SOME APPLICATIONS OF REES PRODUCTS OF POSETS TO EQUIVARIANT GAMMA-POSITIVITY

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Abstract. The Rees product of partially ordered sets was introduced by Björner and Welker. Using the theory of lexicographic shellability, Linusson, Shareshian and Wachs proved formulas, of significance in the theory of gamma-positivity, for the dimension of its homology. Equivariant analogues of these formulas are proven in this paper and are applied to establish the Schur gamma-positivity of certain symmetric functions arising in algebraic and geometric combinatorics.

1. Introduction

The Rees product $P * Q$ of two partially ordered sets (posets, for short) was introduced and studied by Björner and Welker [7] as a combinatorial analogue of the Rees construction in commutative algebra (a precise definition of $P * Q$ can be found in Section 2). The connection of the Rees product of posets to enumerative combinatorics was hinted in [7, Section 5], where it was conjectured that the dimension of the homology of the Rees product of the truncated Boolean algebra $B_n \smallsetminus \{\varnothing\}$ of rank $n-1$ with an n-element chain equals the number of permutations of $[n] := \{1, 2, \dots, n\}$ without fixed points. This statement was generalized in several ways in [13], using enumerative and representation theoretic methods, and in [9], using the theory of lexicographic shellability.

One of the results of [9] proves formulas [9, Corollary 3.8] for the dimension of the homology of the Rees product of an EL-shellable poset P with a contractible poset which generalizes the chain of the same rank as P. This paper provides an equivariant analogue of this result which seems to have enough applications on its own to be of independent interest. To state it, let P be a finite bounded poset, with minimum element $\hat{0}$ and maximum element $\hat{1}$, which is graded of rank $n+1$, with rank function $\rho : P \to \{0, 1, \dots, n+1\}$ (for basic terminology on posets, see [20, Chapter 3]). Fix a field $\mathbf{k}$ and let G be a finite group which acts on P by order preserving bijections. Then, G defines a permutation representation $\alpha_P(S)$ over $\mathbf{k}$ for every $S \subseteq [n]$, induced by the action of G on the set of maximal chains of the rank-selected subposet

$$P_S \;=\; \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\} \qquad (1)$$

Date: July 20, 2018.

Key words and phrases. Rees product, poset homology, group action, Schur gamma-positivity, local face module.

of P. Following [17], we may consider the virtual G-representation

$$\beta_P(S) \ = \ \sum_{T \subseteq S} (-1)^{|S-T|} \, \alpha_P(T), \tag{2}$$

defined equivalently by the equations

$$\alpha_P(T) \ = \ \sum_{S \subseteq T} \beta_P(S) \tag{3}$$

for $T \subseteq [n]$. When P is Cohen–Macaulay over $\mathbf{k}$, $\beta_P(S)$ coincides with the non-virtual G-representation induced on the top homology group of $\bar{P}_S := P_S \smallsetminus \{\hat{0}, \hat{1}\}$; see [17] [25, Section 3.4] for more information.

As in references [9, 13], we write $\beta(\bar{P})$ in place of $\beta_P([n])$ and denote by $T_{t,n}$ the poset whose Hasse diagram is a complete t-ary tree of height n, rooted at the minimum element. We denote by P^-, P_- and $\bar{P}$ the poset obtained from P by removing its maximum element, or minimum element, or both, respectively, and recall from [13] (see also Section 2) that the action of G on P induces actions on $P^- * T_{t,n}$ and $\bar{P} * T_{t,n-1}$ as well. We also write $[a, b] := \{a, a+1, \dots, b\}$ for integers $a \le b$ and denote by $\mathrm{Stab}(\Theta)$ the set of all subsets, called stable, of $\Theta \subseteq \mathbb{Z}$ which do not contain two consecutive integers. The following result reduces to [9, Corollary 3.8], proven in [9] under additional shellability assumptions on P, in the special case of a trivial action.

Theorem 1.1. *Let G be a finite group acting on a finite bounded graded poset P of rank $n+1$ by order preserving bijections. Then,*

$$\beta((P^- * T_{t,n})_-) \ \cong_G \ \sum_{S \in \mathrm{Stab}([n-1])} \beta_P([n] \smallsetminus S) \ t^{|S|} (1+t)^{n-2|S|} \ + \ \sum_{S \in \mathrm{Stab}([n-2])} \beta_P([n-1] \smallsetminus S) \ t^{|S|+1} (1+t)^{n-1-2|S|} \tag{4}$$

and

$$\beta(\bar{P} * T_{t,n-1}) \ \cong_G \ \sum_{S \in \mathrm{Stab}([2,n-2])} \beta_P([n-1] \smallsetminus S) \ t^{|S|+1} (1+t)^{n-2-2|S|} \ + \ \sum_{S \in \mathrm{Stab}([2,n-1])} \beta_P([n] \smallsetminus S) \ t^{|S|} (1+t)^{n-1-2|S|} \tag{5}$$

for every positive integer t, where $\cong_G$ stands for isomorphism of G-representations. If P is Cohen–Macaulay over $\mathbf{k}$, then the left hand-sides of (4) and (5) may be replaced by the G-representations $\widetilde{H}_{n-1}((P^- * T_{t,n})_-; \mathbf{k})$ and $\widetilde{H}_{n-1}(\bar{P} * T_{t,n-1}; \mathbf{k})$, respectively, and all representations which appear in these formulas are non-virtual.

A polynomial in t with real coefficients is said to be *γ-positive* if for some $m \in \mathbb{N}$, it can be written as a nonnegative linear combination of the binomials $t^i (1+t)^{m-2i}$ for $0 \le i \le m/2$. Clearly, all such polynomials have symmetric and unimodal coefficients. Several applications of [9, Corollary 3.8] to γ-positivity appear in [9] [3, Section 3] and are summarized in [5, Section 2.4]. Theorem 1.1 has non-trivial applications to Schur γ-positivity, which we now briefly discuss. Two symmetric function identities due to Gessel (unpublished), stated without proof in [9, Section 4] [14, Theorem 7.3], can be written in the form

$$\frac{1-t}{E(\mathbf{x};tz)-tE(\mathbf{x};z)} \;=\; 1+\sum_{n\geq 2} z^n \sum_{k=0}^{\lfloor (n-2)/2\rfloor} \xi_{n,k}(\mathbf{x})\, t^{k+1}(1+t)^{n-2k-2} \tag{6}$$

and

$$\frac{(1-t)E(\mathbf{x};tz)}{E(\mathbf{x};tz)-tE(\mathbf{x};z)} \;=\; 1+\sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \gamma_{n,k}(\mathbf{x})\, t^{k+1}(1+t)^{n-1-2k}, \tag{7}$$

where $E(\mathbf{x};z)=\sum_{n\geq 0} e_n(\mathbf{x})z^n$ is the generating function for the elementary symmetric functions in $\mathbf{x}=(x_1,x_2,\dots)$ and the $\xi_{n,k}(\mathbf{x})$ and $\gamma_{n,k}(\mathbf{x})$ are Schur-positive symmetric functions, whose coefficients in the Schur basis can be explicitly described (see Corollary 4.1). The coefficients of z^n in the right-hand sides of Equations (6) and (7) are Schur γ-positive symmetric functions, in the sense that their coefficients in the Schur basis are γ-positive polynomials in t with the same center of symmetry. We will show (see Section 4) that these identities can in fact be derived from the special case of Theorem 1.1 in which P^- is the Boolean algebra B_n, endowed with the natural symmetric group action. Moreover, applying the theorem when P^- is a natural signed analogue of B_n, endowed with a hyperoctahedral group action, we obtain new identities of the form

$$\frac{E(\mathbf{x};tz)-tE(\mathbf{x};z)}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} \;=\; \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \xi^+_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-2k}, \tag{8}$$

$$\frac{E(\mathbf{x};z)-E(\mathbf{x};tz)}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} \;=\; \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \xi^-_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-1-2k}, \tag{9}$$

$$\frac{E(\mathbf{x};z)E(\mathbf{x};tz)\,(E(\mathbf{y};tz)-tE(\mathbf{y};z))}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} \;=\; \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \gamma^+_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-2k} \tag{10}$$

and

$$\frac{tE(\mathbf{x};z)E(\mathbf{x};tz)\,(E(\mathbf{y};z)-E(\mathbf{y};tz))}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} \;=\; \sum_{n\geq 1} z^n \sum_{k=1}^{\lfloor (n+1)/2\rfloor} \gamma^-_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n+1-2k}, \tag{11}$$

where the $\xi^\pm_{n,k}(\mathbf{x},\mathbf{y})$ and $\gamma^\pm_{n,k}(\mathbf{x},\mathbf{y})$ are Schur-positive symmetric functions in the sets of variables $\mathbf{x}=(x_1,x_2,\dots)$ and $\mathbf{y}=(y_1,y_2,\dots)$ separately. Note that the left-hand sides of Equations (8) and (11) specialize to those of (6) and (7), respectively, for $\mathbf{x}=0$.

