The local h-polynomial of the edgewise subdivision of the simplex

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Abstract

The r-fold edgewise subdivision is a well studied flag triangulation of the simplex with interesting algebraic, combinatorial and geometric properties. An important enumerative invariant, namely the local h-polynomial, of this triangulation is computed and shown to be γ-nonnegative by providing explicit combinatorial interpretations to the corresponding coefficients. A construction of a flag triangulation of the seven-dimensional simplex whose local h-polynomial is not real-rooted is also described.

Keywords: Simplicial complex, edgewise subdivision, local h-polynomial, Smirnov word, γ-polynomial, real-rooted polynomial.

1 Introduction and results

The r-fold edgewise subdivision is an elegant triangulation of a simplicial complex Δ by which every k-dimensional face of Δ is subdivided into r^k simplices of dimension k. Having arisen in the realm of algebraic topology [11], this construction has appeared in a wide variety of mathematical contexts, such as algebraic K-theory [13], topological cyclic homology [5], discrete and toric geometry [14, 15], combinatorial commutative algebra [6, 7] and enumerative combinatorics [2]. Figure 1 shows the 4-fold edgewise subdivision of a two-dimensional simplex. We will denote by $\mathrm{esd}_r(\Delta)$ the r-fold edgewise subdivision of Δ and refer to Section 2 for a precise definition.

The edgewise subdivision has been studied in its own right and shown to have interesting algebraic, combinatorial and geometric properties; see, for instance, [6, 8, 10, 16]. Its effect on the h-polynomial, a fundamental enumerative invariant of a simplicial complex, is well understood; see Equation (5) in Section 2.

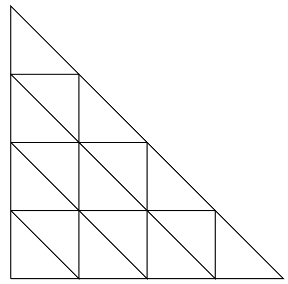

Figure 1: The 4-fold edgewise subdivision of the 2-simplex

The focus of this paper is on another important enumerative invariant, namely the local h-polynomial of the r-fold edgewise subdivision of the simplex, which turns out to have a very elegant combinatorial description. Local h-polynomials were introduced by Stanley [23] as a fundamental tool in his theory of face enumeration for subdivisions of simplicial complexes. To state the definition, recall that the h-polynomial of a simplicial complex Δ of dimension $d-1$ is defined by the formula

$$h(\Delta, x) \ = \ \sum_{i=0}^{d} f_{i-1}(\Delta)\, x^i (1-x)^{d-i}, \tag{1}$$

where $f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. Given a triangulation Γ of the abstract simplex 2^V on an n-element vertex set V, the local h-polynomial $\ell_V(\Gamma, x)$ is defined by the formula

$$\ell_V(\Gamma, x) \ = \sum_{F \subseteq V} (-1)^{n-|F|}\, h(\Gamma_F, x), \tag{2}$$

where Γ_F is the restriction of Γ to the face $F \in 2^V$. The importance of local h-polynomials stems from their appearance in the locality formula [23, Theorem 3.2], which expresses the h-polynomial of a triangulation of a pure simplicial complex Δ as a sum of local contributions, one for each face of Δ. We refer to [3] for a survey of basic properties and outstanding open problems about local h-polynomials and for a discussion of several examples of combinatorial interest.

To state our main results, we need to introduce the following notation and terminology. We will denote by $\mathcal{S}(n, r)$ the set of sequences $w = (w_0, w_1, \dots, w_n) \in \{0, 1, \dots, r-1\}^{n+1}$ having no two consecutive entries equal (known as *Smirnov words*) and satisfying $w_0 = w_n = 0$. We call an index $i \in \{0, 1, \dots, n-1\}$ an *ascent* of such a word w if $w_i < w_{i+1}$ and an index $i \in \{1, 2, \dots, n-1\}$ a *double ascent* of w if $w_{i-1} < w_i < w_{i+1}$ (descents and double descents are defined similarly). We will also denote by E_r the linear operator on the space $\mathbb{R}[x]$ of polynomials in x with real coefficients defined by setting $\mathrm{E}_r(x^m) = x^{m/r}$, if m is divisible by r, and $\mathrm{E}_r(x^m) = 0$ otherwise.

