

A SYMMETRIC UNIMODAL DECOMPOSITION OF THE DERANGEMENT POLYNOMIAL OF TYPE B

CHRISTOS A. ATHANASIADIS AND CHRISTINA SAVVIDOU

ABSTRACT. The derangement polynomial $d_n(x)$ for the symmetric group enumerates derangements by the number of excedances. The derangement polynomial $d_n^B(x)$ for the hyperoctahedral group is a natural type B analogue. A new combinatorial formula for this polynomial is given in this paper. This formula implies that $d_n^B(x)$ decomposes as a sum of two nonnegative, symmetric and unimodal polynomials whose centers of symmetry differ by a half and thus provides a new transparent proof of its unimodality. A geometric interpretation, analogous to Stanley's interpretation of $d_n(x)$ as the local h -polynomial of the barycentric subdivision of the simplex, is given to one of the summands of this decomposition. This interpretation leads to a unimodal decomposition of the Eulerian polynomial of type B whose summands can be expressed in terms of the Eulerian polynomial of type A . The various decomposing polynomials introduced here are also studied in terms of recurrences, generating functions, combinatorial interpretations, expansions and real-rootedness.

1. INTRODUCTION AND RESULTS

The derangement polynomial of order n is an interesting q -analogue of the number of derangements (elements without fixed points) in the symmetric group $\mathfrak{S}_n$. It is defined by the formula

$$(1.1) \quad d_n(x) = \sum_{w \in \mathcal{D}_n} x^{\text{exc}(w)},$$

where $\text{exc}(w)$ is the number of excedances (see Section 2 for missing definitions) of $w \in \mathfrak{S}_n$ and $\mathcal{D}_n$ is the set of derangements in $\mathfrak{S}_n$. The polynomial $d_n(x)$, first studied by Brenti [10] in the context of symmetric functions, has a number of pleasant properties. For instance, it has symmetric and unimodal coefficients [10] (see also [4, Section 4] [23, Section 5] [29]) and only real roots [32]. It can also be expressed as

$$(1.2) \quad d_n(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} A_k(x),$$

where $A_k(x) = \sum_{w \in \mathfrak{S}_k} x^{\text{des}(w)}$ is the k -th Eulerian polynomial.

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We will be concerned with a natural analogue of $d_n(x)$ for the hyperoctahedral group B_n of signed permutations, introduced and studied independently by Chen, Tang and Zhao [15] and by Chow [16]. It is defined by the formula

$$(1.3) \quad d_n^B(x) = \sum_{w \in \mathcal{D}_n^B} x^{\text{exc}_B(w)},$$

where $\text{exc}_B(w)$ is the number of type B excedances of $w \in B_n$, introduced by Brenti [11], and $\mathcal{D}_n^B$ is the set of derangements in B_n .

The derangement polynomial $d_n^B(x)$ shares most of the main properties of $d_n(x)$. For instance, it is real-rooted [15, 16], hence it has unimodal (but not symmetric) coefficients, and satisfies the analogue

$$(1.4) \quad d_n^B(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k(x)$$

of (1.2), where $B_k(x) = \sum_{w \in B_k} x^{\text{des}_B(w)}$ is the k -th Eulerian polynomial of type B . Our first main result is the following combinatorial formula for $d_n^B(x)$.

Theorem 1.1. *We have*

$$(1.5) \quad d_n^B(x) = \sum \binom{n}{r_0, r_1, \dots, r_k} x^{\lfloor \frac{k+1}{2} \rfloor} d_{r_0}(x) A_{r_1}(x) \cdots A_{r_k}(x)$$

for $n \in \mathbb{N}$, where $A_0(x) = 0$, $d_0(x) = 1$ and the sum ranges over all $k \in \mathbb{N}$ and over all sequences $(r_0, r_1, \dots, r_k)$ of nonnegative integers which sum to n .

Chow [16, Section 4] gave an additional proof of the unimodality of $d_n^B(x)$ by expressing it as a sum of certain nonnegative unimodal polynomials, defined by a symmetric function identity, of a common mode. Theorem 1.1 implies that $d_n^B(x)$ can be written as a sum of two polynomials with nonnegative, symmetric and unimodal coefficients, whose centers of symmetry differ by a half, and thus provides a new proof of its unimodality, as we now explain. Since $d_n^B(x)$ has degree n and zero constant term, it can be written uniquely in the form

$$(1.6) \quad d_n^B(x) = f_n^+(x) + f_n^-(x),$$

where $f_n^+(x)$ and $f_n^-(x)$ are polynomials of degrees at most $n-1$ and n , respectively, satisfying

$$(1.7) \quad f_n^+(x) = x^n f_n^+(1/x)$$

$$(1.8) \quad f_n^-(x) = x^{n+1} f_n^-(1/x)$$

(see, for instance, [6, Lemma 2.4] for this elementary fact). For the first few values of n we have

$$f_n^+(x) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n = 1 \\ 3x, & \text{if } n = 2 \\ 7x + 7x^2, & \text{if } n = 3 \\ 15x + 87x^2 + 15x^3, & \text{if } n = 4 \\ 31x + 551x^2 + 551x^3 + 31x^4, & \text{if } n = 5 \\ 63x + 2803x^2 + 8243x^3 + 2803x^4 + 63x^5, & \text{if } n = 6 \\ 127x + 12867x^2 + 84827x^3 + 84827x^4 + 12867x^5 + 127x^6, & \text{if } n = 7 \end{cases}$$

and

$$f_n^-(x) = \begin{cases} 0, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ x + x^2, & \text{if } n = 2 \\ x + 13x^2 + x^3, & \text{if } n = 3 \\ x + 57x^2 + 57x^3 + x^4, & \text{if } n = 4 \\ x + 201x^2 + 761x^3 + 201x^4 + x^5, & \text{if } n = 5 \\ x + 653x^2 + 6333x^3 + 6333x^4 + 653x^5 + x^6, & \text{if } n = 6 \\ x + 2045x^2 + 42757x^3 + 106037x^4 + 42757x^5 + 2045x^6 + x^7, & \text{if } n = 7. \end{cases}$$

The following information for the polynomials $f_n^+(x)$ and $f_n^-(x)$ and for $d_n^B(x)$ can be derived from (1.5).

Corollary 1.2. *We have*

$$(1.9) \quad f_n^+(x) = \sum \binom{n}{r_0, r_1, \dots, r_{2k}} x^k d_{r_0}(x) A_{r_1}(x) \cdots A_{r_{2k}}(x)$$

and

$$(1.10) \quad f_n^-(x) = \sum \binom{n}{r_0, r_1, \dots, r_{2k+1}} x^{k+1} d_{r_0}(x) A_{r_1}(x) \cdots A_{r_{2k+1}}(x)$$

for $n \in \mathbb{N}$, where the sums range over all $k \in \mathbb{N}$ and over all sequences $(r_0, r_1, \dots, r_{2k})$ (respectively, $(r_0, r_1, \dots, r_{2k+1})$) of nonnegative integers which sum to n . Moreover, $f_n^+(x)$ and $f_n^-(x)$ are γ -nonnegative, meaning there exist nonnegative integers $\xi_{n,i}^+$ and $\xi_{n,i}^-$ such that

$$(1.11) \quad f_n^+(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,i}^+ x^i (1+x)^{n-2i}$$

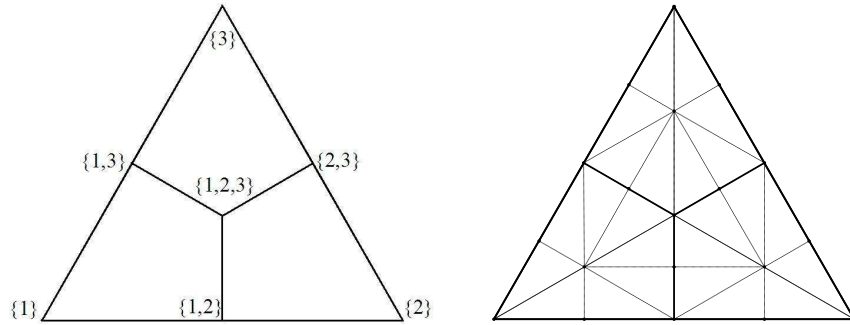


FIGURE 1. The cubical barycentric subdivision of the 2-simplex and its barycentric subdivision K_3

and

$$(1.12) \quad f_n^-(x) = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \xi_{n,i}^- x^i (1+x)^{n+1-2i}.$$

In particular, $f_n^+(x)$ and $f_n^-(x)$ are symmetric and unimodal, with center of symmetry $n/2$ and $(n+1)/2$, respectively, and $d_n^B(x)$ is unimodal with a peak at $\lfloor (n+1)/2 \rfloor$.

Much of the motivation behind this paper comes from the theory of subdivisions and local h -vectors, developed by Stanley [25], and its extension [3]. We recall that the local h -vector is a fundamental enumerative invariant of a simplicial subdivision (triangulation) of the simplex. An example by Stanley (see [25, Proposition 2.4]) shows that $d_n(x)$ is equal to the local h -polynomial of the (first) simplicial barycentric subdivision of the $(n-1)$ -dimensional simplex. This fact gives a geometric interpretation to $d_n(x)$ and another proof of its symmetry and unimodality.

Our second main result provides a type B analogue to this interpretation. To state it, we introduce the following notation. We denote by K_n the simplicial barycentric subdivision of the cubical barycentric subdivision of the $(n-1)$ -dimensional simplex (Figure 1 shows this subdivision for $n=3$). We also introduce the ‘half Eulerian polynomials’

$$(1.13) \quad B_n^+(x) = \sum_{w \in B_n^+} x^{\text{des}_B(w)}$$

and

$$(1.14) \quad B_n^-(x) = \sum_{w \in B_n^-} x^{\text{des}_B(w)}$$

for the group B_n , where B_n^+ and B_n^- are the sets of signed permutations of length n with positive and negative, respectively, last entry, and set $B_0^+(x) = 1$ and $B_0^-(x) = 0$ (the set B_n^+ has appeared in the context of major indices for classical Weyl groups; see [7, page 613]).