Various combinatorial and algebraic-geometric interpretations of the left-hand sides of Equations (6) and (7) are discussed in [14, Section 7] [9, Section 4] [15]. For instance, by [18, Proposition 4.20], the coefficient of z^n in the left-hand side of (6) can be interpreted as the Frobenius characteristic of the symmetric group representation on the local face module of the barycentric subdivision of the $(n-1)$-dimensional simplex, twisted by the sign representation. Thus, the Schur γ-positivity of this coefficient, manifested by Equation (6), is an instance of the local equivariant Gal phenomenon, as discussed in [5, Section 5.2]. Section 5 shows that another instance of this phenomenon follows from the specialization $\mathbf{x} = \mathbf{y}$ of Equation (8). Similarly, setting $\mathbf{y} = 0$ to (10) yields another identity, recently proven by Shareshian and Wachs (see Proposition 3.3 and Theorem 3.4 in [16]) in order to establish the equivariant Gal phenomenon for the n-dimensional stellohedron. Further applications of Theorem 1.1 will appear in [6]. It would be interesting to generalize some of the known interpretations for the left-hand sides of Equations (6) and (7) to those of (8) – (11).

Outline. The proof of Theorem 1.1 is given in Section 3, after the relevant background and definitions are explained in Section 2. This proof is fairly elementary and different from that of [9, Corollary 3.8], given in [9]. Section 4 derives Equations (6) – (11) from Theorem 1.1 and provides explicit combinatorial interpretations, in terms of standard Young (bi-)tableaux and their descents, for the Schur-positive symmetric functions which appear there. Section 5 provides the promised application of Equation (8) to the equivariant γ-positivity of the symmetric group representation on the local face module of a certain triangulation of the simplex.

2. Preliminaries

This section briefly records definitions and background on posets, group representations and (quasi)symmetric functions. For basic notions and more information on these topics, the reader is referred to the sources [11] [17] [20, Chapter 3] [21, Chapter 7] [25]. The symmetric group of permutations of the set $[n] := \{1, 2, \dots, n\}$ is denoted by $\mathfrak{S}_n$ and the cardinality of a finite set S by $|S|$.

Group actions on posets and Rees products. All groups and posets considered here are assumed to be finite. Homological notions for posets always refer to those of their order complex; see [25, Lecture 1]. A poset P has the structure of a *G-poset* if the group G acts on P by order preserving bijections. Then, G induces a representation on every reduced homology group $\widetilde{H}_i(P; \mathbf{k})$, for every field $\mathbf{k}$.

Suppose that P is a G-poset with minimum element $\hat{0}$ and maximum element $\hat{1}$. Sundaram [24] (see also [25, Theorem 4.4.1]) established the isomorphism of G-representations

$$\bigoplus_{k \geq 0} (-1)^k \bigoplus_{x \in P/G} \widetilde{H}_{k-2}((\hat{0}, x); \mathbf{k}) \uparrow_{G_x}^{G} \ \cong_G \ 0. \tag{12}$$

Here P/G stands for a complete set of G-orbit representatives, $(\hat{0}, x)$ denotes the open interval of elements of P lying strictly between $\hat{0}$ and x, G_x is the stabilizer of x and $\uparrow$ denotes induction. Moreover, $\widetilde{H}_{k-2}((\hat{0}, x); \mathbf{k})$ is understood to be the trivial representation 1_{G_x} if $x = \hat{0}$ and $k = 0$, or x covers $\hat{0}$ and $k = 1$.

The *Lefschetz character* of a finite G-poset P (over the field $\mathbf{k}$) is defined as the virtual G-representation

$$L(P;G) \ := \ \bigoplus_{k\geq 0} (-1)^k \, \widetilde{H}_k(P;\mathbf{k}).$$

Note that $L(P;G) = (-1)^r \widetilde{H}_r(P;\mathbf{k})$, if P is Cohen–Macaulay over $\mathbf{k}$ of rank r.

Given finite graded posets P and Q with rank functions ρ_P and ρ_Q, respectively, their *Rees product* is defined in [7] as $P * Q = \{(p,q) \in P \times Q : \rho_P(p) \geq \rho_Q(q)\}$, with partial order defined by setting $(p_1,q_1) \preceq (p_2,q_2)$ if all of the following conditions are satisfied:

- $p_1 \preceq p_2$ holds in P,
- $q_1 \preceq q_2$ holds in Q and
- $\rho_P(p_2) - \rho_P(p_1) \geq \rho_Q(q_2) - \rho_Q(q_1)$.

Equivalently, (p_1,q_1) is covered by (p_2,q_2) in $P*Q$ if and only if (a) p_1 is covered by p_2 in P; and (b) either $q_1 = q_2$, or q_1 is covered by q_2 in Q. We note that the Rees product $P*Q$ is graded with rank function given by $\rho(p,q) = \rho_P(p)$ for $(p,q) \in P*Q$, and that if P is a G-poset, then so is $P*Q$ with the G-action defined by setting $g \cdot (p,q) = (g \cdot p, q)$ for $g \in G$ and $(p,q) \in P*Q$.

A bounded graded G-poset P, with maximum element $\hat{1}$, is said to be *G-uniform* [13, Section 3] if the following hold:

- the intervals $[x,\hat{1}]$ and $[y,\hat{1}]$ in P are isomorphic for all $x,y \in P$ of the same rank,
- the stabilizers G_x and G_y are isomorphic for all $x,y \in P$ of the same rank, and
- there is an isomorphism between $[x,\hat{1}]$ and $[y,\hat{1}]$ that intertwines the actions of G_x and G_y, for all $x,y \in P$ of the same rank.

Given a sequence of groups $G = (G_0, G_1, \dots, G_n)$, a sequence of posets $(P_0, P_1, \dots, P_n)$ is said to be *G-uniform* [13, Section 3] if

- P_k is G_k-uniform of rank k for all k,
- G_k is isomorphic to the stabilizer $(G_n)_x$ for every $x \in P_n$ of rank $n-k$ and all k, and
- there is an isomorphism between P_k and the interval $[x,\hat{1}]$ in P_n that intertwines the actions of G_k and $(G_n)_x$ for every $x \in P_n$ of rank $n-k$ and all k.

Under these assumptions, Shareshian and Wachs [13, Proposition 3.7] established the isomorphism of G-representations

$$1_{G_n} \oplus \bigoplus_{k=0}^{n} W_k(P_n;G_n)\,[k+1]_t \, L((P_{n-k} * T_{t,n-k})_-; G_{n-k}) \uparrow_{G_{n-k}}^{G_n} \ \cong_G \ 0 \tag{13}$$

for every positive integer t, where $W_k(P_n;G_n)$ is the number of G_n-orbits of elements of P_n of rank k and $[k+1]_t := 1 + t + \cdots + t^k$.

Permutations, Young tableaux and symmetric functions. Our notation concerning these topics follows mostly that of [11] [20, Chapter 1] [21, Chapter 7]. In particular, the set of standard Young tableaux of shape λ is denoted by $\mathrm{SYT}(\lambda)$, the descent set

$\{i \in [n-1] : w(i) > w(i+1)\}$ of a permutation $w \in \mathfrak{S}_n$ is denoted by $\mathrm{Des}(w)$ and that of a tableau $Q \in \mathrm{SYT}(\lambda)$, consisting of those entries i for which $i+1$ appears in Q in a lower row than i, is denoted by $\mathrm{Des}(Q)$. We recall that the Robinson–Schensted correspondence is a bijection from the symmetric group $\mathfrak{S}_n$ to the set of pairs $(\mathcal{P}, Q)$ of standard Young tableaux of the same shape and size n. This correspondence has the property [21, Lemma 7.23.1] that $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$ and $\mathrm{Des}(w^{-1}) = \mathrm{Des}(\mathcal{P}(w))$, where $(\mathcal{P}(w), Q(w))$ is the pair of tableaux associated to $w \in \mathfrak{S}_n$.

We will consider symmetric functions in the indeterminates $\mathbf{x} = (x_1, x_2, \dots)$ over the complex field $\mathbb{C}$. We denote by $E(\mathbf{x}; z) := \sum_{n \geq 0} e_n(\mathbf{x}) z^n$ and $H(\mathbf{x}; z) := \sum_{n \geq 0} h_n(\mathbf{x}) z^n$ the generating functions for the elementary and complete homogeneous symmetric functions, respectively, and recall the identity $E(\mathbf{x}; z) H(\mathbf{x}; -z) = 1$. The characteristic map, a $\mathbb{C}$-linear isomorphism of fundamental importance from the space of virtual $\mathfrak{S}_n$-representations to that of homogeneous symmetric functions of degree n, will be denoted by ch. The *fundamental quasisymmetric function* associated to $S \subseteq [n-1]$ is defined as

$$F_{n,S}(\mathbf{x}) \ = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{14}$$

The following well known expansion [21, Theorem 7.19.7]

$$s_\lambda(\mathbf{x}) \ = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{n, \mathrm{Des}(Q)}(\mathbf{x}) \tag{15}$$

of the Schur function $s_\lambda(\mathbf{x})$ associated to $\lambda \vdash n$ will be used in Section 4.

