On the gamma-positivity of the Eulerian transformation

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KTH Combinatorics Seminar

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Contents

1. The Eulerian transformation

Let

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{exc}(w)}$$

be the nth Eulerian polynomial, so

$$A_n(x) = \begin{cases} 1, & n = 0, 1 \\ 1 + x, & n = 2 \\ 1 + 4x + x^2, & n = 3 \\ 1 + 11x + 11x^2 + x^3, & n = 4 \\ 1 + 26x + 66x^2 + 26x^3 + x^4, & n = 5 \end{cases}$$

and

$$A_n^{\circ}(x) = \begin{cases} A_n(x), & n = 0 \\ x A_n(x), & n \geq 1. \end{cases}$$

Definition. The Eulerian transformation is the linear map $\mathcal{A}^{\circ}: \mathbb{R}[x] \to \mathbb{R}[x]$ defined by

$$\mathcal{A}^{\circ}(x^n) = A_n^{\circ}(x)$$

for every $n \in \mathbb{N} := \{0, 1, 2, \dots\}$.

Conjecture (Brenti, 1989) The polynomial $\mathcal{A}^{\circ}(p(x))$ has only real roots for every polynomial $p(x) \in \mathbb{R}[x]$ which has nonnegative coefficients and only real roots.

Recall that $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in$
$\in \mathbb{R}[x]$ is said to be

- unimodal if $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq$
  $\geq a_n$ for some $0 \leq k \leq n$,
- symmetric, with respect to n, if
  $a_i = a_{n-i}$ for $0 \leq i \leq n$.
- γ-positive if

$$p(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

  for some $n \in \mathbb{N}$ and $\gamma_i \geq 0$,
- real-rooted if it has only real roots
  or $p(x) \equiv 0$.

Recall also that $p(x)$ has a unique symmetric decomposition with respect to n (Stapledon, 2009), meaning an expression

$$p(x) = a(x) + x\, b(x)$$

where $a(x)$ and $b(x)$ are symmetric with respect to n and $n-1$, respectively. This decomposition is said to be nonnegative, unimodal, γ-positive or real-rooted if both $a(x)$ and $b(x)$ have the corresponding property.

Note. $p(x)$ has a unimodal symmetric decomposition with respect to n iff

$$a_0 \leq a_n \leq a_1 \leq a_{n-1} \leq \cdots \leq a_{\lfloor (n+1)/2 \rfloor}.$$

Theorem (Brändén - Jochemko, 2021+)

(a) Brenti's conjecture is false.

(b) The polynomial $\mathcal{A}^0(p(x))$ has a unimodal symmetric decomposition with respect to n for every

$$p(x) = \sum_{k=0}^{n} c_k x^{n-k}(1+x)^k$$

with $c_0, c_1, \ldots, c_n \geqslant 0$.

Conjecture (Brändén - Jochemko, 2021+)

The polynomial $\mathcal{A}^0(p(x))$ is real-rooted for every $p(x) \in \mathbb{R}[x]$ as in (b).

Theorem (A, recent) The polynomial $A^{0}(p(x))$ has a γ-positive symmetric decomposition with respect to n for every $p(x) \in \mathbb{R}[x]$ as in (b).

2. The polynomials $\mathring{A}(x^{n-k}(1+x)^k)$

We let

$$q_{n,k}(x) = I_n \mathring{A}(x^{n-k}(1+x)^k)$$

for $k \in \{0,1,\dots,n\}$, where $(I_n p)(x) = x^n \cdot p(1/x)$. Thus, if

$$p(x) = \sum_{k=0}^{n} c_k x^{n-k}(1+x)^k,$$

then

$$I_n \mathring{A}(p(x)) = \sum_{k=0}^{n} c_k q_{n,k}(x).$$

Note.

- $q_{n,0}(x) = A_n(x)$
- $q_{n,n}(x) = \tilde{A}_n(x).$

$n=0$ $\quad 1$

$n=1$ $\quad 1, \quad 1+x$

$n=2$ $\quad 1+x, \quad 1+2x, \quad 1+3x+x^2$

$n=3$ $\quad 1+4x+x^2, \quad 1+5x+2x^2, \quad 1+6x+4x^2,$

$\quad 1+7x+7x^2+x^3$

the $q_{n,k}(x)$ for $n \leq 3$

Conjecture. The $q_{n,k}(x)$ for $0 \leq k \leq n$ form an interlacing sequence of real-rooted polynomials.