Theorem 1.1 *The local h-polynomial of the r-fold edgewise subdivision* $\mathrm{esd}_r(2^V)$ *of the $(n-1)$-dimensional simplex on the vertex set V can be expressed as*

$$\ell_V(\mathrm{esd}_r(2^V), x) \;=\; \mathrm{E}_r\,(x + x^2 + \cdots + x^{r-1})^n \;=\; \sum_{w \in \mathcal{S}(n,r)} x^{\mathrm{asc}(w)}, \qquad (3)$$

where $\mathrm{asc}(w)$ *is the number of ascents of $w \in \mathcal{S}(n,r)$.*

An abstract simplicial complex Δ is called *flag* if every clique in the 1-skeleton of Δ (meaning, every set of vertices of Δ pairwise joined by edges) is a face of Δ. Barycentric and edgewise subdivisions are examples of flag triangulations of the simplex. The following result confirms for edgewise subdivisions a conjecture of the author [1, Conjecture 5.4] [3, Conjecture 3.6], stating that the polynomial $\ell_V(\Gamma, x)$ is γ-nonnegative for every flag triangulation Γ of the simplex, and provides a combinatorial interpretation to the corresponding γ-coefficients.

Corollary 1.2 *For all positive integers n, r we have*

$$\ell_V(\mathrm{esd}_r(2^V), x) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,r,i}\, x^i (1+x)^{n-2i}, \qquad (4)$$

where $\xi_{n,r,i}$ is the number of sequences $w = (w_0, w_1, \dots, w_n) \in \mathcal{S}(n,r)$ with exactly i ascents which have the following property: for every double ascent k of w there exists a double descent $\ell > k$ such that $w_k = w_\ell$ and $w_k \le w_j$ for all $k < j < \ell$.

In particular, $\ell_V(\mathrm{esd}_r(2^V), -1)$ is equal to $(-1)^{n/2}$ times the number of sequences $w \in \mathcal{S}(n,r)$ with exactly $n/2$ ascents, having this property.

The local h-polynomials of flag triangulations of simplices often have only real roots; see [3, Section 4]. For edgewise subdivisions, this was shown recently (using Theorem 1.1) in [17, 18] [25]. The following statement can be viewed as an analogue of a result of Gal [12, Section 3.3], stating that there exists a flag triangulation of the five-dimensional sphere whose h-polynomial is not real-rooted.

Theorem 1.3 *There exists a flag triangulation Γ of a seven-dimensional simplex 2^V such that the local h-polynomial $\ell_V(\Gamma, x)$ is not real-rooted.*

This paper is organized as follows. Section 2 discusses background on simplicial complexes, subdivisions and local h-polynomials. Theorem 1.1 and Corollary 1.2 are proven in Section 3 using techniques from enumerative combinatorics. The proof of Theorem 1.3, given in Section 4, relies on the aforementioned construction of Gal and shows that for every simplicial complex Δ which is the boundary complex of an n-dimensional convex polytope, there exists a (regular) triangulation Γ (which can be chosen to be flag, if so is Δ) of a simplex 2^V of dimension $n+1$ such that $\ell_V(\Gamma, x) = x\, h(\Delta, x)$. This implies, in particular, that Gal's conjecture for polytopal flag triangulations of spheres follows from the validity of [1, Conjecture 5.4] for regular flag triangulations of the simplex. The results of this paper were announced without proof in [3, Section 4].

2 Triangulations

This section briefly reviews the background on simplicial and polytopal complexes, their triangulations and their enumerative invariants which are needed to understand the main results and their proofs. For basic notions and more information on these topics the reader is referred to the sources [4, 9, 24, 26]. All complexes considered here are assumed to be finite. The cardinality and the set of all subsets of a finite set V will be denoted by $|V|$ and 2^V, respectively.

Triangulations. Let Σ' and Σ be geometric simplicial complexes in some Euclidean space $\mathbb{R}^N$ (so the elements of Σ' and Σ are geometric simplices in $\mathbb{R}^N$), with corresponding abstract simplicial complexes Δ' and Δ. We say that Σ' is a *triangulation* of Σ, and that Δ' is a *triangulation* of Δ, if (a) every simplex of Σ' is contained in some simplex of Σ; and (b) the union of the simplices of Σ' is equal to the union of the simplices of Σ. Then, given any simplex $L \in \Sigma$ with corresponding face $F \in \Delta$, the subcomplex Σ'_L of Σ' consisting of all simplices of Σ' contained in L is called the *restriction* of Σ' to L. The subcomplex Δ'_F of Δ' corresponding to Σ'_L is the *restriction* of Δ' to F. Clearly, Δ'_F is a triangulation of the abstract simplex 2^F.