For the applications given there, we need the analogues of these concepts in the representation theory of the hyperoctahedral group of signed permutations of the set $[n]$, denoted here by $\mathcal{B}_n$. We will keep this discussion rather brief and refer to [1, Section 2] for more information.

A *bipartition* of a positive integer n, written $(\lambda, \mu) \vdash n$, is any pair (λ, μ) of integer partitions of total sum n. A *standard Young bitableaux* of shape $(\lambda, \mu) \vdash n$ is any pair $Q = (Q^+, Q^-)$ of Young tableaux such that Q^+ has shape λ, Q^- has shape μ and every element of $[n]$ appears exactly once as an enrty of Q^+ or Q^-. The tableaux Q^+ and Q^- are called the *parts* of Q and the number n is its *size*. The Robinson–Schensted correspondence of type B, as described in [17, Section 6] (see also [1, Section 5]) is a bijection from the group $\mathcal{B}_n$ to the set of pairs $(\mathcal{P}, Q)$ of standard Young bitableaux of the same shape and size n.

The characteristic map for the hyperoctahedral group, denoted by $\mathrm{ch}_\mathcal{B}$, is a $\mathbb{C}$-linear isomorphism from the space of virtual $\mathcal{B}_n$-representations to that of homogeneous symmetric functions of degree n in the sets of indeterminates $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ separately; see, for instance, [22, Section 5]. The following basic properties of $\mathrm{ch}_\mathcal{B}$ will be useful in Section 4:

- $\mathrm{ch}_\mathcal{B}(1_{\mathcal{B}_n}) = h_n(\mathbf{x})$, where $1_{\mathcal{B}_n}$ is the trivial $\mathcal{B}_n$-representation,
- $\mathrm{ch}_\mathcal{B}(\sigma \otimes \tau \uparrow_{\mathcal{B}_k \times \mathcal{B}_{n-k}}^{\mathcal{B}_n}) = \mathrm{ch}_\mathcal{B}(\sigma) \cdot \mathrm{ch}_\mathcal{B}(\tau)$ for all representations σ and τ of $\mathcal{B}_k$ and $\mathcal{B}_{n-k}$, respectively, where $\mathcal{B}_k \times \mathcal{B}_{n-k}$ is a Young subgroup of $\mathcal{B}_n$,

- $\mathrm{ch}_{\mathcal{B}}(\uparrow_{\mathfrak{S}_n}^{\mathcal{B}_n} \rho) = \mathrm{ch}(\rho)(\mathbf{x}, \mathbf{y})$ for every $\mathfrak{S}_n$-representation ρ, where $\mathfrak{S}_n \subset \mathcal{B}_n$ is the standard embedding.

We denote by $E(\mathbf{x}, \mathbf{y}; z) := \sum_{n \geq 0} e_n(\mathbf{x}, \mathbf{y}) z^n$ and $H(\mathbf{x}, \mathbf{y}; z) := \sum_{n \geq 0} h_n(\mathbf{x}, \mathbf{y}) z^n$ the generating function for the elementary and complete homogeneous, respectively, symmetric functions in the variables $(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \dots, y_1, y_2, \dots)$ and note that $E(\mathbf{x}, \mathbf{y}; z) = E(\mathbf{x}; z) E(\mathbf{y}; z)$, since $e_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} e_k(\mathbf{x}) e_{n-k}(\mathbf{y})$, and similarly that $H(\mathbf{x}, \mathbf{y}; z) = H(\mathbf{x}; z) H(\mathbf{y}; z)$.

The *signed descent set* [1, Section 2] [10] of $w \in \mathcal{B}_n$, denoted $\mathrm{sDes}(w)$, records the positions of the increasing (in absolute value) runs of constant sign in the sequence $(w(1), w(2), \dots, w(n))$. Formally, we may define $\mathrm{sDes}(w)$ as the pair $(\mathrm{Des}(w), \varepsilon)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{-, +\}^n$ is the sign vector with ith coordinate equal to the sign of $w(i)$ and $\mathrm{Des}(w)$ consists of the indices $i \in [n-1]$ for which either $\varepsilon_i = +$ and $\varepsilon_{i+1} = -$, or else $\varepsilon_i = \varepsilon_{i+1}$ and $|w(i)| > |w(i+1)|$ (this definition is slightly different from, but equivalent to, the ones given in [1, 10]). The fundamental quasisymmetric function associated to w, introduced by Poirier [10] in a more general setting, is defined as

$$F_{\mathrm{sDes}(w)}(\mathbf{x}, \mathbf{y}) \ = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in \mathrm{Des}(w) \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}, \tag{16}$$

where $z_{i_j} = x_{i_j}$ if $\varepsilon_j = +$, and $z_{i_j} = y_{i_j}$ if $\varepsilon_j = -$. Given a standard Young bitableau Q of size n, one defines the signed descent set $\mathrm{sDes}(Q)$ as the pair $(\mathrm{Des}(Q), \varepsilon)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{-, +\}^n$ is the sign vector with ith coordinate equal to the sign of the part of Q in which i appears and $\mathrm{Des}(Q)$ is the set of indices $i \in [n-1]$ for which either $\varepsilon_i = +$ and $\varepsilon_{i+1} = -$, or else $\varepsilon_i = \varepsilon_{i+1}$ and $i+1$ appears in Q in a lower row than i. The function $F_{s\mathrm{Des}(Q)}(\mathbf{x}, \mathbf{y})$ is then defined by the right-hand side of Equation (16), with w replaced by Q; see [1, Section 2]. The analogue

$$s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) \ = \sum_{Q \in \mathrm{SYT}(\lambda, \mu)} F_{s\mathrm{Des}(Q)}(\mathbf{x}, \mathbf{y}) \tag{17}$$

of the expansion (15) holds ([1, Proposition 4.2]) and the Robinson–Schensted correspondence of type B has the properties that $\mathrm{sDes}(w) = \mathrm{sDes}(Q^B(w))$ and $\mathrm{sDes}(w^{-1}) = \mathrm{sDes}(\mathcal{P}^B(w))$, where $(\mathcal{P}^B(w), Q^B(w))$ is the pair of bitableaux associated to $w \in \mathcal{B}_n$; see [1, Proposition 5.1].

3. Proof of Theorem 1.1

This section proves Theorem 1.1 using only the definition of Rees product and the defining equation (2) of the representations $\beta_P(S)$. For $S = \{s_1, s_2, \dots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \cdots < s_k$ we set

$$\begin{aligned} \varphi_t(S) &:= [s_1 + 1]_t \, [s_2 - s_1 + 1]_t \cdots [s_k - s_{k-1} + 1]_t \\ \psi_t(S) &:= [s_1]_t \, [s_2 - s_1 + 1]_t \cdots [s_k - s_{k-1} + 1]_t, \end{aligned}$$

where $[m]_t := 1 + t + \cdots + t^{m-1}$ for positive integers m.

Lemma 3.1. *Let G be a finite group, P be a finite bounded graded G-poset of rank $n+1$, as in Theorem 1.1, and Q, R be the posets defined by $\bar{Q} = (P^- * T_{t,n})_-$ and $\bar{R} = \bar{P} * T_{t,n-1}$. Then,*

$$\begin{aligned} \alpha_Q(S) &\cong_G \varphi_t(S)\, \alpha_P(S) \\ \alpha_R(S) &\cong_G \psi_t(S)\, \alpha_P(S) \end{aligned}$$

for all positive integers t and $S \subseteq [n]$.

Proof. Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \cdots < s_k$ and let $\rho : T_{t,n} \to \mathbb{N}$ be the rank function of $T_{t,n}$. By the definition of Rees product, the maximal chains in Q_S have the form

$$\hat{0} \prec (p_1, \tau_1) \prec (p_2, \tau_2) \prec \cdots \prec (p_k, \tau_k) \prec \hat{1} \tag{18}$$

where $\hat{0} \prec p_1 \prec p_2 \prec \cdots \prec p_k \prec \hat{1}$ is a maximal chain in P_S and $\tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_k$ is a multichain in $T_{t,n}$ such that

- $0 \le \rho(\tau_1) \le s_1$ and
- $\rho(\tau_j) - \rho(\tau_{j-1}) \le s_j - s_{j-1}$ for $2 \le j \le k$.

Let $m_t(S)$ be the number of these multichains. Since the elements of G act on (18) by fixing the τ_j and acting on the corresponding maximal chain of P_S, we have $\alpha_Q(S) \cong_G m_t(S)\, \alpha_P(S)$. To choose such a multichain $\tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_k$, we need to specify the sequence $i_1 \le i_2 \le \cdots \le i_k$ of ranks of its elements so that $i_j - i_{j-1} \le s_j - s_{j-1}$ for $1 \le j \le k$, where $i_0 := s_0 := 0$, and choose its maximum element τ_k in t^{i_k} possible ways. Summing over all such sequences, we get

$$m_t(S) \;=\; \sum_{(i_1, i_2, \ldots, i_k)} t^{i_k} \;=\; \sum_{0 \le a_j \le s_j - s_{j-1}} t^{a_1 + a_2 + \cdots + a_k} \;=\; \varphi_t(S)$$

and the result for $\alpha_Q(S)$ follows. The same argument applies to $\alpha_R(S)$; one simply has to switch the condition for the rank of τ_1 to $0 \le \rho(\tau_1) \le s_1 - 1$. □

The proof of the following technical lemma will be given after that of Theorem 1.1.