Note. This implies the conjecture of Brändén-Jochemko.

Proposition.

(a) $q_{n,k+1}(x) = q_{n,k}(x) + x\, q_{n-1,k}(x)$

(b)
$$q_{n,k}(x) = \sum_{i=0}^{k} \binom{k}{i} x^i A_{n-i}(x)$$

(c)
$$q_{n,k}(x) = \sum_{w \in \mathfrak{S}_n} (1+x)^{\mathrm{fix}_k(w)}\, x^{\mathrm{exc}(w)}$$

where $\mathrm{fix}_k(w)$ is the number of fixed points of $w \in \mathfrak{S}_n$ which are $\leq k$.

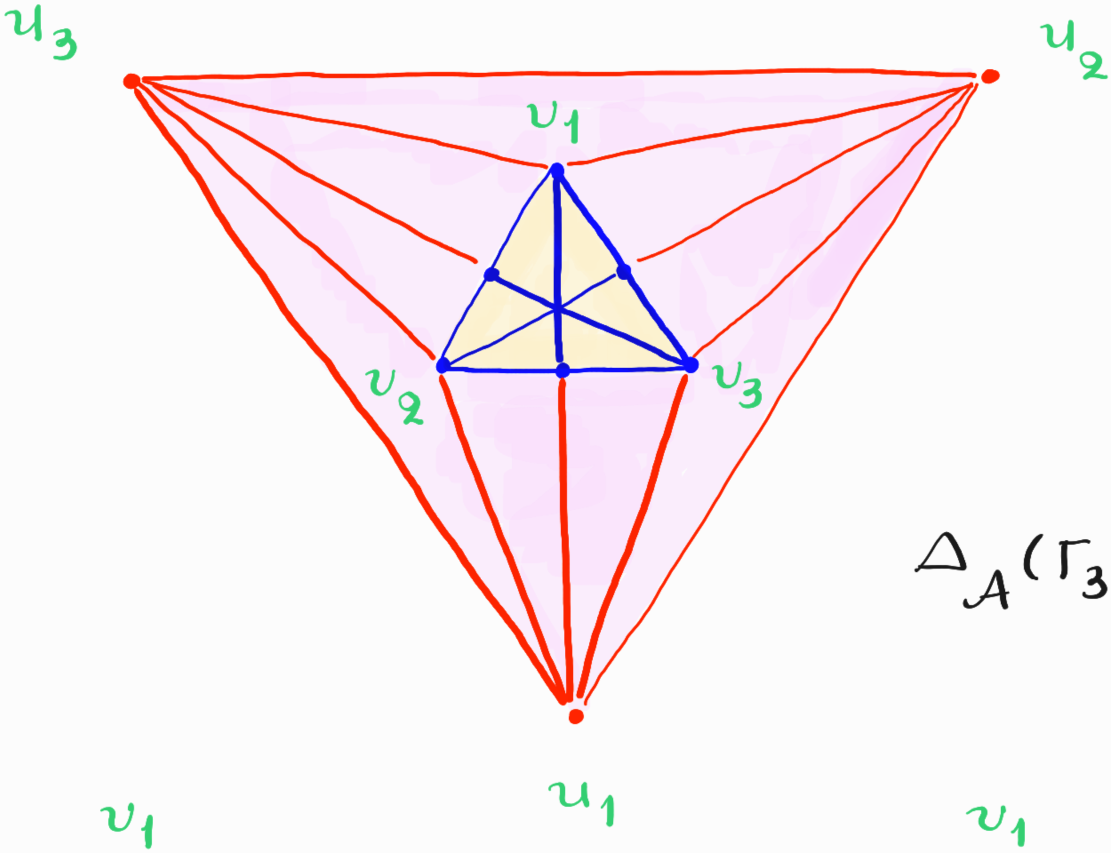
(d) $q_{n,k}(x) = \sum_{i=0}^{k} \binom{k}{i} p_{n-i,k-i}(x)$

where

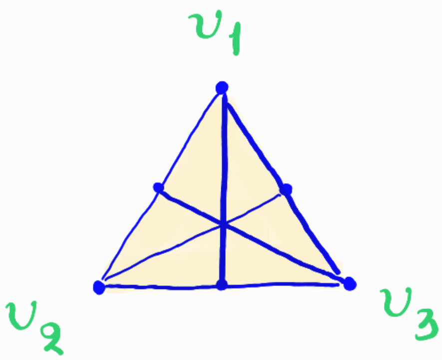
$$p_{n,k}(x) = \sum_{w \in \mathfrak{S}_{n+1} \,:\, w(1) = k+1} x^{\mathrm{des}(w)}$$