Theorem 1.3. *The polynomial $f_n^+(x)$ is equal to the local h -polynomial of the simplicial subdivision K_n (in particular, $f_n^+(x)$ has nonnegative, symmetric and unimodal coefficients). Moreover, we have*

$$(1.15) \quad f_n^+(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k^+(x)$$

and

$$(1.16) \quad f_n^-(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k^-(x)$$

for $n \in \mathbb{N}$.

We should point out that it is Theorem 1.3 and the methods of [3, 25] which led the authors to suspect that formula (1.5) holds. Indeed, it follows from the relevant definitions and some more work (see Section 6) that the local h -polynomial of K_n is equal to the right-hand side of (1.15). By exploiting the symmetry of this polynomial and certain recurrence relations for that and for $d_n^B(x)$ (see Section 7), one can show that the local h -polynomial of K_n is equal to $f_n^+(x)$, as defined by the decomposition (1.6). A formula for the change in the local h -vector of a simplicial subdivision of the simplex after further subdivision [3, Proposition 3.6] (see also Proposition 5.3) can then be used to produce equation (1.9). This suggested that (1.10), and hence (1.5), hold as well.

The structure and other results of this paper are as follows. Section 2 provides the necessary background on (signed) permutations, simplicial complexes and subdivisions. Section 3 proves Theorem 1.1 and Corollary 1.2. A bijective proof of Theorem 1.1, as well as one using generating functions, is given and the exponential generating functions of $f_n^+(x)$ and $f_n^-(x)$ are computed. Section 4 gives a combinatorial interpretation to the coefficients of these polynomials. Section 5 proves the main properties of the relative local h -vector, a generalization of the concept of local h -vector which was introduced in [3, Section 3] (and, in a variant form, in [21]) and derives a monotonicity property for local h -vectors. These results were stated without proof in [3]. As an example (used in one of the proofs of Theorem 1.3), the relative local h -vector of the barycentric subdivision of the simplex is computed. Section 6 gives two proofs of Theorem 1.3. A first step towards these proofs is to interpret $B_n^+(x)$ as the h -polynomial of the simplicial complex K_n (Proposition 6.1). Given that, one proof uses the theory of (relative) local h -vectors, as discussed earlier, while the other uses recurrences and generating functions.

Section 7 studies the polynomials $B_n^+(x)$ and $B_n^-(x)$. A simple relation between the two is shown to hold (Lemma 7.1). Using its interpretation as the h -polynomial of K_n and the theory of local h -vectors, a simple formula for $B_n^+(x)$ (hence one for $B_n^-(x)$ and one for the Eulerian polynomial $B_n(x)$) in terms of the Eulerian polynomial $A_n(x)$ is proven (Proposition 7.2). Using this formula, it is shown that $B_n^+(x)$ and $B_n^-(x)$ are real-rooted, hence unimodal and log-concave, and a new proof of the unimodality of $B_n(x)$ is deduced. Recurrences and generating functions for $B_n^+(x)$ and $B_n^-(x)$, as well as for $f_n^+(x)$ and $f_n^-(x)$, are also given and a third proof of Theorem 1.3 is deduced.

2. PERMUTATIONS AND SUBDIVISIONS

This section fixes notation and includes background material on (signed) permutations, simplicial complexes and their subdivisions. For more information on these topics, the reader is referred to [8, 9, 25, 26, 27].

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers. For each positive integer n we set $[n] := \{1, 2, \dots, n\}$ and $\Omega_n = \{1, -1, 2, -2, \dots, n, -n\}$. We denote by $|S|$ the cardinality, and by 2^S the set of all subsets, of a finite set S .

2.1. Permutations. A *permutation* of a finite set S is a bijective map $w : S \rightarrow S$. We denote by $\mathfrak{S}(S)$ the set of all permutations of S and set $\mathfrak{S}_n := \mathfrak{S}([n])$. Suppose that $S = \{a_1, a_2, \dots, a_n\}$ has n elements, which are totally ordered by $a_1 \prec a_2 \prec \dots \prec a_n$. A permutation $w \in \mathfrak{S}(S)$ can be represented as the sequence $(w(a_1), w(a_2), \dots, w(a_n))$, or as the word $w(a_1)w(a_2)\dots w(a_n)$, or as a disjoint union of cycles [27, Section 1.3]. The *standard cycle form* is defined by requiring that (a) each cycle is written with its largest element (with respect to the total order $\preceq$) first and (b) the cycles are written in increasing order of their largest element [27, page 23].

Given $w \in \mathfrak{S}(S)$, an element $a \in S$ is called an *excedance* of w (with respect to $\preceq$) if $w(a) \succ a$ and an *inverse excedance* if $w(a) \prec a$. The element $a_i \in S$ is called a *descent* (respectively, *ascent*) of w if $i \in [n-1]$ and $w(a_i) \succ w(a_{i+1})$ (respectively, $w(a_i) \prec w(a_{i+1})$). The number of excedances (respectively, inverse excedances, descents or ascents) of w will be denoted by $\text{exc}(w)$ (respectively, by $\text{iexc}(w)$, $\text{des}(w)$ or $\text{asc}(w)$). The n th Eulerian polynomial [27, Section 1.4] is defined by the formulas

$$(2.1) \quad A_n(x) = \sum_{w \in \mathfrak{S}(S)} x^{\text{exc}(w)} = \sum_{w \in \mathfrak{S}(S)} x^{\text{iexc}(w)} = \sum_{w \in \mathfrak{S}(S)} x^{\text{des}(w)} = \sum_{w \in \mathfrak{S}(S)} x^{\text{asc}(w)}.$$

Clearly, these sums depend only on n and not on S or the choice of total order $\preceq$.

The previous definitions apply in particular to $\mathfrak{S}_n$ (with the standard choice of $\preceq$ obtained by setting $a_i = i$ for $1 \leq i \leq n$). We will denote by $\mathcal{D}_n$ the set of all derangements (permutations without fixed points) in $\mathfrak{S}_n$.

2.2. Signed permutations. For the purposes of this paper, it will be convenient to define a *signed permutation* of $[n]$ as a choice of a subset $S = \{a_1, a_2, \dots, a_n\}$ of Ω_n such that $a_i \in \{i, -i\}$ for $1 \leq i \leq n$ and permutation $w \in \mathfrak{S}(S)$. We will represent such a permutation w as the sequence $(w(a_1), w(a_2), \dots, w(a_n))$, or as the word $w(a_1)w(a_2)\dots w(a_n)$, or as a

disjoint union of cycles. We will find it convenient to define the standard cycle form of w using the total order on S which is the reverse of the one inherited from the natural total order on $\mathbb{Z}$. Thus, cycles of w will be written with their *smallest* element first and in *decreasing* order of their smallest element. We will say that w is a *derangement* if there is no $a \in S \cap [n]$ such that $w(a) = a$. We will denote the set of all signed permutations of $[n]$ by B_n and the set of all derangements in B_n by $\mathcal{D}_n^B$.

Given $w \in B_n$ as before, we say that $i \in \{0, 1, \dots, n-1\}$ is a B -descent (respectively, B -ascent) of w if $w(a_i) > w(a_{i+1})$ (respectively, $w(a_i) < w(a_{i+1})$), where $w(a_0) = 0$ by convention. The n th Eulerian polynomial of type B [11, Section 3] can be defined by

$$(2.2) \quad B_n(x) = \sum_{w \in B_n} x^{\text{des}_B(w)} = \sum_{w \in B_n} x^{\text{asc}_B(w)},$$

where $\text{des}_B(w)$ stands for the number of B -descents and $\text{asc}_B(w)$ for the number of B -ascents of $w \in B_n$. Following Brenti [11, p. 431], we say that $a \in S$ is a B -excedance of w if $w(a) > a$, or if $-a \in [n]$ and $w(a) = a$. We say that $a \in S$ is an *inverse B -excedance* of w if $w(a) < a$, or if $-a \in [n]$ and $w(a) = a$. The number of B -excedances of w will be denoted by $\text{exc}_B(w)$ and that of inverse B -excedances by $\text{iexc}_B(w)$. We then have $\text{iexc}_B(w) = \text{exc}_B(w^{-1})$ and (see Theorem 3.15 and Corollary 3.16 in [11])

$$(2.3) \quad B_n(x) = \sum_{w \in B_n} x^{\text{exc}_B(w)}.$$

The n th derangement polynomial of type B is defined by (1.3). Since $\text{exc}_B(w) = \text{iexc}_B(w^{-1})$ and the map which sends a permutation $w \in \mathfrak{S}(S)$ to its inverse w^{-1} induces an involution on B_n which preserves fixed points, we have

$$(2.4) \quad d_n^B(x) = \sum_{w \in \mathcal{D}_n^B} x^{\text{iexc}_B(w)}.$$

For the similar reasons, (1.1) continues to hold if exc is replaced by iexc and (2.3) continues to hold if exc_B is replaced by iexc_B .

2.3. Polynomials. Let $p(x) = \sum_{k \geq 0} a_k x^k = \sum_{k=0}^d a_k x^k$ be a polynomial with real coefficients. We recall that $p(x)$ is unimodal (and has unimodal coefficients) if there exists an index $0 \leq j \leq d$ such that $a_i \leq a_{i+1}$ for $0 \leq i \leq j-1$ and $a_i \geq a_{i+1}$ for $j \leq i \leq d-1$. Such an index is called a *peak*. The polynomial $p(x)$ is said to be log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq d-1$ and to have internal zeros if there exist indices $0 \leq i < j < k \leq d$ such that $a_i, a_k \neq 0$ and $a_j = 0$. We will say that $p(x)$ is symmetric (and that it has symmetric coefficients) if there exists an integer $n \geq d$ such that $a_i = a_{n-i}$ for $0 \leq i \leq n$. The *center of symmetry* of $p(x)$ is then defined to be $n/2$ (this is well-defined provided $p(x)$ is nonzero).

We will say that $p(x)$ is *real-rooted* if all its complex roots are real. It is well-known (see, for instance, [24]) that if $p(x)$ is a real-rooted polynomial with nonnegative coefficients, then $p(x)$ is log-concave and unimodal, with no internal zeros. The following theorem, first proved by Edrei [18], gives a necessary and sufficient condition for a polynomial with nonnegative real coefficients to be real-rooted.