Lemma 3.2. *We have*

$$\sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|}\, \varphi_t(T) \;=\; \begin{cases} 0, & \text{if } [n] \smallsetminus S \text{ is not stable,} \\ t^k (1+t)^{n-2k}, & \text{if } [n] \smallsetminus S \text{ is stable and } n \in S, \\ t^k (1+t)^{n+1-2k}, & \text{if } [n] \smallsetminus S \text{ is stable and } n \notin S \end{cases} \tag{19}$$

and

$$\sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|}\, \psi_t(T) \;=\; \begin{cases} 0, & \text{if } 1 \notin S, \\ 0, & \text{if } [n] \smallsetminus S \text{ is not stable,} \\ t^k (1+t)^{n-1-2k}, & \text{if } [n] \smallsetminus S \text{ is stable and } 1, n \in S, \\ t^k (1+t)^{n-2k}, & \text{if } [n] \smallsetminus S \text{ is stable, } 1 \in S \text{ and } n \notin S \end{cases} \tag{20}$$

for every $S \subseteq [n]$, where $k := n - |S|$.

Proof of Theorem 1.1. Using Equations (2) and (3), as well as Lemma 3.1, we compute that

$$\begin{aligned}\beta_Q([n]) &= \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \alpha_Q(T) \;\cong_G\; \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \, \alpha_P(T) \\ &= \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \sum_{S \subseteq T} \beta_P(S) \\ &= \sum_{S \subseteq [n]} \beta_P(S) \sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T)\end{aligned}$$

and find similarly that

$$\beta_R([n]) \;\cong_G\; \sum_{S \subseteq [n]} \beta_P(S) \sum_{S \subseteq T \subseteq [n]} (-1)^{n-|T|} \, \psi_t(T).$$

The proof follows from these formulas and Lemma 3.2. For the last statement of the theorem one has to note that, as a consequence of [7, Corollary 2], if P is Cohen–Macaulay over $\mathbf{k}$, then so are the Rees products $P^- * T_{t,n}$ and $\bar{P} * T_{t,n-1}$. □

Proof of Lemma 3.2. Let us denote by $\chi_t(S)$ the left-hand side of (19) and write $S = \{s_1, s_2, \ldots, s_k\}$, with $1 \le s_1 < s_2 < \cdots < s_k \le n$. By definition, we have

$$\chi_t(S) \;=\; \chi_t(s_1) \, \chi_t(s_2 - s_1) \cdots \chi_t(s_k - s_{k-1}) \, \omega_t(n - s_k), \tag{21}$$

where

$$\begin{aligned}\chi_t(n) &:= \sum_{n \in T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T) \\ \omega_t(n) &:= \sum_{T \subseteq [n]} (-1)^{n-|T|} \, \varphi_t(T)\end{aligned}$$

for $n \ge 1$ and $\omega_t(0) := 1$. We claim that

$$\chi_t(n) \;=\; \begin{cases} 1+t, & \text{if } n = 1, \\ t, & \text{if } n = 2, \\ 0, & \text{if } n \ge 3 \end{cases} \qquad \text{and} \qquad \omega_t(n) \;=\; \begin{cases} 1, & \text{if } n = 0, \\ t, & \text{if } n = 1, \\ 0, & \text{if } n \ge 2. \end{cases}$$

Equation (19) is a direct consequence of (21) and this claim. To verify the claim, note that the defining equation for $\chi_t(n)$ can be rewritten as

$$\chi_t(n) \;=\; \sum_{(a_1, a_2, \ldots, a_k) \models n} (-1)^{n-k} \, [a_1 + 1]_t \, [a_2 + 1]_t \cdots [a_k + 1]_t,$$

where the sum ranges over all sequences (compositions) $(a_1, a_2, \ldots, a_k)$ of positive integers summing to n. We leave it as a simple combinatorial excercise for the interested reader to show (for instance, by standard generating function methods) that $\chi_t(n) = 0$ for $n \geq 3$. The claim follows from this fact and the obvious recurrence $\omega_t(n) = \chi_t(n) - \omega_t(n-1)$, valid for $n \geq 1$.

Finally, note that Equation (20) is equivalent to (19) in the case $1 \in S$. Otherwise, the terms in the left-hand side can be partitioned into pairs of terms, corresponding to pairs $\{T, T \cup \{1\}\}$ of subsets with $1 \notin T$, cancelling each other. This shows that the left-hand side vanishes. $\square$

4. Symmetric function identities

This section derives Equations (6) – (11) from Theorem 1.1 (Corollaries 4.1, 4.4 and 4.7) and interprets combinatorially the Schur-positive symmetric functions which appear there. We first explain why Gessel's identities are special cases of this theorem. The set of ascents of a permutation $w \in \mathfrak{S}_n$ is defined as $\mathrm{Asc}(w) := [n-1] \smallsetminus \mathrm{Des}(w)$ and, similarly, we have $\mathrm{Asc}(\mathcal{P}) := [n-1] \smallsetminus \mathrm{Des}(\mathcal{P})$ for every standard Young tableau $\mathcal{P}$ of size n.

Corollary 4.1. *Equations (6) and (7) are valid for the functions*

$$\xi_{n,k}(\mathbf{x}) \ = \ \sum_{\lambda \vdash n} c_{\lambda,k} \cdot s_\lambda(\mathbf{x}) \ = \ \sum_{w} F_{n,\mathrm{Des}(w)}(\mathbf{x}) \tag{22}$$

and

$$\gamma_{n,k}(\mathbf{x}) \ = \ \sum_{\lambda \vdash n} d_{\lambda,k} \cdot s_\lambda(\mathbf{x}) \ = \ \sum_{w} F_{n,\mathrm{Des}(w)}(\mathbf{x}), \tag{23}$$

where $c_{\lambda,k}$ (respectively, $d_{\lambda,k}$) stands for the number of tableaux $\mathcal{P} \in \mathrm{SYT}(\lambda)$ for which $\mathrm{Asc}(\mathcal{P}) \in \mathrm{Stab}([2, n-2])$ (respectively, $\mathrm{Asc}(\mathcal{P}) \in \mathrm{Stab}([n-2])$) has k elements and, similarly, $w \in \mathfrak{S}_n$ runs through all permutations for which $\mathrm{Asc}(w^{-1}) \in \mathrm{Stab}([2, n-2])$ (respectively, $\mathrm{Asc}(w^{-1}) \in \mathrm{Stab}([n-2])$) has k elements.

Proof. We will apply Theorem 1.1 when P^- is the Boolean lattice B_n of subsets of $[n]$, partially ordered by inclusion, considered as an $\mathfrak{S}_n$-poset. On the one hand, we have the equality

$$1 \, + \sum_{n \geq 2} \mathrm{ch}\left(\widetilde{H}_{n-1}((B_n \smallsetminus \{\varnothing\}) * T_{t,n-1}; \mathbb{C}) \right) z^n \ = \ \frac{1-t}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)}$$

which, although not explicitly stated in [13], follows as in the proof of its special case $t = 1$ [13, Corollary 5.2]. On the other hand, since B_n has a maximum element, the second summand in the right-hand side of Equation (5) vanishes and hence this equation gives

$$\mathrm{ch}\left(\widetilde{H}_{n-1}((B_n\smallsetminus\{\varnothing\})*T_{t,n-1};\mathbb{C})\right) \;=\; \sum_{S\in\mathrm{Stab}([2,n-2])} \mathrm{ch}\left(\beta_{B_n}([n-1]\smallsetminus S)\right)\, t^{|S|+1}\,(1+t)^{n-2-2|S|}$$

for $n \geq 2$. The representations $\beta_{B_n}(S)$ for $S \subseteq [n-1]$ are known to satisfy (see, for instance, [17, Theorem 4.3])

$$\mathrm{ch}\left(\beta_{B_n}(S)\right) \;=\; \sum_{\lambda\vdash n} c_{\lambda,S}\cdot s_\lambda(\mathbf{x}),$$

where $c_{\lambda,S}$ is the number of standard Young tableaux of shape λ and descent set equal to S. Combining the previous three equalities yields the first equality in Equation (22). The second equality follows from the first by expanding $s_\lambda(\mathbf{x})$ according to Equation (15) to get

$$\xi_{n,k}(\mathbf{x}) \;=\; \sum_{\lambda\vdash n}\sum_{\mathcal{P}}\sum_{Q\in\mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(Q)}(\mathbf{x})$$

where, in the inner sum, $\mathcal{P}$ runs through all tableaux in $\mathrm{SYT}(\lambda)$ for which $\mathrm{Asc}(\mathcal{P}) \in \mathcal{P}_{\mathrm{Stab}}([2,n-2])$ has k elements, and then using the Robinson–Schensted correspondence and its standard properties $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$ and $\mathrm{Des}(w^{-1}) = \mathrm{Des}(\mathcal{P}(w))$ to replace the summations with one running over elements of $\mathfrak{S}_n$, as in the statement of the corollary.