(e) $q_{n,k}(x)$ is equal to the h-polynomial of the induced subcomplex of the antiprism sphere $\Delta_A(\Gamma_n)$ on the vertex set $V \cup \{u_1, u_2, \dots, u_k\}$, where

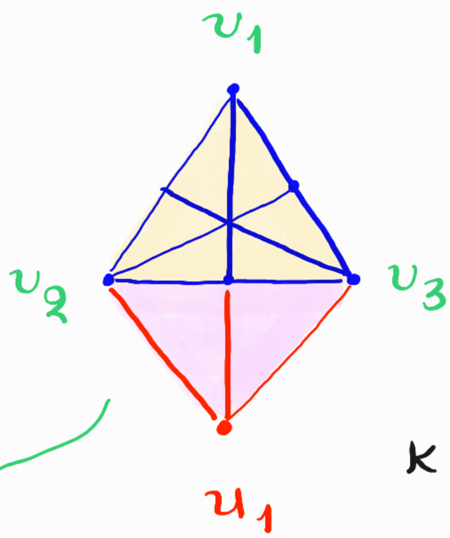
- $V = \{v_1, v_2, \dots, v_n\}$
- Γ_n = barycentric subdivision of the simplex 2^V.



$\Delta_{\mathcal{A}}(\Gamma_3)$

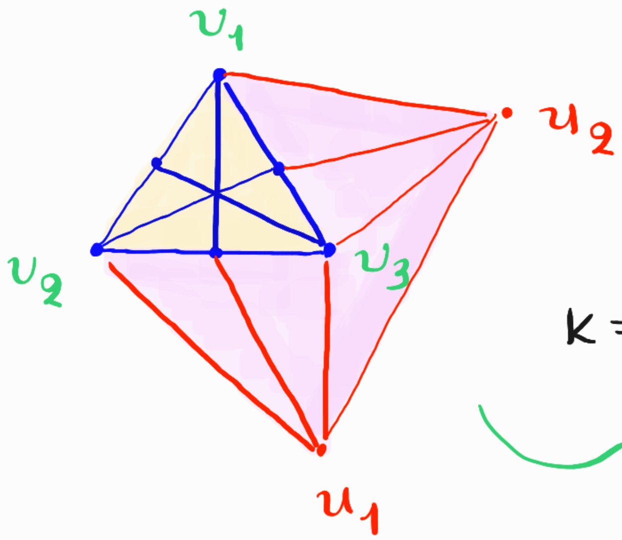


$k=0$



$k=1$

$1+5x+2x^2$



$k=2$

$1+6x+4x^2$

Brändén - Solus considered the linear transformation $\mathcal{D} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ with

$$\mathcal{D}(x^n) = d_n(x) := \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_{n-i}(x)$$

and the polynomials

- $d_{n,k}(x) = \mathcal{D}(x^k(1+x)^{n-k})$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} A_{n-i}(x).$$

Note.
- $d_{n,0}(x) = A_n(x)$
- $d_{n,n}(x) = d_n(x)$

and $d_{n,k+1}(x) = d_{n,k}(x) - d_{n-1,k}(x).$

$n=0$	1			
$n=1$	1	0		
$n=2$	$1+x$	x	x	
$n=3$	$1+4x+x^2$	$3x+x^2$	$2x+x^2$	$x+x^2$

the $d_{n,k}(x)$ for $n \leq 3$

Note. Brändén - Solus (2021) studied

$$\mathcal{D}\left((x+\theta_1)(x+\theta_2)\cdots(x+\theta_n)\right)$$

for $0 \leq \theta_1, \theta_2, \ldots, \theta_n \leq 1$. From their results it follows that for every $n \in \mathbb{N}$, the $d_{n,k}(x)$ for $0 \leq k \leq n$ form an interlacing sequence of real-rooted polynomials.