Theorem 2.1. ([18]) *Let $p(x) = \sum_{k \geq 0} a_k x^k \in \mathbb{R}[x]$ be a polynomial with $a_k \geq 0$ for every $k \in \mathbb{N}$ and set $a_k = 0$ for all negative integers k . Then $p(x)$ is real-rooted if and only if every minor of the lower triangular matrix $(a_{i-j})_{i,j=0}^\infty$ is nonnegative.*

A (nonzero) symmetric polynomial $p(x) \in \mathbb{R}[x]$ can be written (uniquely) in the form

$$(2.5) \quad p(x) = (1+x)^n \gamma \left(\frac{x}{(1+x)^2} \right) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

for some polynomial $\gamma(x) = \sum_{i \geq 0} \gamma_i x^i$. We say that $p(x)$ is γ -nonnegative if $\gamma_i \geq 0$ for every i . Clearly, every γ -nonnegative polynomial is unimodal. For classes of γ -nonnegative polynomials which appear in combinatorics we refer the reader, for instance, to [17] and references therein.

2.4. Simplicial complexes. An (abstract) simplicial complex Δ on the ground set V is a collection Δ of subsets of V such that $F \subseteq G \in \Delta$ implies $F \in \Delta$ (all simplicial complexes considered in this paper will be assumed to be finite). The elements of Δ are called *faces*. The *dimension* of a face is equal to one less than its cardinality. The *dimension* of Δ is the maximum dimension of its faces. Faces of dimension 0 and 1 are called *vertices* and *edges*, respectively. A *facet* of Δ is a face which is maximal with respect to inclusion. The complex Δ is said to be *pure* if all its facets have the same dimension. The *face poset* $\mathcal{F}(\Delta)$ of a simplicial complex Δ is the set of nonempty faces of Δ , partially ordered by inclusion.

The *open star* $\text{st}_\Delta(F)$ of a face $F \in \Delta$ is the collection of all faces of Δ containing F . The *link* of a face F in Δ is the subcomplex of Δ defined as $\text{link}_\Delta(F) = \{G \setminus F : G \in \Delta, F \subseteq G\}$. Suppose that Δ_1 and Δ_2 are simplicial complexes on disjoint ground sets. The *simplicial join* of Δ_1 and Δ_2 is the simplicial complex $\Delta_1 * \Delta_2$ whose faces are the sets of the form $F_1 \cup F_2$, where $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$. The *order complex* [8, Section 9.3] [27, Section 3.8] of a (finite) partially ordered set Q is defined as the simplicial complex of chains (totally ordered subsets) of Q .

All topological properties or invariants of Δ mentioned in the sequel will refer to those of its geometric realization $\|\Delta\|$ [8, Section 9.1]. For example, Δ is a *simplicial ball* if $\|\Delta\|$ is homeomorphic to a ball. For a simplicial d -dimensional ball Δ , we denote by $\partial\Delta$ the subcomplex consisting of all subsets of the $(d-1)$ -dimensional faces which are contained in a unique facet of Δ . We call $\partial\Delta$ the *boundary* and $\text{int}(\Delta) := \Delta \setminus \partial\Delta$ the *interior* of Δ .

2.5. Subdivisions. Let Δ be a simplicial complex. A (topological) *simplicial subdivision* of Δ [25, Section 2] is a simplicial complex Δ' together with a map $\sigma : \Delta' \rightarrow \Delta$ such that the following hold for every $F \in \Delta$: (a) the set $\Delta'_F := \sigma^{-1}(2^F)$ is a subcomplex of Δ' which is a simplicial ball of dimension $\dim(F)$; and (b) the interior of Δ'_F is equal to $\sigma^{-1}(F)$. The subcomplex Δ'_F is called the *restriction* of Δ' to F . The face $\sigma(G) \in \Delta$ is called the *carrier* of $G \in \Delta'$. The subdivision Δ' is called *quasi-geometric* [25, Definition 4.1 (a)] if no face of Δ' has the carriers of its vertices contained in a face of Δ of smaller dimension. Moreover, Δ' is called *geometric* [25, Definition 4.1 (b)] if there exists a geometric realization of Δ' which geometrically subdivides a geometric realization of Δ , in the way prescribed by σ .

Clearly, all geometric subdivisions (such as the barycentric subdivisions considered in this paper) are quasi-geometric.

We now describe two common ways to subdivide a simplicial complex Δ . The order complex of the face poset $\mathcal{F}(\Delta)$, denoted by $\text{sd}(\Delta)$, consists of the chains of nonempty faces of Δ . This complex is naturally a (geometric) simplicial subdivision of Δ , called the *barycentric subdivision*, where the carrier of a chain $\mathcal{C}$ of nonempty faces of Δ is defined as the maximum element of $\mathcal{C}$.

Given a face $F \in \Delta$ we set $\Delta' = (\Delta \setminus \text{st}_\Delta(F)) \cup (\{v\} * \partial(2^F) * \text{lk}_\Delta(F))$, where v is a new vertex added and $\partial(2^F) = 2^F \setminus F$. Then Δ' is a simplicial complex which is a simplicial subdivision of Δ , called the *stellar subdivision* of Δ on F .

2.6. Face enumeration. Let Δ be a $(d-1)$ -dimensional simplicial complex. We denote by $f_i(\Delta)$ the number of i -dimensional faces of Δ . A fundamental enumerative invariant of Δ is the *f-polynomial*, defined by

$$f_\Delta(x) = \sum_{i=0}^{d-1} f_i(\Delta) x^i.$$

The *h-polynomial* of Δ is defined by

$$h_\Delta(x) = \sum_{i=0}^d h_i(\Delta) x^i = (1-x)^d f_\Delta\left(\frac{x}{1-x}\right).$$

For the importance of *h*-polynomials, the reader is referred to [26, Chapter II]. For the simplicial join $\Delta_1 * \Delta_2$ of two simplicial complexes we have $h(\Delta_1 * \Delta_2, x) = h(\Delta_1, x)h(\Delta_2, x)$.

Let Γ be a simplicial subdivision of a $(d-1)$ -dimensional simplex 2^V . The polynomial $\ell_V(\Gamma, x) = \ell_0 + \ell_1 x + \cdots + \ell_d x^d$ defined by

$$(2.6) \quad \ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{d-|F|} h(\Gamma_F, x)$$

is the *local h-polynomial* of Γ (with respect to V) [25, Definition 2.1]. The sequence $\ell_V(\Gamma) = (\ell_0, \ell_1, \dots, \ell_d)$ is the *local h-vector* of Γ (with respect to V).

The following theorem summarizes some of the main properties of local *h*-vectors (see Theorems 3.2 and 3.3 and Corollary 4.7 in [25]). For the definition of regular subdivision we refer the reader to [25, Definition 5.1].

Theorem 2.2. (Stanley [25])

(a) For every simplicial subdivision Δ' of a pure simplicial complex Δ we have

$$(2.7) \quad h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x) h(\text{link}_\Delta(F), x).$$

- (b) The local *h-polynomial* $\ell_V(\Gamma, x)$ is symmetric for every simplicial subdivision Γ of the simplex 2^V , i.e. we have $\ell_i = \ell_{d-i}$ for $0 \leq i \leq d$.
- (c) The local *h-polynomial* $\ell_V(\Gamma, x)$ has nonnegative coefficients for every quasi-geometric simplicial subdivision Γ of the simplex 2^V .

- (d) *The local h -polynomial $\ell_V(\Gamma, x)$ has unimodal coefficients for every regular simplicial subdivision Γ of the simplex 2^V .*

3. PROOF OF THE MAIN FORMULA

This section gives two proofs of Theorem 1.1, one bijective and one using generating functions, and deduces Corollary 1.2. As a byproduct of the second proof, the exponential generating functions of $f_n^+(x)$ and $f_n^-(x)$ are computed.

First proof of Theorem 1.1. Let us denote by $\mathcal{C}_n$ the collection of sequences $(\sigma_0, \sigma_1, \dots, \sigma_k)$ of permutations, where $k \in \mathbb{N}$ and $\sigma_i \in \mathfrak{S}(S_i)$ for $0 \leq i \leq k$, such that $(S_0, S_1, \dots, S_k)$ is a weak ordered partition of $[n]$ with S_i nonempty for $1 \leq i \leq k$ and σ_0 is a derangement of S_0 . We will describe a one-to-one correspondence $\varphi : \mathcal{D}_n^B \rightarrow \mathcal{C}_n$ such that

$$(3.1) \quad \text{iexc}_B(w) = \text{iexc}(\sigma_0) + \sum_{i=1}^k f(\sigma_i) + \lfloor \frac{k+1}{2} \rfloor$$

for every $w \in \mathcal{D}_n^B$, where $(\sigma_0, \sigma_1, \dots, \sigma_k) = \varphi(w)$ and $f(\sigma_i)$ stands for $\text{des}(\sigma_i)$ or $\text{asc}(\sigma_i)$, if i is even or odd, respectively. Given this, using (2.4) and recalling that there are $\binom{n}{r_0, r_1, \dots, r_k}$ weak ordered partitions $(S_0, S_1, \dots, S_k)$ of $[n]$ satisfying $|S_i| = r_i$ for $0 \leq i \leq k$, we get

$$\begin{aligned} d_n^B(x) &= \sum \binom{n}{r_0, r_1, \dots, r_k} x^{\lfloor \frac{k+1}{2} \rfloor} \sum_{\sigma_0 \in \mathcal{D}_{r_0}} x^{\text{iexc}(\sigma_0)} \left(\prod_{i=1}^k \sum_{\sigma_i \in \mathfrak{S}_{r_i}} x^{\text{des}(\sigma_i)} \right) \\ &= \sum \binom{n}{r_0, r_1, \dots, r_k} x^{\lfloor \frac{k+1}{2} \rfloor} d_{r_0}(x) A_{r_1}(x) \cdots A_{r_k}(x) \end{aligned}$$

and the proof follows.