The proof of (23) is entirely similar; one has to use Equation (4) instead of (5), as well as the equality

$$1 + \sum_{n\geq 1} \mathrm{ch}\left(\widetilde{H}_{n-1}((B_n * T_{t,n})_-;\mathbb{C})\right) z^n \;=\; \frac{(1-t)E(\mathbf{x};tz)}{E(\mathbf{x};tz)-tE(\mathbf{x};z)}.$$

The latter follows from the proof of Equation (3.3) in [13, pp. 15–16], where the left-hand side is equal to $-F_t(-z)$, in the notation used in that proof. □

Example 4.2. The coefficient of z^4 in the left-hand sides of Equations (6) and (7) equals

- $e_4(\mathbf{x})\,(t+t^2+t^3) + e_2(\mathbf{x})^2\,t^2$, and
- $e_4(\mathbf{x})\,(t+t^2+t^3+t^4) + e_1(\mathbf{x})e_3(\mathbf{x})\,(t^2+t^3) + e_2(\mathbf{x})^2\,(t^2+t^3)$,

respectively. These expressions may be rewritten as

- $s_{(1,1,1,1)}(\mathbf{x})\,t(1+t)^2 + s_{(2,1,1)}(\mathbf{x})\,t^2 + s_{(2,2)}(\mathbf{x})\,t^2$, and
- $s_{(1,1,1,1)}(\mathbf{x})\,t(1+t)^3 + 2s_{(2,1,1)}(\mathbf{x})\,t^2(1+t) + s_{(2,2)}(\mathbf{x})\,t^2(1+t)$,

respectively, and hence $\xi_{4,0}(\mathbf{x}) = s_{(1,1,1,1)}(\mathbf{x})$, $\xi_{4,1}(\mathbf{x}) = s_{(2,1,1)}(\mathbf{x}) + s_{(2,2)}(\mathbf{x})$, $\gamma_{4,0}(\mathbf{x}) = s_{(1,1,1,1)}(\mathbf{x})$ and $\gamma_{4,1}(\mathbf{x}) = 2s_{(2,1,1)}(\mathbf{x}) + s_{(2,2)}(\mathbf{x})$. We leave it to the reader to verify that these formulas agree with Corollary 4.1. □

We now focus on the identities (8) – (11). We will apply Theorem 1.1 to the collection sB_n of all subsets of $\{1, 2, \ldots, n\} \cup \{-1, -2, \ldots, -n\}$ which do not contain $\{i, -i\}$ for any index i, partially ordered by inclusion. This signed analogue of the Boolean algebra B_n is a graded poset of rank n, having the empty set as its minimum element, on which the hyperoctahedral group $\mathcal{B}_n$ acts in the obvious way [17, Section 6], turning it into a $\mathcal{B}_n$-poset. It is isomorphic to the poset of faces (including the empty one) of the boundary complex of the n-dimensional cross-polytope and hence it is Cohen–Macaulay over $\mathbb{Z}$ and any field. Our computations of the left-hand sides of Equations (4) and (5) for $P^- = sB_n$ follow closely the methods of [13].

Consider the n-element chain $C_n = \{0, 1, \ldots, n-1\}$, with the usual total order. Following [13], we denote by $I_j(B_n)$ the order ideal of elements of $(B_n \smallsetminus \{\varnothing\}) * C_n$ which are strictly less than $([n], j)$. Then $I_j(B_n)$ is an $\mathfrak{S}_n$-poset for every $j \in C_n$ and one of the main results of [13] (see [13, p. 21] [15, Equation (2.5)]) states that

$$1 + \sum_{n \geq 1} z^n \sum_{j=0}^{n-1} t^j \, \mathrm{ch}\left(\widetilde{H}_{n-2}(I_j(B_n); \mathbb{C})\right) = \frac{(1-t)E(\mathbf{x}; z)}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)}. \tag{24}$$

Proposition 4.3. *For the $\mathcal{B}_n$-poset sB_n we have*

$$1 + \sum_{n \geq 1} \mathrm{ch}_{\mathcal{B}}\left(\widetilde{H}_{n-1}((sB_n \smallsetminus \{\varnothing\}) * T_{t,n-1}; \mathbb{C})\right) z^n = \frac{(1-t)E(\mathbf{y}; z)}{E(\mathbf{x}; tz)E(\mathbf{y}; tz) - tE(\mathbf{x}; z)E(\mathbf{y}; z)}. \tag{25}$$

Proof. Following the reasoning in the proof of [13, Corollary 5.2], we apply (12) to the Cohen–Macaulay $\mathcal{B}_n$-poset obtained from $(sB_n \smallsetminus \{\varnothing\}) * T_{t,n-1}$ by adding a minimum and a maximum element. For $0 \leq j < k \leq n$, there are exactly t^j $\mathcal{B}_n$-orbits of elements x of rank k in this poset with second coordinate equal to j and for each one of these, the open interval $(\hat{0}, x)$ is isomorphic to $I_j(B_k)$ and the stabilizer of x is isomorphic to $\mathfrak{S}_k \times \mathcal{B}_{n-k}$. We conclude that

$$\widetilde{H}_{n-1}((sB_n \smallsetminus \{\varnothing\}) * T_{t,n-1}; \mathbb{C}) \cong_{\mathcal{B}_n} \bigoplus_{k=0}^{n} (-1)^{n-k} \bigoplus_{j=0}^{k-1} t^j \left(\widetilde{H}_{k-2}(I_j(B_k); \mathbb{C}) \otimes 1_{\mathcal{B}_{n-k}}\right) \uparrow_{\mathfrak{S}_k \times \mathcal{B}_{n-k}}^{\mathcal{B}_n}.$$

Applying the map $\mathrm{ch}_{\mathcal{B}}$ and using the transitivity $\uparrow_{\mathcal{B}_{n-k} \times \mathfrak{S}_k}^{\mathcal{B}_n} \cong_{\mathcal{B}_n} \uparrow_{\mathcal{B}_{n-k} \times \mathfrak{S}_k}^{\mathcal{B}_{n-k} \times \mathcal{B}_k} \uparrow_{\mathcal{B}_{n-k} \times \mathcal{B}_k}^{\mathcal{B}_n}$ of induction and properties of $\mathrm{ch}_{\mathcal{B}}$ discussed in Section 2, the right-hand side becomes

$$\sum_{k=0}^{n} (-1)^{n-k} \sum_{j=0}^{k-1} t^j \, \mathrm{ch}\left(\widetilde{H}_{k-2}(I_j(B_k); \mathbb{C})\right)(\mathbf{x}, \mathbf{y}) \cdot h_{n-k}(\mathbf{x}).$$

Thus, the left-hand side of Equation (25) is equal to

$$H(\mathbf{x};-z)\cdot\left(1+\sum_{n\geq 1} z^n \sum_{j=0}^{n-1} t^j \,\mathrm{ch}\left(\widetilde{H}_{n-2}(I_j(B_n);\mathbb{C})\right)(\mathbf{x},\mathbf{y})\right)$$

and the result follows from Equation (24) and the identities $E(\mathbf{x};z)H(\mathbf{x};-z)=1$ and $E(\mathbf{x},\mathbf{y};z)=E(\mathbf{x};z)E(\mathbf{y};z)$. □

Recall the definition of the sets $\mathrm{Des}(w)$ and $\mathrm{Des}(\mathcal{P})$ for signed permutations $w\in\mathcal{B}_n$ and standard Young bitableau $\mathcal{P}$ of size n, respectively, from Section 2. Following [17, Section 6], we define the *type B descent set* of $\mathcal{P}=(\mathcal{P}^+,\mathcal{P}^-)$ as $\mathrm{Des}_B(\mathcal{P})=\mathrm{Des}(\mathcal{P})\cup\{n\}$, if n appears in $\mathcal{P}^+$, and $\mathrm{Des}_B(\mathcal{P})=\mathrm{Des}(\mathcal{P})$ otherwise. The complement of $\mathrm{Des}_B(\mathcal{P})$ in the set $[n]$ is called the *type B ascent set* of $\mathcal{P}$ and is denoted by $\mathrm{Asc}_B(\mathcal{P})$. Similarly, we define the *type B descent set* of $w\in\mathcal{B}_n$ as $\mathrm{Des}_B(w)=\mathrm{Des}(w)\cup\{n\}$, if $w(n)$ is positive, and $\mathrm{Des}_B(w)=\mathrm{Des}(w)$ otherwise. The complement of $\mathrm{Des}_B(w)$ in the set $[n]$ is called the *type B ascent set* of w and is denoted by $\mathrm{Asc}_B(w)$. The sets $\mathrm{Des}_B(w)$ and $\mathrm{Des}_B(\mathcal{P})$ depend only on the signed descent sets $\mathrm{sDes}_B(w)$ and $\mathrm{sDes}_B(\mathcal{P})$, respectively, and [1, Proposition 5.1], mentioned at the end of Section 2, implies that $\mathrm{Des}_B(w)=\mathrm{Des}_B(Q^B(w))$ and $\mathrm{Des}_B(w^{-1})=\mathrm{Des}_B(\mathcal{P}^B(w))$ for every $w\in\mathcal{B}_n$.

