrm{sDes}(w)$ is given by Definition 2.2. The group B_{n-1} acts on the vector space V_{n-1} by permuting coordinates and switching their signs.

Theorem 8.2. *For every $n \geq 2$,*

$$\sum_{w \in \mathcal{A}_n^B} F_{n,\mathrm{Des}(w)}(x) \;=\; \mathrm{ch}\left(2\sum_{k=1}^{n-1} \chi^{(k,1^{n-k-1})} \uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \,+\, n\sum_{k=1}^{n} \chi^{(k,1^{n-k})} \right).$$

Theorem 8.3. *For every $n \geq 2$, the set $\mathcal{A}_n^s$ is fine for the B_n-character induced from the exterior algebra $\bigwedge V_{n-1}$. Equivalently,*

$$\sum_{w \in \mathcal{A}_n^s} F_{\mathrm{sDes}(w)}(x,y) \;=\; \mathrm{ch}\left(\sum_{k=0}^{n-1} \chi^{(k),(1^{n-k-1})} \uparrow_{B_{n-1}}^{B_n} \right).$$

Example 8.4. We have

$$\mathcal{A}_2^B = \mathcal{A}_2^s = \{(1,2),(\bar{1},2),(1,\bar{2}),(\bar{1},\bar{2}),(2,1),(\bar{2},1),(2,\bar{1}),(\bar{2},\bar{1})\}.$$

Thus, for $n=2$, Theorem 8.2 states that

$$\begin{aligned} 4F_{2,\varnothing}(x) \,+\, 4F_{2,\{1\}}(x) \;&=\; 4s_{(2)}(x) \,+\, 4s_{(1,1)}(x) \\ &=\; \mathrm{ch}\left(2\chi^{(1)} \uparrow_{\mathfrak{S}_1}^{\mathfrak{S}_2} + 2(\chi^{(1,1)} + \chi^{(2)})\right), \end{aligned}$$

and Theorem 8.3 states that

$$\begin{aligned} &F_{(2)}(x,y) + 2F_{(1,\bar{1})}(x,y) + 2F_{(\bar{1},1)}(x,y) + F_{(\bar{2})}(x,y) + F_{(1,1)}(x,y) + F_{(\bar{1},\bar{1})}(x,y) \\ &\quad=\; s_{(2)}(x) \,+\, s_{(2)}(y) \,+\, s_{(1,1)}(x) \,+\, s_{(1,1)}(y) \,+\, 2s_{(1)}(x)s_{(1)}(y) \\ &\quad=\; \mathrm{ch}\left(\left(\chi^{\varnothing,(1)} \,+\, \chi^{(1),\varnothing}\right) \uparrow_{B_1}^{B_2}\right), \end{aligned}$$

where we have indexed the functions $F_\sigma(x,y)$ by the corresponding signed descent compositions.

For the proof of Theorems 8.1 and 8.2 we need one more definition. A permutation $w \in \mathfrak{S}_n$ is called *left-unimodal* (respectively, *right-unimodal*) if the set $\{w(1), w(2), \dots, w(n)\}$ (respectively, $\{w(i), w(i+1), \dots, w(n)\}$) is an interval in $\mathbb{Z}$ for every $i \in [n]$. Clearly, all left-unimodal and all right-unimodal permutations are arc permutations. The set of left-unimodal permutations in $\mathfrak{S}_n$ will be denoted by $\mathcal{L}_n$, and that of right-unimodal permutations by $\mathcal{R}_n$.

Proposition 8.5. *For $n \geq 1$,*

$$\sum_{w \in \mathcal{L}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \sum_{w \in \mathcal{R}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \mathrm{ch}\left(\sum_{k=1}^{n} \chi^{(k,1^{n-k})}\right).$$

Proof. As explained in the proofs of Propositions 4 and 6 in [17], from the definition of left and right unimodality we get

$$\sum_{w \in \mathcal{L}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \sum_{w \in \mathcal{R}_n} F_{n,\mathrm{Des}(w)}(x) \;=\; \sum_{S \subseteq [n-1]} F_{n,S}(x).$$