Given a simplicial complex Δ and an element v not in its vertex set, the *cone* of Δ over v is defined as $\operatorname{cone}(\Delta) = \Delta \cup \{F \cup \{v\} : F \in \Delta\}$. We note that if Δ' triangulates Δ, then $\operatorname{cone}(\Delta')$ naturally triangulates $\operatorname{cone}(\Delta)$. The *link* of $F \in \Delta$ is the subcomplex of Δ defined as $\operatorname{link}_\Delta(F) = \{G \smallsetminus F : G \in \Delta,\ F \subseteq G\}$. The *stellar subdivision* of Δ on its edge $e = \{a, b\}$ is defined as the simplicial complex obtained from Δ by removing all faces which contain e and adding the sets of the form $F \cup \{v\}$, $F \cup \{v, a\}$ and $F \cup \{v, b\}$ for all $F \in \operatorname{link}_\Delta(e)$, where (as before) v is a new vertex added. This complex is naturally a triangulation of Δ.

The notions of triangulation of a simplicial complex and restriction to a face can easily be extended to polytopal complexes (see [26, Section 5.1] for the definition and examples of polytopal complexes). The boundary complex of a convex polytope Q (consisting of all proper faces of Q) will be denoted by $\partial(Q)$.

Edgewise subdivisions. Consider the simplex 2^V on the vertex set $V = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of coordinate vectors in $\mathbb{R}^n$. For $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ set $\iota(\mathbf{a}) = (a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_n)$. The r-fold edgewise subdivision of 2^V is the abstract simplicial complex $\operatorname{esd}_r(2^V)$ on the vertex set $\Omega_r = \{(i_1, i_2, \dots, i_n) \in \mathbb{N}^n : i_1 + i_2 + \cdots + i_n = r\}$ of which a set $G \subseteq \Omega_r$ is a face if $\iota(\mathbf{u}) - \iota(\mathbf{v}) \in \{0,1\}^n$, or $\iota(\mathbf{v}) - \iota(\mathbf{u}) \in \{0,1\}^n$, for all $\mathbf{u}, \mathbf{v} \in G$. This is a flag simplicial complex which can be realized as a triangulation of 2^V; see, for instance, [7, Section 6]. To be more precise, this triangulation is defined by the dissection of the geometric simplex which is the convex hull of the set Ω_r into smaller geometric simplices by the affine hyperplanes in $\mathbb{R}^n$ of the form $x_i + x_{i+1} + \cdots + x_j = k$ for $i \le j$ and $k \in \{0, 1, \dots, r\}$. The restriction of $\operatorname{esd}_r(2^V)$ to a face $F \in 2^V$ coincides with $\operatorname{esd}_r(2^F)$; it has exactly $r^{\dim(F)}$ faces of the same dimension as F. The r-fold edgewise subdivision of an arbitrary simplicial complex Δ may be defined so that its restriction to any face $F \in \Delta$ is combinatorially isomorphic to $\operatorname{esd}_r(2^F)$; see [6, Section 4] [7, Section 6].

Face enumeration. Let Δ be a $(d-1)$-dimensional simplicial complex and let $f_i(\Delta)$ be the number of i-dimensional faces of Δ. The *h-polynomial* of Δ, defined by Equation (1), is a convenient way to record the information provided by the numbers $f_i(\Delta)$. For example, for the two-dimensional complex of Figure 1 we have $f_{-1}(\Delta) = 1$, $f_0(\Delta) = 15$, $f_1(\Delta) = 30$, $f_2(\Delta) = 16$ and $h(\Delta, x) = (1-x)^3 + 15x(1-x)^2 + 30x^2(1-x) + 16x^3 = 1 + 12x + 3x^2$. For the importance of h-polynomials, see [24, Chapter II].