Corollary 4.4. *Equations (8) and (9) are valid for the functions*

$$\xi^+_{n,k}(\mathbf{x},\mathbf{y}) \;=\; \sum_{(\lambda,\mu)\vdash n} c^+_{(\lambda,\mu),k}\cdot s_\lambda(\mathbf{x})s_\mu(\mathbf{y}) \;=\; \sum_w F_{\mathrm{sDes}(w)}(\mathbf{x},\mathbf{y}) \tag{26}$$

and

$$\xi^-_{n,k}(\mathbf{x},\mathbf{y}) \;=\; \sum_{(\lambda,\mu)\vdash n} c^-_{(\lambda,\mu),k}\cdot s_\lambda(\mathbf{x})s_\mu(\mathbf{y}) \;=\; \sum_w F_{\mathrm{sDes}(w)}(\mathbf{x},\mathbf{y}), \tag{27}$$

where $c^+_{(\lambda,\mu),k}$ (respectively, $c^-_{(\lambda,\mu),k}$) stands for the number of bitableaux $\mathcal{P}\in\mathrm{SYT}(\lambda,\mu)$ for which $\mathrm{Asc}_B(\mathcal{P})\in\mathrm{Stab}([2,n])$ has k elements and contains (respectively, does not contain) n and where, similarly, $w\in\mathcal{B}_n$ runs through all signed permutations for which $\mathrm{Asc}_B(w^{-1})\in\mathrm{Stab}([2,n])$ has k elements and contains (respectively, does not contain) n.

Proof. We apply the second part of Theorem 1.1 for $P^-=sB_n$, thought of as a $\mathcal{B}_n$-poset. The representations $\beta_P(S)$ for $S\subseteq[n]$ were computed in this case in [17, Theorem 6.4], which implies that

$$\mathrm{ch}_{\mathcal{B}}\left(\beta_{B_n}(S)\right) \;=\; \sum_{(\lambda,\mu)\vdash n} c_{(\lambda,\mu),S}\cdot s_\lambda(\mathbf{y})s_\mu(\mathbf{x})$$

for $S\subseteq[n]$, where $c_{(\lambda,\mu),S}$ is the number of standard Young bitableaux $\mathcal{P}$ of shape (λ,μ) such that $\mathrm{Des}_B(\mathcal{P})=S$. Switching the roles of $\mathbf{x}$ and $\mathbf{y}$ and combining this result with the second part of Theorem 1.1 and Proposition 4.3 we get

$$\frac{(1-t)E(\mathbf{x};z)}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)} = \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \xi^+_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-2k} \ +$$

$$\sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \xi^-_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-1-2k}, \tag{28}$$

where the $\xi^{\pm}_{n,k}(\mathbf{x},\mathbf{y})$ are given by the first equalities in (26) and (27). We now note that the left-hand side of Equation (28) is equal to the sum of the left-hand sides, say $\Xi^+(\mathbf{x},\mathbf{y},t;z)$ and $\Xi^-(\mathbf{x},\mathbf{y},t;z)$, of Equations (8) and (9). Since, as one can readily verify, $\Xi^+(\mathbf{x},\mathbf{y},t;z)$ is left invariant under replacing t with $1/t$ and z with tz, while $\Xi^-(\mathbf{x},\mathbf{y},t;z)$ is multiplied by t after these substitutions, the coefficient of z^n in $\Xi^+(\mathbf{x},\mathbf{y},t;z)$ (respectively, $\Xi^-(\mathbf{x},\mathbf{y},t;z)$) is a symmetric polynomial in t with center of symmetry $n/2$ (respectively, $(n-1)/2$) for every $n \in \mathbb{N}$. Since the corresponding properties are clear for the coefficient of z^n in the two summands in the right-hand side of Equation (28) and because of the uniqueness of the decomposition of a polynomial $f(t)$ as a sum of two symmetric polynomials with centers of symmetry $n/2$ and $(n-1)/2$ (see [5, Section 5.1]), we conclude that (26) and (27) follow from the single equation (28).

The second equalities in (26) and (27) follow by expanding $s_\lambda(\mathbf{x})s_\mu(\mathbf{y})$ according to Equation (17) and then using the Robinson–Schensted correspondence of type B and its properties $\mathrm{sDes}(w) = \mathrm{sDes}(Q^B(w))$ and $\mathrm{Des}_B(w^{-1}) = \mathrm{Des}_B(\mathcal{P}^B(w))$, exactly as in the proof of Corollary 4.1. □

Example 4.5. The coefficient of z^2 in the left-hand side of Equations (8) and (9) equals

- $e_1(\mathbf{x})e_1(\mathbf{y})\,t + e_2(\mathbf{y})\,t = s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})\,t + s_{(1,1)}(\mathbf{y})\,t$, and
- $e_2(\mathbf{x})\,(1+t) = s_{(1,1)}(\mathbf{x})\,(1+t)$,

respectively and hence $\xi^+_{2,0}(\mathbf{x},\mathbf{y}) = 0$, $\xi^+_{2,1}(\mathbf{x},\mathbf{y}) = s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y}) + s_{(1,1)}(\mathbf{y})$ and $\xi^-_{2,0}(\mathbf{x},\mathbf{y}) = s_{(1,1)}(\mathbf{x})$, in agreement with Corollary 4.4. □

Proposition 4.6. *For the $\mathcal{B}_n$-poset sB_n we have*

$$1 + \sum_{n\geq 1} \mathrm{ch}_{\mathcal{B}}\left(\widetilde{H}_{n-1}((sB_n * T_{t,n})_-;\mathbb{C})\right) z^n = \frac{(1-t)E(\mathbf{y};z)E(\mathbf{x};tz)E(\mathbf{y};tz)}{E(\mathbf{x};tz)E(\mathbf{y};tz)-tE(\mathbf{x};z)E(\mathbf{y};z)}. \tag{29}$$

Proof. Following the reasoning in the proof of [13, Equation (3.3)], we set

$$L_n(\mathbf{x},\mathbf{y};t) := \mathrm{ch}_{\mathcal{B}}\left(L((sB_n * T_{t,n})_-;\mathcal{B}_n)\right),$$

where $L(P;G)$ denotes the Lefschetz character of the G-poset P over $\mathbb{C}$ (see Section 2). Since $(sB_n * T_{t,n})_-$ is Cohen–Macaulay over $\mathbb{C}$ of rank $n-1$, we have

$$\mathrm{ch}_{\mathcal{B}}\left(\widetilde{H}_{n-1}((sB_n * T_{t,n})_-;\mathbb{C})\right) \;=\; (-1)^{n-1}\,L_n(\mathbf{x},\mathbf{y};t).$$

Thus, the left-hand side of (29) is equal to $-\sum_{n\geq 0} L_n(\mathbf{x},\mathbf{y};t)(-z)^n$. The sequence of posets $(sB_0, sB_1, \ldots, sB_n)$ can easily be verified to be $(\mathcal{B}_0\times\mathfrak{S}_n, \mathcal{B}_1\times\mathfrak{S}_{n-1},\ldots,\mathcal{B}_n\times\mathfrak{S}_0)$-uniform (see Section 2). Moreover, there is a single $\mathcal{B}_n$-orbit of elements of sB_n of rank k for each $k\in\{0,1,\ldots,n\}$. Thus, applying (13) to this sequence gives

$$1_{\mathcal{B}_n}\oplus\bigoplus_{k=0}^{n}[k+1]_t\,L((sB_{n-k}*T_{t,n-k})_-;\mathcal{B}_{n-k}\times\mathfrak{S}_k)\uparrow_{\mathcal{B}_{n-k}\times\mathfrak{S}_k}^{\mathcal{B}_n}\;\cong_{\mathcal{B}_n}\;0.$$

Applying the characteristic map $\mathrm{ch}_{\mathcal{B}}$, as in the proof of Proposition 4.3, gives

$$\sum_{k=0}^{n}[k+1]_t\,h_k(\mathbf{x},\mathbf{y})\,L_{n-k}(\mathbf{x},\mathbf{y};t)\;=\;-h_n(\mathbf{x}).$$

Standard manipulation with generating functions, as in the proof of [13, Equation (3.3)], results in the formula

$$\sum_{n\geq 0} L_n(\mathbf{x},\mathbf{y};t)z^n \;=\; -\,\frac{H(\mathbf{x};z)}{\sum_{n\geq 0}[n+1]_t\,h_n(\mathbf{x},\mathbf{y})z^n}\;=\;-\,\frac{(1-t)H(\mathbf{x};z)}{H(\mathbf{x},\mathbf{y};z)-tH(\mathbf{x},\mathbf{y};tz)}.$$

The proof now follows by switching z to $-z$ and using the identities $E(\mathbf{x};z)H(\mathbf{x};-z)=1$ and $E(\mathbf{x},\mathbf{y};z)=E(\mathbf{x};z)E(\mathbf{y};z)$. □