Proposition.

(a)

$$d_{n,k}(x) = \sum_{\substack{w \in \mathfrak{S}_n : \\ \mathrm{Fix}(w) \subseteq [n-k]}} x^{\mathrm{exc}(w)}$$

(b)

$$d_{n,k}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^{+}_{n,k,i} \, x^{i} (1+x)^{n-2i} +$$

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi^{-}_{n,k,i} \, x^{i} (1+x)^{n-1-2i}$$

where

- $\xi^{+}_{n,k,i}$ = # $w \in \mathfrak{S}_n$ with $w(1) > n-k$ which have i decreasing runs, all of length > 1

- $\xi^{-}_{n,k,i}$ = # $w \in \mathfrak{S}_n$ with $w(1) \leq n-k$ which have i decreasing runs, all but possibly the first of length > 1.

Corollary. Since

$$q_{n,k}(x) = \sum_{i=0}^{k} \binom{k}{i} (1+x)^i \, d_{n-i,k-i}(x),$$

In $q_{n,k}(x)$ has a γ-positive symmetric decomposition with respect to n.

Note. More conjectures about $q_{n,k}(x)$ are possible. For instance, for $0 \leq q \leq 1$ let

$$p_{n,k}(x;q) = \sum_{i=0}^{k} \binom{k}{i} (x-q)^i A_{n-i}(x),$$

so

- $p_{n,k}(x;1) = p_{n,k}(x)$
- $p_{n,k}(x;0) = q_{n,k}(x)$

& $p_{n,k+1}(x;q) = p_{n,k}(x;q) + (x-q)\, p_{n-1,k}(x;q)$.

Conjecture. $p_{n,k}(x;q)$ interlaces $p_{n,k}(x;q')$ for $0 \leq q < q' \leq 1$. In particular, $q_{n,k}(x)$ interlaces $p_{n,k}(x)$.

3. Uniform triangulations

Let $h_0(x), h_1(x), h_2(x), \ldots \in \mathbb{R}[x]$ be such that

- $h_n(x)$ is real-rooted with nonnegative coefficients
- $h_n(x)$ interlaces $h_{n+1}(x)$

for every $n \in \mathbb{N}$ and set

$$h_{n,k}(x) = \sum_{i=0}^{k} \binom{k}{i} x^i h_{n-i}(x)$$

for $k \in \{0, 1, \ldots, n\}$.

- $h_0(x)$
- $h_1(x)$, $h_1(x) + x h_0(x)$
- $h_2(x)$, $h_2(x) + x h_1(x)$, $h_2(x) + 2x h_1(x) + x^2 h_0(x)$
- $h_3(x)$, $\quad \ldots$

Question. What additional conditions on the $h_n(x)$ guarantee that for every $n \in \mathbb{N}$, $(h_{n,k}(x))_{0 \le k \le n}$ is an interlacing sequence of real-rooted polynomials?

Computations suggest this may be the case for the following sequences of polynomials :

- $A_n(x)$ = n^{th} Eulerian polynomial
- $B_n(x)$ = n^{th} Eulerian polynomial of type B
- $A_{n,r}(x)$ = n^{th} r-colored Eulerian polynomial
- $B_n^+(x) = \sum_{w \in B_n : w(n) > 0} x^{des(w)}$

- $h_{n,r}(x)$ = the h-polynomial or interior h-polynomial of the r-fold edgewise subdivision of the (n-1)-dimensional simplex, for $r \geqslant n$
- $h_n(x)$ = the h-polynomial or interior h-polynomial of the antiprism triangulation of the (n-1)-dimensional simplex.

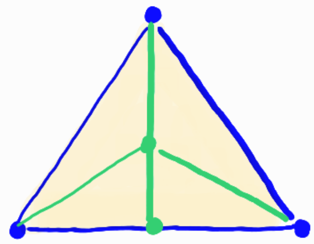
Note. This fails for $h_n(x) = (1+x)^n$

$n=0$ $\quad 1$

$n=1$ $\quad 1+x, \qquad 1+2x$

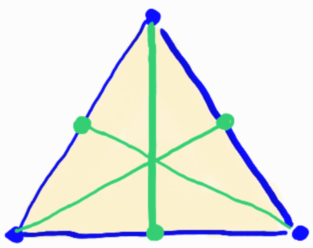
$n=2$ $\quad (1+x)^2 \qquad (1+x)(1+2x), \quad (1+2x)^2$

although $(1+x)^n$ is equal to the h-polynomial of a triangulation of the n-dimensional simplex.