An explicit formula for the h-polynomial of the r-fold edgewise subdivision of Δ can be given. Indeed, combining [7, Corollary 6.8] with [6, Corollary 1.2] (see also [2, Equation (21)]) one gets

$$h(\mathrm{esd}_r(\Delta), x) \;=\; \mathrm{E}_r\left((1 + x + x^2 + \cdots + x^{r-1})^d \, h(\Delta, x)\right), \tag{5}$$

where $\mathrm{E}_r : \mathbb{R}[x] \to \mathbb{R}[x]$ is the linear operator defined in the introduction and $d-1$ is the dimension of Δ. In particular,

$$h(\mathrm{esd}_r(2^V), x) \;=\; \mathrm{E}_r\left((1 + x + x^2 + \cdots + x^{r-1})^n\right) \tag{6}$$

for every n-element set V.

Given a triangulation Γ of an $(n-1)$-dimensional simplex 2^V, the *local h-polynomial*, denoted $\ell_V(\Gamma, x)$, of Γ is defined [23, Definition 2.1] by Equation (2), where Γ_F is the restriction of Γ to the face $F \in 2^V$. For the triangulation of Figure 1 we have $h(\Gamma_V, x) = 1 + 12x + 3x^2$, $h(\Gamma_F, x) = 1 + 3x$ if F has two elements and $h(\Gamma_F, x) = 1$ otherwise, so that $\ell_V(\Gamma, x) = (1 + 12x + 3x^2) - 3(1 + 3x) + 3 - 1 = 3x + 3x^2$. The polynomial $\ell_V(\Gamma, x)$ has degree at most $n-1$ (unless $V = \varnothing$, in which case $\ell_V(\Gamma, x) = 1$) and nonnegative and symmetric coefficients, in the sense that $x^n \, \ell_V(\Gamma, 1/x) = \ell_V(\Gamma, x)$. For examples and further properties of local h-polynomials, see [3] [23, Part I] [24, Section II.10].

Barycentric subdivisions. Consider a polytopal complex $\mathcal{K}$. Choose a point p_G in the relative interior of each face G of $\mathcal{K}$ (and note that the chosen points include all vertices of $\mathcal{K}$). The *barycentric subdivision* of $\mathcal{K}$ is the unique triangulation $\mathrm{sd}(\mathcal{K})$ of $\mathcal{K}$ with the following properties: (a) the vertices of $\mathrm{sd}(\mathcal{K})$ are exactly the chosen points, and; (b) the restriction of $\mathrm{sd}(\mathcal{K})$ to any face G of $\mathcal{K}$ of positive dimension is the cone over p_G of the restriction of $\mathrm{sd}(\mathcal{K})$ to the boundary of G. A more general construction appears in Section 4 (see Lemma 4.2).

3 Proof of Theorem 1.1 and Corollary 1.2

This section proves Theorem 1.1 and Corollary 1.2 using methods of enumerative combinatorics. The proof of the latter uses a variant of the valley hopping technique of Foata, Schützenberger and Strehl; see, for instance, [21, Section 4.2] and references therein.

Proof of Theorem 1.1. We set $\Gamma = \mathrm{esd}_r(2^V)$ and note that the restriction Γ_F is the r-fold edgewise subdivision $\mathrm{esd}_r(2^F)$ for every $F \subseteq V$. Thus, by Equation (6) we have

$$h(\Gamma_F, x) \;=\; \mathrm{E}_r\left((1 + x + x^2 + \cdots + x^{r-1})^{|F|}\right)$$

for every $F \subseteq V$. The defining Equation (2) then yields

$$\begin{aligned} \ell_V(\Gamma, x) &= \mathrm{E}_r \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1 + x + x^2 + \cdots + x^{r-1})^k \right) \\ &= \mathrm{E}_r \left((x + x^2 + \cdots + x^{r-1})^n \right). \end{aligned}$$

This proves the first equality in (3). For the second equality, we extend the action of the operator E_r on the space of formal power series in x with real coefficients in the obvious way and note that

$$\begin{aligned} \frac{h(\Gamma, x)}{(1-x)^n} &= \frac{\mathrm{E}_r \left((1 + x + x^2 + \cdots + x^{r-1})^n \right)}{(1-x)^n} = \mathrm{E}_r \left(\frac{(1 + x + x^2 + \cdots + x^{r-1})^n}{(1-x^r)^n} \right) \\ &= \mathrm{E}_r \left(\frac{1}{(1-x)^n} \right) = \sum_{m \ge 0} \binom{n + rm - 1}{n-1} x^m. \end{aligned}$$