To define φ , consider a derangement $w \in \mathcal{D}_n^B$ and let $C_1 C_2 \cdots C_m$ be the standard cycle form of w . Then there is an index $j \in \{0, 1, \dots, m\}$ such that all elements of $C_1, C_2, \dots, C_j$ are positive and the first (smallest) element of C_{j+1} is negative. We define σ_0 as the product of $C_1, C_2, \dots, C_j$ and S_0 as the set of all elements which appear in these cycles, so that $\sigma_0 \in \mathfrak{S}(S_0)$ is a derangement. The remaining cycles $C_{j+1}, \dots, C_m$ form a word u whose first element is negative. This word decomposes uniquely as a product $u = u_1 u_2 \cdots u_k$ of subwords u_i so that for $1 \leq i \leq k$, all elements of u_i are negative if i is odd and positive if i is even. We define S_i as the set of absolute values of the elements of u_i and $\sigma_i \in \mathfrak{S}(S_i)$ as the permutation which corresponds to the word u_i . For instance, if $n = 9$ and $w = (3\ 7)(1\ 4)(-5\ 9\ -2)(-8\ -6)$ in standard cycle form, then $\sigma_0 = (1\ 4)(3\ 7)$ in cycle form, $k = 3$ and $\sigma_1 = (5)$, $\sigma_2 = (9)$, $\sigma_3 = (2, 8, 6)$, as sequences. We set $\varphi(w) = (\sigma_0, \sigma_1, \dots, \sigma_k)$ and leave it to the reader to verify that the map $\varphi : \mathcal{D}_n^B \rightarrow \mathcal{C}_n$ is a well defined bijection.

To verify (3.1) we let $w \in \mathcal{D}_n^B$ with $\varphi(w) = (\sigma_0, \sigma_1, \dots, \sigma_k)$ and $u = a_1 a_2 \cdots a_p$ be the word defined in the previous paragraph. Then, by the definitions of standard cycle form and (inverse) B -excedance, $a \in \Omega_n$ is an inverse B -excedance of w if and only if a is an inverse excedance of σ_0 , or $a = a_i$ for some index $1 \leq i < p$ with $a_i > a_{i+1}$, or $a = a_p$. Thus, equation (3.1) follows. $\square$

For the second proof of Theorem 1.1 we set

$$(3.2) \quad \mathcal{A}(t) := \sum_{n \geq 1} A_n(x) \frac{t^n}{n!} = \frac{e^t - e^{xt}}{e^{xt} - xe^t}$$

and (see [10, Proposition 5])

$$(3.3) \quad \mathcal{D}(t) := \sum_{n \geq 0} d_n(x) \frac{t^n}{n!} = \frac{1 - x}{e^{xt} - xe^t},$$

where $d_0(x) = 1$. We also recall (see [15, Theorem 3.3] [16, Theorem 3.2]) that

$$(3.4) \quad \sum_{n \geq 0} d_n^B(x) \frac{t^n}{n!} = \frac{(1 - x)e^{xt}}{e^{2xt} - xe^{2t}},$$

where $d_0^B(x) = 1$.

Second proof of Theorem 1.1. We denote by $S_n(x)$ (respectively, by $S_n^+(x)$ and $S_n^-(x)$) the right-hand side of (1.5) (respectively, of (1.9) and (1.10)), so that $S_n(x) = S_n^+(x) + S_n^-(x)$ for $n \in \mathbb{N}$. We compute that

$$\begin{aligned} \sum_{n \geq 0} S_n^+(x) \frac{t^n}{n!} &= \sum_{k, r_i \geq 0} x^k d_{r_0}(x) \frac{t^{r_0}}{r_0!} A_{r_1}(x) \frac{t^{r_1}}{r_1!} \cdots A_{r_{2k}}(x) \frac{t^{r_{2k}}}{r_{2k}!} \\ &= \sum_{n \geq 0} d_n(x) \frac{t^n}{n!} \sum_{k \geq 0} x^k \left(\sum_{r \geq 1} A_r(x) \frac{t^r}{r!} \right)^{2k} \\ &= \frac{\mathcal{D}(t)}{1 - x(\mathcal{A}(t))^2} \end{aligned}$$

and similarly that

$$\begin{aligned} \sum_{n \geq 0} S_n^-(x) \frac{t^n}{n!} &= \sum_{k, r_i \geq 0} x^{k+1} d_{r_0}(x) \frac{t^{r_0}}{r_0!} A_{r_1}(x) \frac{t^{r_1}}{r_1!} \cdots A_{r_{2k+1}}(x) \frac{t^{r_{2k+1}}}{r_{2k+1}!} \\ &= \sum_{n \geq 0} d_n(x) \frac{t^n}{n!} \sum_{k \geq 0} x^{k+1} \left(\sum_{r \geq 1} A_r(x) \frac{t^r}{r!} \right)^{2k+1} \\ &= \mathcal{D}(t) \cdot \frac{x\mathcal{A}(t)}{1 - x(\mathcal{A}(t))^2} \end{aligned}$$

and conclude that

$$\sum_{n \geq 0} S_n(x) \frac{t^n}{n!} = \mathcal{D}(t) \cdot \frac{1 + x\mathcal{A}(t)}{1 - x(\mathcal{A}(t))^2}.$$

Combining the previous equation with (3.2) and (3.3) we get, after some straightforward algebraic manipulations, that

$$\sum_{n \geq 0} S_n(x) \frac{t^n}{n!} = \frac{(1-x)e^{xt}}{e^{2xt} - xe^{2t}} = \sum_{n \geq 0} d_n^B(x) \frac{t^n}{n!}$$

and the proof follows. $\square$

Proof of Corollary 1.2. As in the second proof of Theorem 1.1, we denote by $S_n^+(x)$ and $S_n^-(x)$ the right-hand side of (1.9) and (1.10), respectively.

Theorem 1.1 shows that $d_n^B(x) = S_n^+(x) + S_n^-(x)$ for every $n \in \mathbb{N}$. From the symmetry properties $A_n(x) = x^{n-1}A_n(1/x)$ and $d_n(x) = x^n d_n(1/x)$ of the Eulerian and derangement polynomials for $\mathfrak{S}_n$ it follows that $S_n^+(x)$ and $S_n^-(x)$ satisfy (1.7) and (1.8), respectively. The uniqueness of the defining properties of $f_n^+(x)$ and $f_n^-(x)$ imply that $f_n^+(x) = S_n^+(x)$ and $f_n^-(x) = S_n^-(x)$ for every $n \in \mathbb{N}$. This proves equations (1.9) and (1.10).

The γ -nonnegativity of $f_n^+(x)$ and $f_n^-(x)$ follows from equations (1.9) and (1.10) and the γ -nonnegativity of $A_n(x)$ and $d_n(x)$ (see Proposition 3.1 in the sequel). The last statement in the corollary follows from (1.11) and (1.12). $\square$

Since the polynomials $A_n(x)$ and $d_n(x)$ have nonnegative and symmetric coefficients and only real roots, we can write

$$(3.5) \quad A_n(x) = (1+x)^{n-1} \gamma_n \left(\frac{x}{(1+x)^2} \right)$$

and

$$(3.6) \quad d_n(x) = (1+x)^n \xi_n \left(\frac{x}{(1+x)^2} \right)$$

for some polynomials $\gamma_n(x)$ and $\xi_n(x)$ with nonnegative coefficients. Explicit combinatorial interpretations to these coefficients are known (see, for instance, [19, Theorem 5.6] and [4, Section 4]). Equations (1.9), (1.10), (3.5) and (3.6) imply explicit combinatorial formulas for the polynomials $\xi_n^+(x) = \sum \xi_{n,i}^+ x^i$ and $\xi_n^-(x) = \sum \xi_{n,i}^- x^i$, appearing in Corollary 1.2, which we record in the following proposition.

Proposition 3.1. *We have*

$$(3.7) \quad f_n^+(x) = (1+x)^n \xi_n^+ \left(\frac{x}{(1+x)^2} \right)$$

and

$$(3.8) \quad f_n^-(x) = (1+x)^{n+1} \xi_n^- \left(\frac{x}{(1+x)^2} \right),$$

where

$$(3.9) \quad \xi_n^+(x) = \sum \binom{n}{r_0, r_1, \dots, r_{2k}} x^k \xi_{r_0}(x) \gamma_{r_1}(x) \cdots \gamma_{r_{2k}}(x),$$

$$(3.10) \quad \xi_n^-(x) = \sum \binom{n}{r_0, r_1, \dots, r_{2k+1}} x^{k+1} \xi_{r_0}(x) \gamma_{r_1}(x) \cdots \gamma_{r_{2k+1}}(x),$$

the sums in the previous equations range as in (1.9) and (1.10), respectively, and $\xi_0(x) = 1$, $\gamma_0(x) = 0$. $\square$

For the first few values of n we have

$$\xi_n^+(x) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n = 1 \\ 3x, & \text{if } n = 2 \\ 7x, & \text{if } n = 3 \\ 15x + 57x^2, & \text{if } n = 4 \\ 31x + 458x^2, & \text{if } n = 5 \\ 63x + 2551x^2 + 2763x^3, & \text{if } n = 6 \\ 127x + 12232x^2 + 46861x^3, & \text{if } n = 7 \end{cases}$$

and

$$\xi_n^-(x) = \begin{cases} 0, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ x, & \text{if } n = 2 \\ x + 11x^2, & \text{if } n = 3 \\ x + 54x^2, & \text{if } n = 4 \\ x + 197x^2 + 361x^3, & \text{if } n = 5 \\ x + 648x^2 + 4379x^3, & \text{if } n = 6 \\ x + 2039x^2 + 34586x^3 + 24611x^4, & \text{if } n = 7. \end{cases}$$

We are not aware of any combinatorial interpretations for the coefficients of $\xi_n^+(x)$ or $\xi_n^-(x)$.

The second proof of Theorem 1.1 and the proof of Corollary 1.2 yield the following explicit formulas for the exponential generating functions of $f_n^+(x)$ and $f_n^-(x)$.