A simple application of Proposition 2.6 shows that

$$\sum_{S \subseteq [n-1]} F_{n,S}(x) \;=\; \sum_{k=1}^{n} s_{(k,1^{n-k})}(x)$$

and the proof follows. $\square$

Proof of Theorem 8.1. We may assume that $n \geq 4$. Then [17, Theorem 5] may be rephrased as stating that

$$\sum_{w \in \mathcal{A}_n \setminus (\mathcal{L}_n \cup \mathcal{R}_n)} F_{n,\mathrm{Des}(w)}(x) \;=\; \mathrm{ch}\left(\sum_{k=2}^{n-2} \chi^{(k,2,1^{n-k-2})}\right).$$

The second statement in the theorem follows from this equation, Proposition 8.5 and the fact that $\mathcal{L}_n \cap \mathcal{R}_n$ consists of the identity permutation and its reverse. The equivalence to the first statement is explained in the proof of [17, Theorem 6]. $\square$

Proof of Theorem 8.2. The proposed equation may be rewritten as

$$\sum_{w\in\mathcal{A}_n^B} F_{n,\mathrm{Des}(w)}(x)$$
$$= \mathrm{ch}\left((n+2)(\chi^{(n)} + \chi^{(1^n)}) + (n+4)\sum_{k=2}^{n-1}\chi^{(k,1^{n-k})} + 2\sum_{k=2}^{n-2}\chi^{(k,2,1^{n-k-2})}\right).$$

To prove this equation, and in view of Theorem 8.1 and Proposition 8.5, it suffices to find a subset $\widehat{\mathcal{A}}_n$ of $\mathcal{A}_n^B$ for which there exist an n-to-one descent preserving map from $\widehat{\mathcal{A}}_n$ to the set $\mathcal{R}_n$ of right-unimodal permutations and a two-to-one descent preserving map from $\mathcal{A}_n^B \smallsetminus \widehat{\mathcal{A}}_n$ to the set $\mathcal{A}_n$ of arc permutations. We will verify these properties when $\widehat{\mathcal{A}}_n$ is the set of $w \in \mathcal{A}_n^B$ such that $w(i) = 1$ for some $i \in [n]$. For the first property, we observe that $\widehat{\mathcal{A}}_n$ is the disjoint union of the sets $\widehat{\mathcal{A}}_{n,k}$, consisting of the elements $w \in \mathcal{A}_n^B$ for which $(w(1), w(2), \dots, w(n))$ is a permutation of $\{\overline{k+1}, \dots, \bar{n}, 1, \dots, k\}$, for $1 \le k \le n$ and that there is a natural descent preserving bijection from each $\widehat{\mathcal{A}}_{n,k}$ to $\mathcal{R}_n$.

For the second property, we note that $\bar{\mathcal{A}}_n := \mathcal{A}_n^B \smallsetminus \widehat{\mathcal{A}}_n$ is the set of $w \in \mathcal{A}_n^B$ for which $(w(1), w(2), \dots, w(n))$ is a permutation of $\{k+1, \dots, n, \bar{1}, \dots, \bar{k}\}$ for some $k \in [n]$ and consider the map $f : \bar{\mathcal{A}}_n \to \mathcal{A}_n$ which simply forgets the bars, meaning that $f(w) = (|w(1)|, |w(2)|, \dots, |w(n)|)$ for $w \in \bar{\mathcal{A}}_n$. This map is clearly descent preserving. We leave to the reader to verify that the preimages of a given arc permutation $(a_1, a_2, \dots, a_n) \in \mathcal{A}_n$ under f are

- $(\bar{1}, a_2, \dots, a_n)$ and $(\bar{1}, \bar{a}_2, \dots, \bar{a}_n)$, if $a_1 = 1$, and
- $(a_1, b_2, \dots, b_n)$ and $(\bar{a}_1, b_2, \dots, b_n)$ otherwise, where $b_i = a_i$ if $a_i > a_1$ and $b_i = \bar{a}_i$ if $a_i < a_1$, for $i \ge 2$.

This shows that the map f is also two-to-one. $\square$

Remark 8.6. Using the reasoning in Remark 6.5, it can be verified that $\mathcal{A}_n^B$ is not a fine set for any B_n-character for $n \ge 3$.