We now use the identity

$$\sum_{m \ge 0} \binom{n + rm}{n} x^m = \frac{\sum_{w \in \{0,1\ldots,r-1\}^n} x^{\mathrm{asc}(w)}}{(1-x)^{n+1}}, \qquad (7)$$

where $\mathrm{asc}(w)$ stands for the number of indices $i \in \{0, 1, \ldots, n-1\}$ such that $w_i < w_{i+1}$ for $w = (w_1, w_2, \ldots, w_n) \in \{0, 1 \ldots, r-1\}^n$, with the convention that $w_0 = 0$. This identity follows from [22, Corollary 8], derived in the context of Ehrhart theory, although a simple combinatorial proof (for instance, one analogous to the proof of [20, Theorem 1]) can be given.

Replacing n with $n-1$ in (7) and comparing with our computation of $h(\Gamma, x)/(1-x)^n$ yields that

$$h(\Gamma, x) = \sum_{w \in \{0,1\ldots,r-1\}^{n-1}} x^{\mathrm{asc}(w)}. \qquad (8)$$

Let us consider the sequences which appear in the right-hand side of this formula as having length $n+1$ and first and last coordinate zero and set $V := \{v_1, v_2 \ldots, v_n\}$. Then, for every $F \subseteq V$, we may interpret $h(\Gamma_F, x)$ as

$$h(\Gamma_F, x) = \sum x^{\mathrm{asc}(w)},$$

where the summation runs over all words $w = (w_0, w_1, \ldots, w_n) \in \{0, 1 \ldots, r-1\}^{n+1}$ satisfying $w_{i-1} = w_i$ for all $i \in \{1, 2, \ldots, n\}$ with $v_i \notin F$ and $w_0 = w_n = 0$. The defining Equation (2) and an application of the principle of inclusion-exclusion then yield

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x) = \sum_{w \in \mathcal{S}(n,r)} x^{\mathrm{asc}(w)}$$

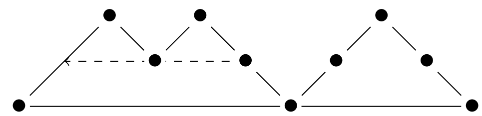

Figure 2: The equivalence class of $(0,2,1,2,1,0,1,2,1,0)$

and the proof follows. $\square$

Proof of Corollary 1.2. We define an equivalence relation on the set $\mathcal{S}(n,r)$ as follows. Let $w = (w_0, w_1, \dots, w_n) \in \mathcal{S}(n,r)$ and $k \in \{1, 2, \dots, n-1\}$ be a double descent of w. Since $w_0 = 0$, we may consider the largest index $1 \le \ell < k$ such that $w_{\ell-1} < w_k$. We then have $w_\ell \ge w_k$. Assuming $w_\ell > w_k$, we define the *left match* of w with respect to k as the sequence w' which is obtained from w by first deleting w_k and then inserting it between $w_{\ell-1}$ and w_ℓ. Formally, we define $w' = (w'_0, w'_1, \dots, w'_n)$ by setting $w'_\ell = w_k$, $w'_{i+1} = w_i$ for $\ell \le i < k$ and $w'_i = w_i$ for all other values of i. We note that the left match w' is an element of $\mathcal{S}(n,r)$ with one more ascent than w. Similarly, suppose $k \in \{1, 2, \dots, n-1\}$ is a double ascent of $w = (w_0, w_1, \dots, w_n) \in \mathcal{S}(n,r)$. Since $w_n = 0$, we may consider the smallest index $k < \ell < n$ such that $w_{\ell+1} < w_k$. We must then have $w_k \le w_\ell$. Assuming $w_k < w_\ell$, we define the *right match* of w with respect to k as the sequence obtained from w by first deleting w_k and then inserting it between w_ℓ and $w_{\ell+1}$ and note that this is an element of $\mathcal{S}(n,r)$ with one less ascent than w. For example, if $n = 9$, $r = 3$ and $w = (0,2,1,2,1,0,1,2,1,0)$, then $k = 4$ is a double descent of w and the corresponding left match of w is the sequence $(0,1,2,1,2,0,1,2,1,0)$, obtained from w by deleting its fifth entry and inserting it between the first two; see Figure 2. Note that if w' is the left match of w with respect to its double descent k, with w_k inserted between $w_{\ell-1}$ and w_ℓ as above, then the right match of w' with respect to its double ascent ℓ is equal to w and similarly for right matches of w.