Proposition 3.2. *We have*

$$(3.11) \quad \sum_{n \geq 0} f_n^+(x) \frac{t^n}{n!} = \frac{e^{xt} - xe^t}{e^{2xt} - xe^{2t}}$$

and

$$(3.12) \quad \sum_{n \geq 0} f_n^-(x) \frac{t^n}{n!} = \frac{x(e^t - e^{xt})}{e^{2xt} - xe^{2t}}.$$

Proof. We noticed in the proof of Corollary 1.2 that $f_n^+(x) = S_n^+(x)$ and $f_n^-(x) = S_n^-(x)$. Thus, the result follows from the formulas in the second proof of Theorem 1.1 and straightforward algebraic manipulations. $\square$

4. A COMBINATORIAL INTERPRETATION

This section gives a combinatorial interpretation to the coefficients of $f_n^+(x)$ and $f_n^-(x)$ by exploiting the first proof of Theorem 1.1, given in Section 3.

Consider a signed permutation $w \in \mathfrak{S}(S)$, where $S = \{a_1, a_2, \dots, a_n\}$ is as in Section 2.2. We denote by m_w the minimum element of S with respect to the natural total order inherited from $\mathbb{Z}$ and set $B_n^* = \{w \in B_n : w(m_w) > 0\}$.

Proposition 4.1. *We have*

$$(4.1) \quad f_n^+(x) = \sum_{w \in \mathcal{D}_n^B \cap B_n^*} x^{\text{exc}_B(w)}$$

and

$$(4.2) \quad f_n^-(x) = \sum_{w \in \mathcal{D}_n^B \setminus B_n^*} x^{\text{exc}_B(w)}$$

for every $n \geq 1$.

Proof. We will follow the setup of the first proof of Theorem 1.1. Given $w \in \mathcal{D}_n^B$ with $\varphi(w) = (\sigma_0, \sigma_1, \dots, \sigma_k)$, we observe that k is even if and only if the last element in the standard cycle form of w is positive. Therefore, equation (1.9) and the argument in the proof of Theorem 1.1 show that

$$f_n^+(x) = \sum x^{\text{iexc}_B(w)},$$

where the sum ranges over all $w \in \mathcal{D}_n^B$ for which the last element in the standard cycle form is positive. Since this element equals $w^{-1}(m_w)$, we get

$$f_n^+(x) = \sum_{w \in \mathcal{D}_n^B : w^{-1}(m_w) > 0} x^{\text{iexc}_B(w)} = \sum_{w \in \mathcal{D}_n^B : w(m_w) > 0} x^{\text{iexc}_B(w^{-1})} = \sum_{w \in \mathcal{D}_n^B \cap B_n^*} x^{\text{exc}_B(w)}.$$

Equation (4.2) follows from (4.1) and (1.3), or by a similar argument. $\square$

5. THE RELATIVE LOCAL h -VECTOR

This section reviews the definition of the relative local h -polynomial of a simplicial subdivision of a simplex, introduced in [3, Section 3] and, independently (in a different level of generality), in [21], and establishes some of its main properties (most of them stated without proof in [3, Section 3]). The relative local h -polynomial of the barycentric subdivision of the simplex is also computed (Example 5.2). This computation will be used in Section 6.

We will fix a field $\mathbf{k}$ in this section and work with the notion of a homology (rather than topological) simplicial subdivision over $\mathbf{k}$, as in [3]. Thus, in the definition of a subdivision $\sigma : \Delta' \rightarrow \Delta$ we require that the subcomplex $\Delta'_F := \sigma^{-1}(2^F)$ of Δ' is a homology (rather than topological) ball over $\mathbf{k}$ of dimension $\dim(F)$, for every $F \in \Delta$; see [3, Section 2] for details. The following concept was introduced in [3, Remark 3.7] and (for regular triangulations of polytopes) in [21].

Definition 5.1. ([3, Section 3]) Let Γ be a homology subdivision of a $(d-1)$ -dimensional simplex 2^V , with subdivision map $\sigma : \Gamma \rightarrow 2^V$, and let $E \in \Gamma$. The polynomial

$$(5.1) \quad \ell_V(\Gamma, E, x) = \sum_{\sigma(E) \subseteq F \subseteq V} (-1)^{d-|F|} h(\text{link}_{\Gamma_F}(E), x)$$

is the *relative local h -polynomial* of Γ (with respect to V) at E .

Thus, $\ell_V(\Gamma, E, x)$ reduces to the local h -polynomial $\ell_V(\Gamma, x)$ for $E = \emptyset$.

Example 5.2. Let $\Gamma = \text{sd}(2^V)$ be the barycentric subdivision of an $(n-1)$ -dimensional simplex 2^V and $E = \{S_1, S_2, \dots, S_k\}$ be a face of Γ , where $S_1 \subset S_2 \subset \dots \subset S_k \subseteq V$ are nonempty sets. We will show that

$$(5.2) \quad \ell_V(\Gamma, E, x) = d_{r_0}(x) A_{r_1}(x) A_{r_2}(x) \cdots A_{r_k}(x),$$

where $r_0 = |V \setminus S_k|$ and $r_i = |S_i \setminus S_{i-1}|$ for $1 \leq i \leq k$ (with the convention $S_0 = \emptyset$).

We recall from Section 2.5 that the carrier of E in Γ is given by $\sigma(E) = S_k$. Thus the right-hand side of (5.1) is a sum over all $S_k \subseteq F \subseteq V$. The restriction Γ_F is the barycentric subdivision of 2^F and the link of E in this restriction satisfies $\text{link}_{\Gamma_F}(E) = \Delta_0 * \Delta_1 * \dots * \Delta_k$, where Δ_i is the simplicial complex of all chains of subsets of V which strictly contain S_{i-1} and are strictly contained in S_i , for $1 \leq i \leq k$, and Δ_0 is the simplicial complex of all chains of subsets of V which strictly contain S_k and are strictly contained in F . As a result, we have

$$\begin{aligned} h(\text{link}_{\Gamma_F}(E), x) &= h(\Delta_0, x) h(\Delta_1, x) \cdots h(\Delta_k, x) \\ &= A_{|F \setminus S_k|}(x) A_{r_1}(x) A_{r_2}(x) \cdots A_{r_k}(x). \end{aligned}$$

Multiplying this equation with $(-1)^{d-|F|}$, summing over all $S_k \subseteq F \subseteq V$ and using (1.2) we get (5.2). $\square$

Our motivation for introducing the relative local h -polynomial comes from the following statement (for another motivation, see [21, Section 3]).

Proposition 5.3. ([3, Proposition 3.6]) *For every homology subdivision Γ of the simplex 2^V and every homology subdivision Γ' of Γ we have*

$$(5.3) \quad \ell_V(\Gamma', x) = \sum_{E \in \Gamma} \ell_E(\Gamma'_E, x) \ell_V(\Gamma, E, x).$$

We now confirm that the polynomial $\ell_V(\Gamma, E, x)$ shares two of the main properties of $\ell_V(\Gamma, x)$ and deduce a monotonicity property of local h -vectors. These results were stated without proof in [3, Remark 3.7]. Here we will sketch the proof, which follows closely ideas of [25] and their refinements in [2]. For that reason, we will assume familiarity with the corresponding proofs in [2, 25].

Theorem 5.4. *Let V be a set with d elements.*

- (a) *The relative local h -polynomial $\ell_V(\Gamma, E, x)$ has symmetric coefficients, in the sense that*

$$(5.4) \quad x^{d-|E|} \ell_V(\Gamma, E, 1/x) = \ell_V(\Gamma, E, x),$$

for every homology subdivision Γ of the simplex 2^V and every $E \in \Gamma$.

- (b) The relative local h -polynomial $\ell_V(\Gamma, E, x)$ has nonnegative coefficients for every quasi-geometric homology subdivision Γ of the simplex 2^V and every $E \in \Gamma$.

Proof. (a) The proof of [2, Theorem 4.2] can be adapted as follows. Using the defining equation (5.1) and [2, Proposition 2.1], we find that

$$\begin{aligned} x^{d-|E|} \ell_V(\Gamma, E, 1/x) &= \sum_{\sigma(E) \subseteq F \subseteq V} (-1)^{d-|F|} x^{d-|E|} h(\text{link}_{\Gamma_F}(E), 1/x) \\ &= \sum_{\sigma(E) \subseteq F \subseteq V} (-x)^{d-|F|} h(\text{int}(\text{link}_{\Gamma_F}(E)), x). \end{aligned}$$

An inclusion-exclusion argument, similar to the one in the proof of [2, (4.3)], shows that

$$h(\text{int}(\text{link}_{\Gamma_F}(E)), x) = \sum_{\sigma(E) \subseteq G \subseteq F} (x-1)^{|F|-|G|} h(\text{link}_{\Gamma_G}(E), x).$$

Replacing $h(\text{int}(\text{link}_{\Gamma_F}(E)), x)$ in the first formula by the right-hand side of the previous equation and changing the order of summation, as in the proof of [2, Theorem 4.2], results in (5.4).

(b) The special case $E = \emptyset$ is equivalent to part (iii) of [3, Theorem 3.3] (essentially, part (c) of Theorem 2.2). The general case follows by the argument in the proof of [2, Theorem 5.1] (generalizing that in the proof of [25, Theorem 4.6]), where the role of Δ in that proof is played by $\text{link}_{\Gamma}(E)$, the role of d is played by $d - |E| = \dim \text{link}_{\Gamma}(E) + 1$ and the role of e is played by the rank $d - |\sigma(E)|$ of the interval $[\sigma(E), V]$ in the lattice of subsets of V . $\square$

For polynomials $p(x), q(x) \in \mathbb{R}[x]$ we write $p(x) \geq q(x)$ if the difference $p(x) - q(x)$ has nonnegative coefficients.

Corollary 5.5. *For every quasi-geometric homology subdivision Γ of the simplex 2^V and every quasi-geometric homology subdivision Γ' of Γ , we have $\ell_V(\Gamma', x) \geq \ell_V(\Gamma, x)$.*

Proof. The right-hand side of (5.3) reduces to $\ell_V(\Gamma, x)$ for $E = \emptyset$. The other terms in the sum are nonnegative by Theorems 2.2 (c) and 5.4 (b) and the proof follows. $\square$

6. A GEOMETRIC INTERPRETATION

This section formally defines the simplicial subdivision K_n and gives two proofs of Theorem 1.3, one using the theory of (relative) local h -vectors (specifically, Proposition 5.3) and another using generating functions.