Even though $\mathcal{A}_n^s$ is not a union of Knuth classes of type B, the Robinson–Schensted correspondence of type B will be used in the proof of Theorem 8.3. We recall from Section 2.2 that we think of signed permutations $w \in B_n$ as written in the form $(w(1), w(2), \dots, w(n))$.

Proof of Theorem 8.3. The proposed equation may be rewritten as

$$\sum_{w \in \mathcal{A}_n^s} F_{\mathrm{sDes}(w)}(x,y) \;=\; s_{(n)}(x) \;+\; s_{(1^n)}(y) \;+\; \sum_{k=1}^{n-1} s_{(k,1)}(x) s_{(1^{n-k-1})}(y) \;+\; \sum_{k=1}^{n-1} s_{(k-1)}(x) s_{(2,1^{n-k-1})}(y) \;+\; 2\sum_{k=1}^{n-1} s_{(k)}(x) s_{(1^{n-k})}(y).$$

By Corollary 3.7 (or Proposition 4.2), it suffices to show that the distribution of sDes over $\mathcal{A}_n^s$ is equal to its distribution over all standard Young bitableaux of the following multiset of shapes:

(i) shapes $((n), \varnothing)$ and $(\varnothing, (1^n))$ with multiplicity one;
(ii) all shapes $((k,1),(1^{n-k-1}))$ and $((n-k-1),(2,1^{k-1}))$, for $1 \le k \le n-1$, with multiplicity one; and
(iii) all shapes $((k),(1^{n-k}))$, for $1 \le k \le n-1$, with multiplicity two.

Given our discussion of the Robinson–Schensted correspondence for B_n in Section 5 (and, in particular, since the map $Q^B(\cdot)$ preserves the signed descent set), it suffices to show that the restriction of $Q^B(\cdot)$ to $\mathcal{A}_n^s$ induces a two-to-one map from a subset $\widehat{\mathcal{A}}_n^s$ of $\mathcal{A}_n^s$ to the set of standard Young bitableaux of shapes of type (iii) and a bijection from $\mathcal{A}_n^s \smallsetminus \widehat{\mathcal{A}}_n^s$ to the set of standard Young bitableaux of shapes of types (i) and (ii).

Let $e \in B_n$ be the identity permutation and $w_\circ \in B_n$ be defined by $w_\circ(i) = \overline{n-i+1}$ for $i \in [n]$. We note that $\mathcal{A}_n^s$ is invariant under right multiplication by $w_\circ$. For $k \in [n-1]$, we denote by $\mathcal{A}_{n,k}$ the set consisting of all signed permutations which are shuffles of the sequences $(\bar{k}, \overline{k-1}, \dots, \bar{1})$ and $(k+1, \dots, n)$. Then $\mathcal{A}_{n,k} \subseteq \mathcal{A}_n^s$ and $\mathcal{A}_{n,k} w_\circ$ consists of all shuffles of $(\bar{n}, \overline{n-1}, \dots, \overline{k+1})$ with $(1, 2, \dots, k)$. Clearly, $\mathcal{A}_{n,k}$ and $\mathcal{A}_{n,k} w_\circ$ are disjoint for $k \in [n-1]$ and $\mathcal{A}_{n,i} \sqcup \mathcal{A}_{n,i} w_\circ$ and $\mathcal{A}_{n,j} \sqcup \mathcal{A}_{n,j} w_\circ$ are disjoint for $i \ne j$. We choose $\widehat{\mathcal{A}}_n^s$ to be the disjoint union

$$\widehat{\mathcal{A}}_n^s \;:=\; \bigsqcup_{k=1}^{n-1} (\mathcal{A}_{n,k} \sqcup \mathcal{A}_{n,k} w_\circ).$$

For $w \in \mathcal{A}_{n,k}$, we note that $Q^B(w)$ is the standard Young bitableau of shape $((n-k),(1^k))$ whose entries in (1^k) are the positions of the barred letters in w. As a result, $Q^B(\cdot)$ induces a bijection from $\mathcal{A}_{n,k}$ to $\mathrm{SYT}((n-k),(1^k))$. Similarly, $Q^B(\cdot)$ induces a bijection from $\mathcal{A}_{n,k} w_\circ$ to $\mathrm{SYT}((k),(1^{n-k}))$. Thus, $Q^B(\cdot)$

induces a two-to-one map from $\widehat{\mathcal{A}}^s_n$ to the set of standard Young bitableaux of shapes of type (iii).