We say that two elements of $\mathcal{S}(n,r)$ are equivalent if one can be obtained from the other by a sequence of matchings, as just described. Our previous discussion shows that this defines an equivalence relation on $\mathcal{S}(n,r)$ and that the equivalence class of $u \in \mathcal{S}(n,r)$ has exactly $2^{m(u)}$ elements, where $m(u)$ is the total number of (left or right) matches of u. Moreover, each equivalence class contains a unique element with no right match; these representatives are exactly the elements $w \in \mathcal{S}(n,r)$ which have the property that for every double ascent k of w there exists a double descent $\ell > k$ such that $w_k = w_\ell$ and $w_k \le w_j$ for all $k < j < \ell$. We leave it to the reader to verify that $m(w) = \mathrm{des}(w) - \mathrm{asc}(w) = n - 2\mathrm{asc}(w)$ for every such element w. Since there are $\binom{m(w)}{i}$ ways to choose an element u in the equivalence class $O(w)$ of w by applying i left matchings to w and since every such element u has exactly $\mathrm{asc}(w) + i$ ascents, we conclude that

$$\sum_{u \in O(w)} x^{\mathrm{asc}(u)} \ = \ \sum_{i=0}^{m(w)} \binom{m(w)}{i} x^{\mathrm{asc}(w)+i} \ = \ x^{\mathrm{asc}(w)} \, (1+x)^{m(w)} \ = \ x^{\mathrm{asc}(w)} \, (1+x)^{n-2\mathrm{asc}(w)}.$$

For our example $w = (0,2,1,2,1,0,1,2,1,0)$, this expression is equal to $x^4(1+x)$; see

Figure 3: A partial barycentric subdivision

Figure 2 for an attempt to draw the equivalence class of this sequence. Summing over all equivalence classes and taking Theorem 1.1 into account, we get the desired expression for $\ell_V(\mathrm{esd}_r(2^V), x)$. □

4 Proof of Theorem 1.3

This section uses enumerative and geometric arguments, as well as the construction of flag triangulations of the five-dimensional sphere whose h-polynomials are not real-rooted by Gal [12], to prove Theorem 1.3. Since it will be crucial in the proof, we note that the spheres constructed by Gal are easily seen to be polytopal, meaning they are boundary complexes of simplicial polytopes. We prepare for the proof with a couple of lemmas.

Lemma 4.1 *For every triangulation Γ of an $(n-1)$-dimensional simplex 2^V having an interior vertex p, there exists a triangulation Γ' of an n-dimensional simplex $2^{V'}$ such that $\ell_{V'}(\Gamma', x) = x\, h(\mathrm{link}_\Gamma(p), x)$. If Γ is flag, then Γ' can be chosen to be flag as well.*

Proof. Consider the cone of Γ over a new vertex v and let Γ' be the stellar subdivision of this cone on the edge $e = \{v, p\}$. Then Γ' is a triangulation of the simplex $2^{V'}$, where $V' = V \cup \{v\}$. The effect of stellar subdivisions on edges on the local h-polynomial was studied in [1, Section 6]. In particular, from the first displayed equation in the proof of Proposition 6.1 in that reference, the definition [1, Equation (3-5)] of the relative local h-polynomial and the fact that e is an interior face of cone(Γ) (or by direct computation), we get

$$\ell_{V'}(\Gamma', x) \ = \ \ell_{V'}(\mathrm{cone}(\Gamma), x) \ + \ x\, h(\mathrm{link}_{\mathrm{cone}(\Gamma)}(e), x) \ = \ x\, h(\mathrm{link}_\Gamma(p), x).$$

For the second equality we have used that fact (see the discussion after Proposition 4.14 in [23]) that the local h-polynomial of a cone vanishes and the obvious equality $\mathrm{link}_{\mathrm{cone}(\Gamma)}(e) = \mathrm{link}_\Gamma(p)$. The last statement of the lemma follows from the previous construction and the fact (see, for instance, [12, Proposition 2.4.6]) that conings and stellar subdivisions on edges preserve flagness. □

The following lemma constructs a partial barycentric subdivision of a polytopal complex $\mathcal{K}$ with respect to a simplicial subcomplex $\mathcal{F}$, which coincides with the usual barycentric subdivision when $\mathcal{F}$ consists only of vertices of $\mathcal{K}$. Figure 3 shows this subdivision when $\mathcal{K}$ consists of two squares (along with their faces) sharing a common edge and $\mathcal{F}$

is the boundary complex of one of them. This classical construction is usually called the *barycentric subdivision of $\mathcal{K}$ relative to $\mathcal{F}$*; see, for instance, [19, Definition 2.5.7].