Let Δ be a simplicial complex. The *cubical barycentric subdivision* (see, for instance, [5, Section 2.3]) of Δ , denoted $\text{sd}_c(\Delta)$, is defined as the set of all nonempty closed intervals $[F, G]$ in the face poset $\mathcal{F}(\Delta)$, partially ordered by inclusion. It follows from [31, Theorem 6.1 (a)] and [27, Equation (3.24)] that the order complex, say Δ' , of $\text{sd}_c(\Delta)$ is homeomorphic to Δ . Moreover, Δ' is naturally a simplicial subdivision of Δ : the carrier of a face of Δ' is the maximum element of the largest of the intervals in the corresponding

chain of intervals of $\mathcal{F}(\Delta)$. We will denote by K_n the order complex of $\text{sd}_c(2^{[n]})$, so that K_n is a simplicial subdivision of the simplex $2^{[n]}$ (see Figure 1 for the case $n = 3$). We note that K_n is the special case $N = 1$ of a subdivision of the simplex considered in [14, p. 414].

The following statement is an essential step for both proofs of Theorem 1.3 which will be given in this section.

Proposition 6.1. *We have $h(K_n, x) = B_n^+(x)$ for $n \in \mathbb{N}$.*

Proof. The poset $\text{sd}_c(2^{[n]})$ consists of all intervals of the form $[A, B]$, where $\emptyset \neq A \subseteq B \subseteq [n]$, partially ordered by inclusion. To describe this poset differently, we consider the following poset $(P_n, \preceq)$. The elements of P_n are the subsets of Ω_n which contain at least one positive number and at most one number from each set $\{i, -i\}$ for $i \in \{1, 2, \dots, n\}$; the partial order is reverse inclusion. We observe that the map $\varphi : \text{sd}_c(2^{[n]}) \rightarrow P_n$ defined by $\varphi([A, B]) = A \cup (-([n] \setminus B))$ is a poset isomorphism. Thus, we may identify K_n with the order complex of P_n .

For $S = \{s_1, s_2, \dots, s_k\} \subseteq [n]$ with $s_1 < s_2 < \dots < s_k$, we define $\alpha_{P_n}(S)$ as the number of chains $F_1 \prec F_2 \prec \dots \prec F_k$ in P_n such that $|F_i| = s_{k-i+1}$ for $i \in \{1, 2, \dots, k\}$. The map $\alpha_{P_n} : 2^{[n]} \rightarrow \mathbb{N}$ is the flag f -vector of P_n ; see [27, Section 3.13]. The chains of P_n enumerated by $\alpha_{P_n}(S)$ are in one-to-one correspondence with the elements $w \in B_n^+$ for which $\text{Des}_B(w) \subseteq n - S := \{n - s : s \in S\}$. Indeed, given such a chain, the corresponding element of B_n^+ consists of the elements of $[n] \setminus \{|s| : s \in F_1\}$ in increasing order, followed by those of $F_1 \setminus F_2$ in increasing order and so on, followed at the end by the elements of F_k in increasing order.

Recall that the flag h -vector $\beta_{P_n} : 2^{[n]} \rightarrow \mathbb{Z}$ of P_n is defined by

$$\beta_{P_n}(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_{P_n}(T),$$

for $S \subseteq [n]$, or equivalently, by

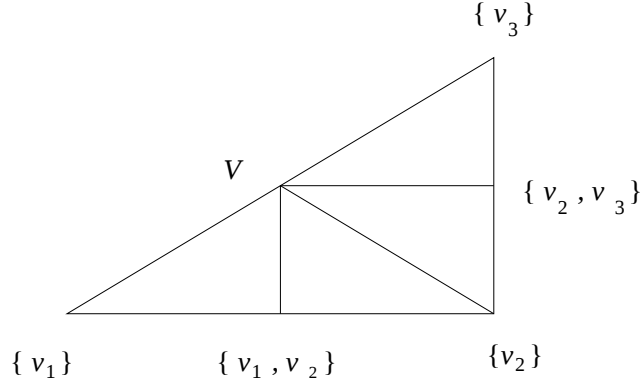
$$\alpha_{P_n}(S) = \sum_{T \subseteq S} \beta_{P_n}(T)$$

for $S \subseteq [n]$. Since $\alpha_{P_n}(S)$ enumerates signed permutations $w \in B_n^+$ for which $\text{Des}_B(w) \subseteq n - S$, by the Principle of Inclusion-Exclusion we get that $\beta_{P_n}(S)$ enumerates signed permutation $w \in B_n^+$ for which $\text{Des}_B(w) = n - S$. The result follows from this interpretation by recalling [27, Section 3.13] that

$$h_k(K_n) = \sum_{S \subseteq [n], |S|=k} \beta_{P_n}(S)$$

and switching S to $n - S$ in the previous equation. $\square$

Our first proof of Theorem 1.3 will be based on the fact that K_n can be viewed as a subdivision of the barycentric subdivision $\text{sd}(2^{[n]})$. To explain how, we consider the following setup. Let $V = \{v_1, v_2, \dots, v_d\}$ be a set totally ordered by $v_1 < v_2 < \dots < v_d$. We recall that $\text{sd}_c(V)$ denotes the poset of intervals in V of the form $[v_i, v_j] = \{v_i, v_{i+1}, \dots, v_j\}$ for $1 \leq i \leq j \leq d$, partially ordered by inclusion. We denote by Γ the order complex of

FIGURE 2. The subdivision Γ for $d = 3$

$\text{sd}_c(V)$, consisting of all chains of such intervals. For such a chain $G \in \Gamma$, we define $\sigma(G)$ as the set of all endpoints of the intervals in G . Thus we have a well defined map $\sigma : \Gamma \rightarrow 2^V$.

Lemma 6.2. *Under the previous assumptions and notation, the map $\sigma : \Gamma \rightarrow 2^V$ turns Γ into a geometric subdivision of 2^V . The number of facets of Γ is equal to 2^{d-1} , where d is the number of elements of V .*

Proof. Let Σ_V be a geometric $(d-1)$ -dimensional simplex whose vertices are labeled by the singleton subsets of $V = \{v_1, v_2, \dots, v_d\}$. We will construct a geometric simplicial subdivision (triangulation) Γ_V of Σ_V whose vertices are labeled (in a one-to-one fashion) with the closed intervals in the total order V , so that: (a) the singleton intervals label the vertices of Σ_V ; (b) the point labeled by a non-singleton interval $I = [v_i, v_j] \in \text{sd}_c(V)$ lies in the relative interior of the edge of Σ_V whose endpoints are labeled by $\{v_i\}$ and $\{v_j\}$; and (c) the faces of Γ_V correspond to the chains of intervals (see Figure 2 for the case $d = 3$).

We proceed by induction on d . The triangulation Γ_V is a single point for $d = 1$ and the triangulation of a line segment with one interior point (labeled by $\{v_1, v_2\}$) for $d = 2$. We assume $d \geq 3$ and set $U = V \setminus \{v_d\}$ and $W = V \setminus \{v_1\}$. We choose the simplices Σ_U and Σ_W as the codimension one faces of Σ_V which correspond to U and W and, using the inductive hypothesis, triangulations Γ_U and Γ_W of these two simplices having properties (a), (b) and (c) with respect to the totally ordered subsets U and W of V , respectively. Clearly, we may choose these triangulations to have the same restriction on the face $\Sigma_U \cap \Sigma_W$ of Σ_V . We then label by V an arbitrary point p in the relative interior of the edge of Σ_V whose endpoints are labeled with $\{v_1\}$ and $\{v_d\}$ and define Γ_V as the collection consisting of all simplices in $\Gamma_U \cup \Gamma_W$ and the cones of these on the vertex p . We leave it to the reader to verify that Γ_V has properties (a), (b) and (c) and that it realizes an abstract simplicial subdivision of 2^V with the required properties. $\square$

We now recall that K_n consists of all chains of intervals of the form $[A, B]$, where $\emptyset \neq A \subseteq B \subseteq [n]$. We define the carrier of such a chain $\mathcal{C}$ as the set of all endpoints of the intervals in $\mathcal{C}$ and note that this set is a chain in the poset $\mathcal{F}(2^{[n]})$ of nonempty subsets of $[n]$ and hence belongs to the barycentric subdivision $\text{sd}(2^{[n]})$. Applying Lemma 6.2 to

an arbitrary chain $V \in \text{sd}(2^{[n]})$ we conclude that K_n is a subdivision of $\text{sd}(2^{[n]})$ and that the restriction of this subdivision to a nonempty face $V \in \text{sd}(2^{[n]})$ of dimension $d - 1$ has exactly 2^{d-1} facets.

Lemma 6.3. *Let Γ be a quasi-geometric simplicial subdivision of a $(d - 1)$ -dimensional simplex 2^V . If the restriction Γ_F has exactly $2^{\dim(F)}$ facets for every nonempty face F of 2^V , then*

$$\ell_V(\Gamma, x) = \begin{cases} x^{d/2}, & \text{if } d \text{ is even} \\ 0, & \text{if } d \text{ is odd.} \end{cases}$$

Proof. We recall that the number of facets of a simplicial complex Δ is equal to the value of the h -polynomial $h(\Delta, x)$ at $x = 1$. Thus, setting $x = 1$ in the defining equation (2.6) and using the assumption on Γ , we find that

$$\ell_V(\Gamma, 1) = (-1)^d + \sum_{k=1}^d (-1)^{d-k} \binom{d}{k} 2^{k-1} = \begin{cases} 1, & \text{if } d \text{ is even} \\ 0, & \text{if } d \text{ is odd} \end{cases}$$

and the result follows from parts (b) and (c) of Theorem 2.2. $\square$

First proof of Theorem 1.3. Let us denote by $\ell_n^+(x)$ the local h -polynomial of K_n . To compute this polynomial, we will apply Proposition 5.3 to $\Gamma' = K_n$ and $\Gamma = \text{sd}(2^{[n]})$. Let $E = \{S_1, S_2, \dots, S_k\}$ be a face of Γ with k elements, where $S_1 \subset S_2 \subset \dots \subset S_k \subseteq [n]$ are nonempty sets. We have already noted that the restriction Γ'_E satisfies the assumptions of Lemma 6.3. Thus, by Lemma 6.3 we have

$$(6.1) \quad \ell_E(\Gamma'_E, x) = \begin{cases} x^{k/2}, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

The relative local h -vector of Γ was computed in Example 5.2. Thus, in view of (6.1) and (5.2), Proposition 5.3 yields that

$$\ell_n^+(x) = \sum \binom{n}{r_0, r_1, \dots, r_k} x^{k/2} d_{r_0}(x) A_{r_1}(x) \cdots A_{r_k}(x),$$

where the sum ranges over all even numbers $k \in \mathbb{N}$ and over all sequences $(r_0, r_1, \dots, r_k)$ of nonnegative integers which sum to n . This equation and (1.9) imply that $\ell_n^+(x) = f_n^+(x)$ and the first statement of Theorem 1.3 follows.