Finally, we need to show that $Q^B(\cdot)$ induces a bijection from $\mathcal{A}^s_n \smallsetminus \widehat{\mathcal{A}}^s_n$ to the set of all standard Young bitableaux of shapes of types (i) and (ii). Let us denote by $\mathcal{C}_{n,k}$ the set of all shuffles of $(n-k+1,\ldots,n,1)$ with $(\overline{n-k},\overline{n-k-1},\ldots,\bar{2})$ and note that $\mathcal{C}_{n,k}$ is a subset of $\mathcal{A}^s_n$. Denoting by c the positive cycle $(1\ 2\ \cdots\ n)(\bar{1}\ \bar{2}\ \cdots\ \bar{n})$ and letting

$$\mathcal{B}_{n,k} \ := \ \bigsqcup_{r=0}^{k-1} c^r \mathcal{C}_{n,k},$$

we may express $\mathcal{A}^s_n \smallsetminus \widehat{\mathcal{A}}^s_n$ as a disjoint union

$$\mathcal{A}^s_n \smallsetminus \widehat{\mathcal{A}}^s_n \ = \ \{e, w\} \sqcup \bigsqcup_{k=1}^{n-1} \mathcal{B}_{n,k} \sqcup \bigsqcup_{k=1}^{n-1} \mathcal{B}_{n,k} w_\circ.$$

Clearly, $Q^B(e)$ is the only standard Young bitableau of shape $((n), \varnothing)$ and $Q^B(w_\circ)$ is the only standard Young bitableau of shape $(\varnothing, (1^n))$. We leave to the reader to verify that $Q^B(\cdot)$ induces bijective maps from $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n,k} w_\circ$ to the set of standard Young bitableaux of shape $((k,1),(1^{n-k-1}))$ and $((n-k,1),(2,1^{k-1}))$, respectively. This completes the proof. $\square$

9. Remarks

Character formulas for the hyperoctahedral group were studied in this paper. A generalization to the corresponding Iwahori-Hecke algebra would be most desirable.

Problem 9.1. Find an appropriate q-analogue of Definition 3.5 which will provide character formulas for the Iwahori–Hecke algebra of type B.

Most of the results in this paper may be naturally generalized to the wreath product group $G(r,n) := \mathbb{Z}_r \wr S_n$ of r-colored permutations, where r is an arbitrary positive integer (this paper has focused on the special case of $G(2,n) = B_n$). For example, the B_n-analogue of the fundamental quasisymmetric functions, considered here, was introduced and studied in the general setting of r-colored permutations by Poirier [32]. However, it should be noted that generalizations to $G(r,n)$ are not always straightforward. The following problem is challenging.

Problem 9.2. Generalize the concept of signed arc permutations and Theorem 8.3 to the groups $G(r,n)$.

The situation is more complicated for general complex reflection groups.

Problem 9.3. Generalize the setting and results of this paper to the classical complex reflection groups $G(r,p,n)$.

For example, the generalization of Theorems 7.7 and 7.8 is related to the problem of characterizing the finite groups G for which the function $r_k : G \to \mathbb{N}$, assigning to $w \in G$ the number of k-roots of w, is a non-virtual character; see, for instance, [26, Chapter 4] [39].

The group of even signed permutations $D_n = G(2,2,n)$ is of special interest.

Another possible direction is to extend the setting of this paper to other (not necessarily classical or finite) Coxeter groups.

Problem 9.4. Generalize the concept of fine set to arbitrary Coxeter groups.

This objective may require a different approach. For example, we believe that the results in Section 6 may be generalized to classical complex reflection groups. This hope is based on the current good understanding of the flag-major index and its role in the combinatorics of the coinvariant and diagonal invariant algebras of classical complex reflection groups; see, for instance, [13]. Unfortunately, exceptional Weyl groups are still mysterious and the problem of finding a "correct" flag-major index on these groups is widely open.

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