Lemma 4.2 *Let $\mathcal{K}$ be a polytopal complex and $\mathcal{F}$ be a simplicial subcomplex of $\mathcal{K}$. Choose a point p_G in the relative interior of each face G of $\mathcal{K}$, which is not a face of $\mathcal{F}$ of positive dimension. Then there exists a unique triangulation $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ of $\mathcal{K}$ satisfying the following conditions:*

(i) *the vertices of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ are exactly the chosen points p_G, and*

(ii) *the restriction of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ to any face G of $\mathcal{K}$ of positive dimension which is not a face of $\mathcal{F}$ is the cone over p_G of the restriction of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ to the boundary of G.*

Moreover, $\mathcal{F}$ is a subcomplex of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ and if $\mathcal{F}$ is flag, then so is $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$.

Proof. The existence and uniqueness of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ are known and follow by induction on the dimension of $\mathcal{K}$. That $\mathcal{F}$ is a subcomplex of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ follows from condition (i). Finally, assume that $\mathcal{F}$ is flag. To show that $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ is flag, we consider a clique C in the 1-skeleton of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ and verify that C is the vertex set of some face of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ as follows. We define the *rank* of a chosen point p_G as the dimension of G and proceed by induction on the maximum rank m of the elements of C. Assume first that $m = 0$, meaning that C consists of vertices of $\mathcal{K}$. Since two vertices of $\mathcal{K}$ are joined by an edge in $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ only if they are both vertices of $\mathcal{F}$, the clique C consists of vertices of $\mathcal{F}$ and therefore forms the vertex set of a face of $\mathcal{F}$, hence of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ as well, by flagness of $\mathcal{F}$. Suppose now that $m \ge 1$ and let p_G be an element of C of rank m. By the construction of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$, all other elements of C lie in G and have rank less than m. By the induction hypothesis, they form the vertex set of a face of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ which is contained in G and hence, by the construction of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$, the same is true for the elements of C. □

Proof of Theorem 1.3. Gal [12, Section 3.3] has constructed a six-dimensional flag simplicial polytope Q for which the polynomial $h(\partial(Q), x)$ has at least one non-real root. Thus, in view of Lemma 4.1, it suffices to prove the existence of a flag triangulation Γ of a six-dimensional simplex Σ which has an interior vertex p such that $\mathrm{link}_\Gamma(p)$ is combinatorially isomorphic to $\partial(Q)$.

Consider the hyperplane H in $\mathbb{R}^7$ consisting of all points having last coordinate $x_0 = 0$ and the natural projection $\pi : \mathbb{R}^7 \to H$. Let Σ be a large six-dimensional simplex within H and let R be a copy of Q, embedded in the hyperplane $x_0 = 1$ of $\mathbb{R}^7$, which projects into the interior of Σ under the map π. Let P be the convex hull of $\Sigma \cup R$. Then R is a facet of the polytope P. Projecting the faces of P other than R and Σ under π we get a polytopal complex $\mathcal{K}$ which contains $\partial(\pi(R))$ as a subcomplex. Since the latter is affinely isomorphic to $\partial(Q)$, and is therefore flag, applying Lemma 4.2 to $\mathcal{K}$ and $\mathcal{F} := \partial(\pi(R))$ we get a flag triangulation $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ of $\mathcal{K}$ which contains $\partial(\pi(R))$ as a subcomplex. This triangulation can be completed to a flag triangulation Γ of Σ by choosing a point p in the relative interior of $\pi(R)$ and defining Γ as the union of $\mathrm{sd}_{\mathcal{F}}(\mathcal{K})$ with the set of all cones of the faces of $\partial(\pi(R))$ over p. Then $\mathrm{link}_\Gamma(p) = \partial(\pi(R))$, which is affinely isomorphic to $\partial(Q)$, and hence the triangulation Γ constructed has the desired properties. □

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