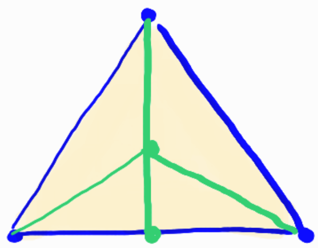
$\Gamma =$ 

$h(\Gamma, x) = (1+x)^2$

Definition (A, 2020) A triangulation Δ' of a simplicial complex Δ is said to be **uniform** if the f-vector of the restriction of Δ' to a face $F \in \Delta$ depends only on $\dim(F)$.



uniform



not uniform

we denote by

$$\mathcal{F} = (f(i,j))$$

the triangular array of numbers in which $(f(i,j))_{0 \leq i \leq j}$ is the f-vector of the restriction of Δ' to a $(j-1)$-dimensional face of Δ.

Let

$$\sigma_n = (n-1)\text{-dimensional simplex.}$$

The data $\mathcal{F}$ determines the polynomials

- $f_{\mathcal{F}}(\sigma_n, x)$ = f-polynomial
- $f^{\circ}_{\mathcal{F}}(\sigma_n, x)$ = interior f-polynomial
- $h_{\mathcal{F}}(\sigma_n, x)$ = h-polynomial
- $h^{\circ}_{\mathcal{F}}(\sigma_n, x)$ = interior h-polynomial
- $\ell_{\mathcal{F}}(\sigma_n, x)$ = local h-polynomial

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} h_{\mathcal{F}}(\sigma_k, x)$$

- $\theta_{\mathcal{F}}(\sigma_n, x)$ = theta polynomial

$$= h_{\mathcal{F}}(\sigma_n, x) - h_{\mathcal{F}}(\partial\sigma_n, x)$$

of any $\mathcal{F}$-uniform triangulation of σ_n.

Definition. We say that

(a) $\mathcal{F}$ has the strong interlacing property if each $h_{\mathcal{F}}(\sigma_n, x)$

- is real-rooted
- has a nonnegative, real-rooted symmetric decomposition with respect to $n-1$, each part of which is interlaced by $h_{\mathcal{F}}(\sigma_{n-1}, x)$.

(b) $\mathcal{F}$ is theta γ-positive if $\theta_{\mathcal{F}}(\sigma_n, x)$ is γ-positive for every n.

Given $\mathcal{F}$, we may now consider the linear operators

$$\mathcal{H}_{\mathcal{F}}, \mathcal{H}^{\circ}_{\mathcal{F}} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$$

defined by

- $\mathcal{H}_{\mathcal{F}}(x^n) = h_{\mathcal{F}}(\sigma_n, x)$
- $\mathcal{H}^{\circ}_{\mathcal{F}}(x^n) = h^{\circ}_{\mathcal{F}}(\sigma_n, x)$

for $n \in \mathbb{N}$.

Note. In the special case of barycentric subdivision

$$\mathcal{H}^{\circ}_{\mathcal{F}}(x^n) = \begin{cases} 1, & n=0 \\ x A_n(x), & n \geq 1 \end{cases} = \mathcal{A}^{\circ}(x^n).$$

Conjecture. Suppose $\mathcal{F}$ has the strong interlacing property. Then $\mathcal{H}(p(x))$ and $\mathcal{H}^{\circ}(p(x))$ are real-rooted for every

$$p(x) = \sum_{k=0}^{n} c_k \, x^{n-k} (1+x)^k$$

with $c_0, c_1, \ldots, c_n \geqslant 0$.

Theorem (A, recent) Suppose that $\mathcal{F}$ is theta γ-positive. Then $\mathcal{H}^{\circ}(p(x))$ has a γ-positive symmetric decomposition with respect to n for every $p(x) \in \mathbb{R}[x]$ as above.