Corollary 4.7. *Equations (10) and (11) are valid for the functions*

$$\gamma^+_{n,k}(\mathbf{x},\mathbf{y})\;=\;\sum_{(\lambda,\mu)\vdash n} d^+_{(\lambda,\mu),k}\cdot s_\lambda(\mathbf{x})s_\mu(\mathbf{y})\;=\;\sum_w F_{\mathrm{sDes}(w)}(\mathbf{x},\mathbf{y}) \tag{30}$$

and

$$\gamma^-_{n,k}(\mathbf{x},\mathbf{y})\;=\;\sum_{(\lambda,\mu)\vdash n} d^-_{(\lambda,\mu),k}\cdot s_\lambda(\mathbf{x})s_\mu(\mathbf{y})\;=\;\sum_w F_{\mathrm{sDes}(w)}(\mathbf{x},\mathbf{y}), \tag{31}$$

where $d^+_{(\lambda,\mu),k}$ (respectively, $d^-_{(\lambda,\mu),k}$) is the number of bitableaux $\mathcal{P}\in\mathrm{SYT}(\lambda,\mu)$ for which $\mathrm{Asc}_B(\mathcal{P})\in\mathrm{Stab}([n])$ has k elements and does not contain (respectively, contains) n and, similarly, $w\in\mathcal{B}_n$ runs through all signed permutations for which $\mathrm{Asc}_B(w^{-1})\in\mathrm{Stab}([n])$ has k elements and does not contain (respectively, contains) n.

Proof. This statement follows by the same reasoning as in the proof of Corollary 4.4, provided one appeals to the first part of Theorem 1.1 and Proposition 4.6 instead. □

Example 4.8. The coefficient of z^2 in the left-hand side of Equations (10) and (11) equals

- $e_2(\mathbf{x})\,(1+t+t^2)+e_1(\mathbf{x})^2\,t+e_1(\mathbf{x})e_1(\mathbf{y})\,t = s_{(1,1)}(\mathbf{x})\,(1+t)^2+s_{(2)}(\mathbf{x})\,t+s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})\,t$,
- $e_1(\mathbf{x})e_1(\mathbf{y})\,(t+t^2)+e_2(\mathbf{y})\,(t+t^2) = s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})\,t(1+t)+s_{(1,1)}(\mathbf{y})\,t(1+t)$,

respectively and hence we have $\gamma^+_{2,0}(\mathbf{x},\mathbf{y}) = s_{(1,1)}(\mathbf{x})$, $\gamma^+_{2,1}(\mathbf{x},\mathbf{y}) = s_{(2)}(\mathbf{x}) + s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y})$ and $\gamma^-_{2,1}(\mathbf{x},\mathbf{y}) = s_{(1)}(\mathbf{x})s_{(1)}(\mathbf{y}) + s_{(2)}(\mathbf{y})$, in agreement with Corollary 4.7. □

5. An instance of the local equivariant Gal phenomenon

This section uses Equation (8) to verify an equivariant analogue of Gal's conjecture [8] for the local face module of a certain triangulation of the simplex with interesting combinatorial properties. Background and any undefined terminology on simplicial complexes can be found in [19].

To explain the setup, let $V_n = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ be the set of unit coordinate vectors in $\mathbb{R}^n$ and Σ_n be the geometric simplex on the vertex set V_n. Consider a triangulation Γ of Σ_n (meaning, a geometric simplicial complex which subdivides Σ_n) with vertex set V_Γ and the polynomial ring $S = \mathbb{C}[x_v : v \in V_\Gamma]$ in indeterminates which are in one-to-one correspondence with the vertices of Γ. The *face ring* [19, Chapter II] of Γ is defined as the quotient ring $\mathbb{C}[\Gamma] = S/I_\Gamma$, where I_Γ is the ideal of S generated by the square-free monomials which correspond to the non-faces of Γ. Following [18, p. 824], we consider the linear forms

$$\theta_i \;=\; \sum_{v \in V_\Gamma} \langle v, \varepsilon_i \rangle x_v \tag{32}$$

for $i \in [n]$, where $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^n$, and denote by Θ the ideal in $\mathbb{C}[\Gamma]$ generated by $\theta_1, \theta_2, \dots, \theta_n$. This sequence is a special linear system of parameters for $\mathbb{C}[\Gamma]$, in the sense of [18, Definition 4.2]. As a result, the quotient ring $\mathbb{C}(\Gamma) = \mathbb{C}[\Gamma]/\Theta$ is a finite dimensional, graded $\mathbb{C}$-vector space and so is the *local face module* $L_{V_n}(\Gamma)$, defined [18, Definition 4.5] as the image in $\mathbb{C}(\Gamma)$ of the ideal of $\mathbb{C}[\Gamma]$ generated by the square-free monomials which correspond to the faces of Γ lying in the interior of Σ_n. The Hilbert polynomials $\sum_{i=0}^n \dim_{\mathbb{C}}(\mathbb{C}(\Gamma))_i t^i$ and $\sum_{i=0}^n \dim_{\mathbb{C}}(L_{V_n}(\Gamma))_i t^i$ of $\mathbb{C}(\Gamma)$ and $L_{V_n}(\Gamma)$ are two important enumerative invariants of Γ, namely the h-polynomial [19, Section II.2] and the local h-polynomial [18, Section 2] [19, Section III.10], respectively.

Suppose that G is a subgroup of the automorphism group $\mathfrak{S}_n$ of Σ_n which acts simplicially on Γ. Then, G acts on the polynomial ring S and (as discussed on [23, p. 250]) leaves the $\mathbb{C}$-linear span of $\theta_1, \theta_2, \dots, \theta_n$ invariant. Therefore, G acts on the graded $\mathbb{C}$-vector spaces $\mathbb{C}(\Gamma)$ and $L_{V_n}(\Gamma)$ as well and the polynomials $\sum_{i=0}^n (\mathbb{C}(\Gamma))_i t^i$ and $\sum_{i=0}^n (L_{V_n}(\Gamma))_i t^i$, whose coefficients lie in the representation ring of G, are equivariant generalizations of the h-polynomial and local h-polynomial of Γ, respectively. The pair (Γ, G) is said (see also [5, Section 5.2]) to satisfy the *local equivariant Gal phenomenon* if

$$\sum_{i=0}^{n} (L_{V_n}(\Gamma))_i \, t^i \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} M_k \, t^k (1+t)^{n-2k} \tag{33}$$

for some non-virtual G-representations M_k. This is an analogue for local face modules of the equivariant Gal phenomenon, formulated by Shareshian and Wachs [16, Section 5] for group actions on (flag) triangulations of spheres as an equivariant version of Gal's conjecture [8, Conjecture 2.1.7]. For trivial actions on flag triangulations of simplices, the validity of the local equivariant Gal phenomenon was conjectured in [2] and has been verified in many special cases; see [4, Section 4] [5, Section 3.2] and references therein.

Although it would be too optimistic to expect that the local equivariant Gal phenomenon holds for all group actions on flag triangulations of Σ_n, the case $G = \mathfrak{S}_n$ deserves special attention. We then use the notation

$$\begin{aligned} \mathrm{ch}\,(\mathbb{C}(\Gamma), t) &:= \sum_{i=0}^{n} \mathrm{ch}\,(\mathbb{C}(\Gamma))_i\, t^i, \\ \mathrm{ch}\,(L_{V_n}(\Gamma), t) &:= \sum_{i=0}^{n} \mathrm{ch}\,(L_{V_n}(\Gamma))_i\, t^i. \end{aligned}$$

For the (first) barycentric subdivision of Σ_n we have the following result of Stanley.

Proposition 5.1. ([18, Proposition 4.20]) *For the $\mathfrak{S}_n$-action on the barycentric subdivision Γ_n of the simplex Σ_n, we have*

$$1 + \sum_{n \geq 1} \mathrm{ch}\,(L_{V_n}(\Gamma_n), t)\, z^n \;=\; \frac{1-t}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)}. \tag{34}$$

Combining this result with Gessel's identity (6) gives

$$\mathrm{ch}\,(L_{V_n}(\Gamma_n), t) \;=\; \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \omega\, \xi_{n,k}(\mathbf{x})\, t^{k+1} (1+t)^{n-2k-2},$$

where ω is the standard involution on symmetric functions exchanging $e_\lambda(\mathbf{x})$ and $h_\lambda(\mathbf{x})$ for every λ, whence it follows that $(\Gamma_n, \mathfrak{S}_n)$ satisfies the local equivariant Gal phenomenon for every n.

The combinatorics of the barycentric subdivision Γ_n is related to the symmetric group $\mathfrak{S}_n$. We now consider a triangulation K_n of the simplex Σ_n, studied in [12, Chapter 3] (see also [4, Remark 4.5] [5, Section 3.3]) and shown on the right of Figure 1 for $n = 3$, the combinatorics of which is related to the hyperoctahedral group $\mathcal{B}_n$. The triangulation K_n can be defined as the barycentric subdivision of the standard cubical subdivision of Σ_n, shown on the left of Figure 1 for $n = 3$, whose faces are in inclusion-preserving bijection with the nonempty closed intervals in the truncated Boolean lattice $B_n \smallsetminus \{\varnothing\}$. Thus, the faces of K_n correspond bijectively to chains of nonempty closed intervals in $B_n \smallsetminus \{\varnothing\}$ and $\mathfrak{S}_n$ acts simplicially on K_n in the obvious way.