We leave to the reader to verify that K_n can be obtained from the trivial subdivision of the simplex by successive stellar subdivisions. This implies that K_n is a regular subdivision. The claim that $f_n^+(x)$ has nonnegative, symmetric and unimodal coefficients follows from the main properties of local h -polynomials [25] (see Theorem 2.2). Equation (1.15) follows from the fact that $f_n^+(x) = \ell_n^+(x)$, the defining equation (2.6) of local h -polynomials and Proposition 6.1. Given that $d_n^B(x) = f_n^+(x) + f_n^-(x)$ and $B_n(x) = B_n^+(x) + B_n^-(x)$ for every n , equation (1.16) is a consequence of (1.4) and (1.15). $\square$

For the second proof of Theorem 1.3 we will need the exponential generating functions of $B_n^+(x)$ and $B_n^-(x)$. These will be computed in Section 7.

Second proof of Theorem 1.3. Let us denote by $\ell_n^+(x)$ and $\ell_n^-(x)$ the right-hand side of (1.15) and (1.16), respectively. Proposition 6.1 and (2.6) imply that $\ell_n^+(x)$ is equal to the local h -polynomial of K_n . Thus, we need to show that $\ell_n^+(x) = f_n^+(x)$ and $\ell_n^-(x) = f_n^-(x)$ for every n . From the definition of $\ell_n^+(x)$ and $\ell_n^-(x)$ and Proposition 7.7 we get

$$\sum_{n \geq 0} \ell_n^+(x) \frac{t^n}{n!} = e^{-t} \sum_{n \geq 0} B_n^+(x) \frac{t^n}{n!} = \frac{e^{xt} - xe^t}{e^{2xt} - xe^{2t}}$$

and

$$\sum_{n \geq 0} \ell_n^-(x) \frac{t^n}{n!} = e^{-t} \sum_{n \geq 0} B_n^-(x) \frac{t^n}{n!} = \frac{x(e^t - e^{xt})}{e^{2xt} - xe^{2t}}.$$

The result follows from these equations and Proposition 3.2. $\square$

7. A DECOMPOSITION OF THE EULERIAN POLYNOMIAL OF TYPE B

This section studies the decomposition of the Eulerian polynomial $B_n(x)$ as a sum of $B_n^+(x)$ and $B_n^-(x)$. First, it is observed that a simple relation between the two summands holds. Then, using the theory of local h -vectors and results of Section 6, a simple formula for $B_n^+(x)$ in terms of the Eulerian polynomial $A_n(x)$ is proven (Proposition 7.2). From this formula, it is deduced that $B_n^+(x)$ and $B_n^-(x)$ are real-rooted (Corollary 7.5), hence unimodal and log-concave, and a new proof of the unimodality of $B_n(x)$ is derived. Finally, recurrences and generating functions for $B_n^+(x)$ and $B_n^-(x)$ are given. These lead to recurrences and generating functions for $f_n^+(x)$ and $f_n^-(x)$ and to yet another proof of Theorem 1.3.

We recall that $B_n^+(x)$ and $B_n^-(x)$ are defined by (1.13) and (1.14). For the first few values of n we have

$$B_n^+(x) = \begin{cases} 1, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 1 + 3x, & \text{if } n = 2 \\ 1 + 16x + 7x^2, & \text{if } n = 3 \\ 1 + 61x + 115x^2 + 15x^3, & \text{if } n = 4 \\ 1 + 206x + 1056x^2 + 626x^3 + 31x^4, & \text{if } n = 5 \\ 1 + 659x + 7554x^2 + 11774x^3 + 2989x^4 + 63x^5, & \text{if } n = 6 \end{cases}$$

and

$$B_n^-(x) = \begin{cases} 0, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ 3x + x^2, & \text{if } n = 2 \\ 7x + 16x^2 + x^3, & \text{if } n = 3 \\ 15x + 115x^2 + 61x^3 + x^4, & \text{if } n = 4 \\ 31x + 626x^2 + 1056x^3 + 206x^4 + x^5, & \text{if } n = 5 \\ 63x + 2989x^2 + 11774x^3 + 7554x^4 + 659x^5 + x^6, & \text{if } n = 6. \end{cases}$$

The previous data suggest the following statement.

Lemma 7.1. *We have $B_n^-(x) = x^n B_n^+(1/x)$ for $n \geq 1$.*

Proof. Given a signed permutation $w = (w(a_1), w(a_2), \dots, w(a_n)) \in B_n$, where the notation is as in Section 2.2, we set $-w := (-w(-a_1), -w(-a_2), \dots, -w(-a_n)) \in B_n$. Then the induced map $\varphi : B_n^+ \rightarrow B_n^-$ defined by $\varphi(w) = -w$ is a bijection. Moreover, for every $w \in B_n^+$, an index $i \in \{0, 1, \dots, n-1\}$ is a B -ascent of w if and only if i is a B -descent of $\varphi(w)$ and the proof follows. $\square$

To prove the formula for $B_n^+(x)$ promised, we will use the construction of the r th edgewise subdivision $\Delta^{(r)}$ of a simplicial complex Δ . We refer the reader to [13, 12] for the definition and history of this subdivision and recall the following known facts. First, the restriction $\Delta_F^{(r)}$ of $\Delta^{(r)}$ has exactly $r^{\dim(F)}$ facets for every nonempty face $F \in \Delta$. Second, combining [13, Corollary 6.8] with [12, Corollary 1.2], one gets the explicit formula

$$(7.1) \quad h(\Delta^{(r)}, x) = E_r((1 + x + \dots + x^{r-1})^d h(\Delta, x))$$

for the h -polynomial of $\Delta^{(r)}$, where $d-1$ is the dimension of Δ and E_r is the operator on polynomials (more generally, on formal power series) defined by

$$E_r\left(\sum_{k \geq 0} c_k x^k\right) = \sum_{k \geq 0} c_{rk} x^k = c_0 + c_r x + c_{2r} x^2 + \dots.$$

Figure 3 shows the second edgewise subdivision of the barycentric subdivision of the 2-dimensional simplex.

Proposition 7.2. *We have $B_n^+(x) = E_2((1+x)^n A_n(x))$ for every $n \geq 1$.*

Proof. We consider the subdivision K_n and the second edgewise subdivision K'_n of the barycentric subdivision $\text{sd}(2^{[n]})$ (see Figures 1 and 3 for the special case $n = 3$). Applying (2.7) for $\Delta' = K_n$ or K'_n , respectively, and $\Delta = \text{sd}(2^{[n]})$ we get

$$\begin{aligned} h(K_n, x) &= \sum_{F \in \Delta} \ell_F((K_n)_F, x) h(\text{link}_\Delta(F), x), \\ h(K'_n, x) &= \sum_{F \in \Delta} \ell_F((K'_n)_F, x) h(\text{link}_\Delta(F), x). \end{aligned}$$

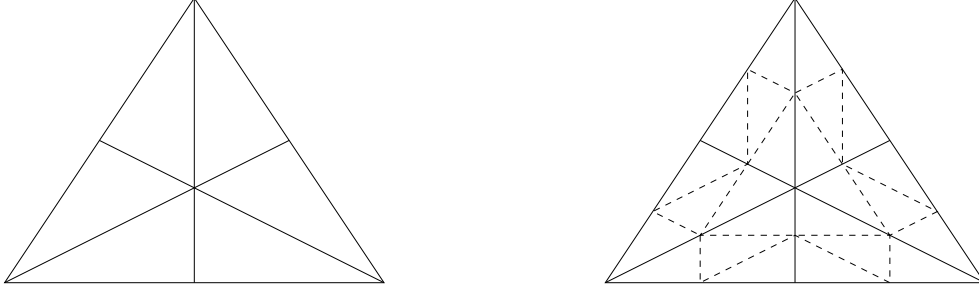


FIGURE 3. The barycentric subdivision of the 2-simplex and its second edge-wise subdivision K'_3

Since both restrictions $(K_n)_F$ and $(K'_n)_F$ have exactly $2^{\dim(F)}$ facets for every nonempty face $F \in \Delta$, it follows from the previous formulas and Lemma 6.3 that $h(K_n, x) = h(K'_n, x)$. Combining this equality with Proposition 6.1 we get $B_n^+(x) = h(K'_n, x)$ for every $n \geq 1$. Formula (7.1) implies that

$$B_n^+(x) = E_2((1+x)^n h(\Delta, x)) = E_2((1+x)^n A_n(x))$$

for $n \geq 1$ and the proof follows. $\square$

Remark 7.3. We thank Mirkó Visontai for informing us that a formula similar to the one in Proposition 7.2 can be derived from [1, Theorem 4.4], for which a bijective proof was given in [20].

We will use the following lemma to deduce the real-rootedness of $B_n^+(x)$ and $B_n^-(x)$.