4. Some generalized local h-polynomials

A crucial role in the proof is played by the following concept. Recall that

- $$h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta)\, x^i (1-x)^{n-i}$$

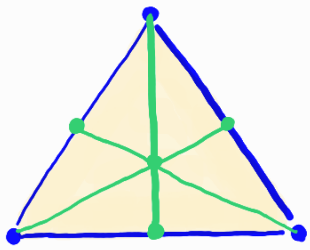
- $$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|}\, h(\Gamma_F, x)$$

are the definitions of the h-polynomial of a simplicial complex Δ and the local h-polynomial of a triangulation Γ of the simplex 2^V, where $n = \dim(\Delta) + 1 = |V|$.

Example. For the barycentric subdivision Γ_n of the $(n-1)$-dimensional simplex 2^V

- $h(\Gamma_n, x) = A_n(x)$
- $\ell_V(\Gamma_n, x) = d_n(x)$

$$= \sum_{\substack{w \in \mathfrak{S}_n : \\ \mathrm{Fix}(w) = \emptyset}} x^{\mathrm{exc}(w)}.$$



$n = 3$

- $h(\Gamma_3, x) = (1-x)^3 + 7x(1-x)^2 + 12x^2(1-x) + 6x^3 = 1 + 4x + x^2$
- $\ell_V(\Gamma_3, x) = (1+4x+x^2) - 3(1+x) + 3 - 1 = x + x^2.$

Definition. Given a triangulation Γ of 2^V and $E \subseteq V$, we define

$$\ell_{V,E}(\Gamma, x) = \sum_{E \subseteq F \subseteq V} (-1)^{|V \setminus F|} h(\Gamma_F, x).$$

Note.

- $\ell_{V,V}(\Gamma, x) = h(\Gamma, x)$
- $\ell_{V,\emptyset}(\Gamma, x) = \ell_V(\Gamma, x)$.

Example. For the barycentric subdivision Γ_n,

$$\ell_{V,E}(\Gamma_n, x) = d_{n,n-k}(x),$$

where $k = |E|$.

Note. The $\ell_{V,E}(\Gamma, x)$ are relevant because in the case of uniform triangulations.

- $\mathcal{H}^{\circ}_{\mathcal{F}}\left(x^{n-k} (1+x)^{k} \right) =$

$$= \sum_{E \subseteq F \subseteq V} \ell_{F,E}(\Gamma, x)\, (1+x)^{n-|F|},$$

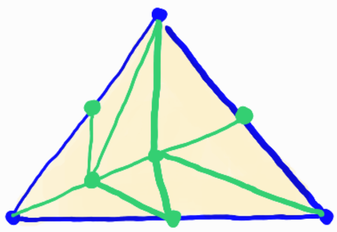
where $|E| = n-k$, in analogy with the corresponding formula for the $q_{n,k}(x)$.

Definition. We define

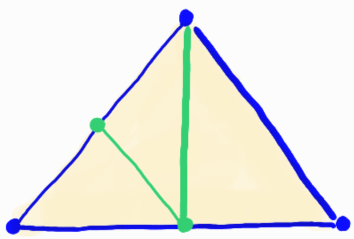
$$\theta(\Gamma, x) = h(\Gamma, x) - h(\partial\Gamma, x)$$

and say that Γ is theta γ-positive if $\theta(\Gamma_F, x)$ is γ-positive for every $F \subseteq V$.

Conjecture (A, 2022+, Chudnovsky-Nevo 2020) The polynomial $\theta(\Gamma, x)$ is γ-positive for every flag triangulation Γ of the simplex 2^V such that $\partial\Gamma$ is a vertex-induced subcomplex of Γ.



induced



not induced

Note.

- $$h(\Gamma, x) = \sum_{F \subseteq V} \theta(\Gamma_F, x) A_{|V \setminus F|}(x)$$

  (A, 2022+)

- $$\ell_V(\Gamma, x) = \sum_{F \subseteq V} \theta(\Gamma_F, x) d_{|V \setminus F|}(x)$$

  (Kubitzke-Murai-Sieg, 2019)

Proposition.

$$\ell_{V,E}(\Gamma, x) = \sum_{F \subseteq V} \theta(\Gamma_F, x) d_{|V \setminus F|, |V \setminus (E \cup F)|}(x).$$

In particular, the theta γ-positivity of Γ implies the γ-positivity of the symmetric decompositions of $\ell_{V,E}(\Gamma, x)$ and $I_n \mathcal{H}_F^{\circ}(x^{n-k}(1+x)^k)$.