Proposition 5.2. *For the $\mathfrak{S}_n$-action on K_n we have*

$$1 + \sum_{n \geq 1} \mathrm{ch}\,(\mathbb{C}(K_n), t)\, z^n \;=\; \frac{H(\mathbf{x}; z)\,(H(\mathbf{x}; tz) - tH(\mathbf{x}; z))}{H(\mathbf{x}; tz)^2 - tH(\mathbf{x}; z)^2} \tag{35}$$

and

$$1 + \sum_{n \geq 1} \mathrm{ch}\,(L_{V_n}(K_n), t)\, z^n \;=\; \frac{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)}{H(\mathbf{x}; tz)^2 - tH(\mathbf{x}; z)^2}. \tag{36}$$

Moreover, the pair $(K_n, \mathfrak{S}_n)$ satisfies the local equivariant Gal phenomenon for every n.

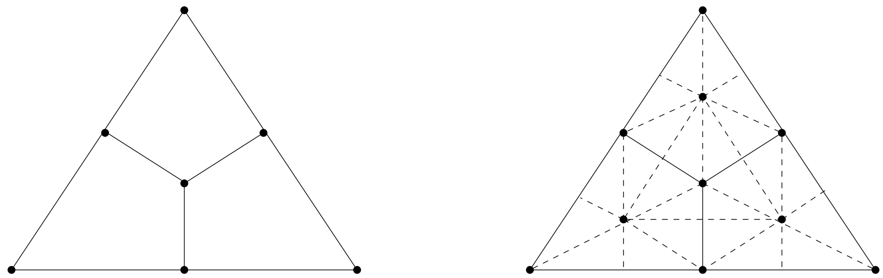

FIGURE 1. The triangulation K_3

The proof relies on methods developed by Stembridge [23] to study representations of Weyl groups on the cohomology of the toric varieties associated to Coxeter complexes. To prepare for it, we recall that the *h-polynomial* of a simplicial complex Δ of dimension $n-1$ is defined as

$$h(\Delta, t) \;=\; \sum_{i=0}^{n} f_{i-1}(\Delta)\, t^i (1-t)^{n-i},$$

where $f_i(\Delta)$ stands for the number of i-dimensional faces of Δ. Consider a pair (Γ, G), consisting of a triangulation Γ of Σ_n and a subgroup G of $\mathfrak{S}_n$ acting on Γ, as discussed earlier. Following [23, Section 1], we call the action of G on Γ *proper* if w fixes all vertices of every face $F \in \Delta$ which is fixed by w, for every $w \in G$. Note that group actions, such as the $\mathfrak{S}_n$-actions on Γ_n and K_n, on the order complex (simplicial complex of chains) of a poset which are induced by an action on the poset itself, are always proper. Under this assumption, the set Γ^w of faces of Γ which are fixed by w forms an induced subcomplex of Γ, for every $w \in G$.

Although Stembridge [23] deals with triangulations of spheres, rather than simplices, his methods apply to our setting and his Theorem 1.4, combined with the considerations of Section 6 in [23], imply that

$$\mathrm{ch}\,(\mathbb{C}(\Gamma), t) \;=\; \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \frac{h(\Gamma^w, t)}{(1-t)^{1+\dim(\Gamma^w)}} \prod_{i \ge 1} (1 - t^{\lambda_i(w)})\, p_{\lambda_i(w)}(\mathbf{x}) \tag{37}$$

for every proper $\mathfrak{S}_n$-action on Γ, where $\lambda_1(w) \ge \lambda_2(w) \ge \cdots$ are the sizes of the cycles of $w \in \mathfrak{S}_n$ and $p_k(\mathbf{x})$ is a power sum symmetric function.

Proof of Proposition 5.2. To prove Equation (35), we follow the analogous computation in [23, Section 6] for the barycentric subdivision of the boundary complex of the simplex. We first note that $(K_n)^w$ is combinatorially isomorphic to $K_{c(w)}$ for every $w \in \mathfrak{S}_n$, where $c(w)$ is the number of cycles of w. Furthermore, it was shown in [12, Section 3.6] that, in the notation of Section 3, $h(K_n, t)$ is the 'half $\mathcal{B}_n$-Eulerian polynomial'

$$B_n^+(t) \;=\; \sum_{w \in \mathcal{B}_n^+} t^{|\mathrm{Des}_B(w)|}, \tag{38}$$

where $\mathcal{B}_n^+$ consists of the signed permutations $w \in \mathcal{B}_n$ with *negative* first coordinate. These remarks and Equation (37) imply that

$$\mathrm{ch}\,(\mathbb{C}(K_n), t)\, z^n \;=\; \sum_{\lambda=(\lambda_1,\lambda_2,\dots)\vdash n} m_\lambda^{-1} \frac{B^+_{\ell(\lambda)}(t)}{(1-t)^{\ell(\lambda)}} \prod_{i\geq 1} (1-t^{\lambda_i})\, p_{\lambda_i}(\mathbf{x}) z^{\lambda_i},$$

where $n!/m_\lambda$ is the cardinality of the congugacy class of $\mathfrak{S}_n$ which corresponds to $\lambda \vdash n$ and $\ell(\lambda)$ is the number of parts of λ. The polynomials $B_n^+(t)$ are known (see, for instance, [12, Equation 3.7.5]) to satisfy

$$\frac{B_n^+(t)}{(1-t)^n} \;=\; \sum_{k\geq 0} \left((2k+1)^n - (2k)^n\right) t^k \tag{39}$$

and hence, we may rewrite the previous formula as

$$\mathrm{ch}\,(\mathbb{C}(K_n), t)\, z^n \;= \sum_{k\geq 0} t^k \sum_{\lambda=(\lambda_1,\lambda_2,\dots)\vdash n} m_\lambda^{-1} \left((2k+1)^{\ell(\lambda)} - (2k)^{\ell(\lambda)}\right) \prod_{i\geq 1} (1-t^{\lambda_i})\, p_{\lambda_i}(\mathbf{x}) z^{\lambda_i}.$$

Summing over all $n \geq 1$ and using the standard identities

$$H(\mathbf{x}; z) \;=\; \sum_{\lambda} m_\lambda^{-1} p_\lambda(\mathbf{x}) z^{|\lambda|} \;=\; \exp\left(\sum_{n\geq 1} p_n(\mathbf{x}) z^n/n\right)$$

just as in the proof of [23, Theorem 6.2] (one considers the p_n as algebraically independent indeterminates and replaces first each p_n with $(2k+1)(1-t^n)p_n$, then with $(2k)(1-t^n)p_n$), we conclude that

$$\begin{aligned} 1 \,+\, \sum_{n\geq 1} \mathrm{ch}\,(\mathbb{C}(K_n), t)\, z^n \;&=\; 1 \,+\, \sum_{k\geq 0} t^k \left(\frac{H(\mathbf{x}; z)^{2k+1}}{H(\mathbf{x}; tz)^{2k+1}} - \frac{H(\mathbf{x}; z)^{2k}}{H(\mathbf{x}; tz)^{2k}}\right) \\ &=\; 1 \,+\, \left(\frac{H(\mathbf{x}; z)}{H(\mathbf{x}; tz)} - 1\right) \left(1 - t\,\frac{H(\mathbf{x}; z)^2}{H(\mathbf{x}; tz)^2}\right)^{-1} \\ &=\; \frac{H(\mathbf{x}; z)\,(H(\mathbf{x}; tz) - tH(\mathbf{x}; z))}{H(\mathbf{x}; tz)^2 - tH(\mathbf{x}; z)^2} \end{aligned}$$

and the proof of (35) follows. To prove (35), it suffices to observe that

$$1 \,+\, \sum_{n\geq 1} \mathrm{ch}\,(L_{V_n}(K_n), t)\, z^n \;=\; E(\mathbf{x}, -z) \left(1 \,+\, \sum_{n\geq 1} \mathrm{ch}\,(\mathbb{C}(K_n), t)\, z^n\right).$$

The latter follows exactly as the corresponding identity for the barycentric subdivision Γ_n, shown in the proof of [18, Proposition 4.20]. Finally, from Equations (8) and (36) we deduce that

$$\mathrm{ch}\,(L_{V_n}(K_n), t) \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} \omega\, \xi^+_{n,k}(\mathbf{x}, \mathbf{x})\, t^k (1+t)^{n-2k}.$$

This expression, Corollary 4.4 and the well known fact that $s_\lambda(\mathbf{x}) s_\mu(\mathbf{x})$ is Schur-positive for all partitions λ, μ imply that $\mathrm{ch}\,(L_{V_n}(K_n), t)$ is Schur γ-positive for every n, as claimed in the last statement of the proposition. □

Acknowledgements. The author wishes to thank Eric Katz and Kalle Karu for some helpful e-discussions about the local equivariant Gal phenomenon.

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