Lemma 7.4. *Let $p(x)$ be a polynomial with real coefficients and let r be a positive integer.*

- (a) *If $p(x)$ has unimodal coefficients, then so does $E_r(p(x))$.*
- (b) *If $p(x)$ has nonnegative and log-concave coefficients, with no internal zeros, then so does $E_r(p(x))$.*
- (c) *If $p(x)$ is real-rooted, then so is $E_r(p(x))$.*

Proof. Part (a) is trivial and part (b) can be left as an exercise. For part (c) we set $p(x) = \sum_{k \geq 0} a_k x^k$ and note that the matrix $(a_{ri-rj})_{i,j=0}^\infty$ is a submatrix of $(a_{i-j})_{i,j=0}^\infty$. Therefore, every minor of the former is also a minor of the latter and the result follows from Theorem 2.1. $\square$

Corollary 7.5. *The polynomials $B_n^+(x)$ and $B_n^-(x)$ are real-rooted for every $n \geq 1$. They are unimodal with peaks at $\lfloor n/2 \rfloor$ and $\lfloor (n+1)/2 \rfloor$, respectively, for every $n \geq 2$.*

Proof. The first statement follows from Lemma 7.1, Proposition 7.2 and the fact that the Eulerian polynomial $A_n(x)$ is real-rooted for $n \geq 1$, via part (c) of Lemma 7.4. The second statement follows from Lemma 7.1, Proposition 7.2 and the fact that $(1+x)^n A_n(x)$ is a polynomial of degree $2n-1$ with symmetric and unimodal coefficients, via part (a) of Lemma 7.4. $\square$

Remark 7.6. Since $B_n(x) = B_n^+(x) + B_n^-(x)$, Lemma 7.1 and Proposition 7.2 express the Eulerian polynomial $B_n(x)$ as a sum of two unimodal polynomials with peaks which differ by at most one (see Corollary 7.5). This decomposition shows that the unimodality of $B_n(x)$ is a consequence of the unimodality of $A_n(x)$. For a γ -nonnegativity proof of the unimodality of $B_n(x)$, see [22, Proposition 4.16]. For an equation relating the Eulerian polynomials of types A , B and D , see [30, Lemma 9.1].

We will now give recurrences and generating functions for $B_n^+(x)$ and $B_n^-(x)$.

Proposition 7.7. *We have*

$$(7.2) \quad B_n^+(x) = 2(n-1)x B_{n-1}^+(x) + 2x(1-x) \frac{\partial B_{n-1}^+}{\partial x}(x) + B_{n-1}(x)$$

for every $n \geq 1$,

$$(7.3) \quad \sum_{n \geq 0} B_n^+(x) \frac{t^n}{n!} = \frac{e^t(e^{xt} - xe^t)}{e^{2xt} - xe^{2t}}$$

and

$$(7.4) \quad \sum_{n \geq 0} B_n^-(x) \frac{t^n}{n!} = \frac{xe^t(e^t - e^{xt})}{e^{2xt} - xe^{2t}}.$$

Proof. Let $w = u_1 u_2 \cdots u_{n-1} \in B_{n-1}$ be a signed permutation, represented as a word. For $i \in \{1, 2, \dots, n\}$, we will denote by w_i (respectively, w_{-i}) the signed permutation in B_n obtained from w by inserting n (respectively, $-n$) between u_{i-1} and u_i . For $1 \leq i \leq n-1$ we have $w_i \in B_n^+$ (respectively, $w_{-i} \in B_n^+$) if and only if $w \in B_{n-1}^+$. On the other hand, $w_n \in B_n^+$ and $w_{-n} \in B_n^-$ for every $w \in B_{n-1}$. Moreover, for $1 \leq i \leq n-1$ we have

$$\text{des}_B(w_{\pm i}) = \begin{cases} \text{des}_B(w), & \text{if } i-1 \in \text{Des}_B(w) \\ \text{des}_B(w) + 1, & \text{if } i-1 \notin \text{Des}_B(w) \end{cases}$$

and $\text{des}_B(w_n) = \text{des}_B(w)$. Thus, we compute that

$$\begin{aligned} B_n^+(x) &= \sum_{\sigma \in B_n^+} x^{\text{des}_B(\sigma)} = \sum_{i=1}^{n-1} \left(\sum_{w \in B_{n-1}^+} x^{\text{des}_B(w_i)} + x^{\text{des}_B(w_{-i})} \right) + \sum_{w \in B_{n-1}} x^{\text{des}_B(w_n)} \\ &= 2 \sum_{w \in B_{n-1}^+} (\text{des}_B(w) x^{\text{des}_B(w)} + (n-1 - \text{des}_B(w)) x^{\text{des}_B(w)+1}) + B_{n-1}(x) \\ &= 2(n-1) \sum_{w \in B_{n-1}^+} x^{\text{des}_B(w)+1} + 2(1-x) \sum_{w \in B_{n-1}^+} \text{des}_B(w) x^{\text{des}_B(w)} + B_{n-1}(x) \\ &= 2(n-1)x B_{n-1}^+(x) + 2x(1-x) \frac{\partial B_{n-1}^+}{\partial x}(x) + B_{n-1}(x), \end{aligned}$$

which proves (7.2). We now claim that

$$(7.5) \quad \frac{B_n^+(x)}{(1-x)^n} = \sum_{i \geq 0} ((2i+1)^n - (2i)^n) x^i.$$

Given that $B_0^+(x) = 1$, equation (7.3) then follows by straightforward computations. To prove (7.5), denote by $a_n(i)$ the coefficient of x^i in the expansion of $B_n^+(x)/(1-x)^n$ as a formal power series. Dividing (7.2) by $(1-x)^n$ and using the equality

$$\frac{\partial}{\partial x} \left(\frac{B_{n-1}^+(x)}{(1-x)^{n-1}} \right) = \frac{\frac{\partial B_{n-1}^+(x)}{\partial x}}{(1-x)^{n-1}} + (n-1) \frac{B_{n-1}^+(x)}{(1-x)^n}$$

we find that

$$\frac{B_n^+(x)}{(1-x)^n} = 2x \frac{\partial}{\partial x} \left(\frac{B_{n-1}^+(x)}{(1-x)^{n-1}} \right) + \frac{B_{n-1}^+(x)}{(1-x)^n}.$$

Comparing the coefficients of x^i in the two sides of the previous equation and using [11, Theorem 3.4 (ii)], we get $a_n(i) = 2ia_{n-1}(i) + (2i+1)^{n-1}$. The claim then follows by induction on n .

Equation (7.4) follows from (7.3) and Lemma 7.1. Alternatively, it follows from (7.4) and the formula for the exponential generating function of $B_n(x)$ [11, Theorem 3.4 (iv)]. $\square$

We now deduce recurrence relations for $f_n^+(x)$ and $f_n^-(x)$.

Proposition 7.8. *For $n \geq 1$ we have*

$$f_n^+(x) = (2(n-1)x - 1) f_{n-1}^+(x) + 2x(1-x) \frac{\partial f_{n-1}^+(x)}{\partial x} + 2(n-1)x f_{n-2}^+(x) + d_{n-1}^B(x).$$

Proof. Using equation (1.15), we compute that

$$\begin{aligned} f_n^+(x) &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k^+(x) \\ &= \sum_{k=1}^n (-1)^{n-k} \binom{n-1}{k-1} B_k^+(x) + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} B_k^+(x) \\ &= \sum_{k=1}^n (-1)^{n-k} \binom{n-1}{k-1} B_k^+(x) - f_{n-1}^+(x). \end{aligned}$$

Substituting for $B_k^+(x)$ the right-hand side of (7.2), setting

$$S_n(x) = 2x \sum_{k=1}^n (-1)^{n-k} (k-1) \binom{n-1}{k-1} B_{k-1}^+(x)$$

and using (1.4), we get

$$\begin{aligned}
f_n^+(x) &= S_n(x) + 2x(1-x) \sum_{k=1}^n (-1)^{n-k} \binom{n-1}{k-1} \frac{\partial B_{k-1}^+}{\partial x}(x) \\
&\quad + \sum_{k=1}^n (-1)^{n-k} \binom{n-1}{k-1} B_{k-1}(x) - f_{n-1}^+(x) \\
&= S_n(x) + 2x(1-x) \frac{\partial f_{n-1}^+}{\partial x}(x) + d_{n-1}^B(x) - f_{n-1}^+(x).
\end{aligned}$$

Finally, using equation (1.15), we compute that

$$\begin{aligned}
S_n(x) &= 2x \sum_{k=1}^n (-1)^{n-k} k \binom{n-1}{k-1} B_{k-1}^+(x) - 2x \sum_{k=1}^n (-1)^{n-k} \binom{n-1}{k-1} B_{k-1}^+(x) \\
&= 2x \sum_{k=1}^n (-1)^{n-k} k \binom{n}{k} B_{k-1}^+(x) - 2x \sum_{k=1}^{n-1} (-1)^{n-k} k \binom{n-1}{k} B_{k-1}^+(x) \\
&\quad - 2x f_{n-1}^+(x) \\
&= 2nx \sum_{k=1}^n (-1)^{n-k} \binom{n-1}{k-1} B_{k-1}^+(x) - 2(n-1)x \sum_{k=1}^n (-1)^{n-k} \binom{n-2}{k-1} B_{k-1}^+(x) \\
&\quad - 2x f_{n-1}^+(x) \\
&= 2(n-1)x f_{n-1}^+(x) + 2(n-1)x f_{n-2}^+(x)
\end{aligned}$$

and the proof follows. $\square$

We will denote by $a_{n,k}^+$, $a_{n,k}^-$ and $d_{n,k}^B$ the coefficient of x^k in $f_n^+(x)$, $f_n^-(x)$ and $d_n^B(x)$, respectively. The following recurrence relations can be derived from Proposition 7.8 and [15, Corollary 4.3].

Corollary 7.9. *For $n \geq 2$ and $k \geq 1$ we have*

$$(7.6) \quad a_{n,k}^+ = (2k-1)a_{n-1,k}^+ + 2(n-k)a_{n-1,k-1}^+ + 2(n-1)a_{n-2,k-1}^+ + d_{n-1,k}^B$$

and

$$(7.7) \quad a_{n,k}^- = (2k-1)a_{n-1,k}^- + 2(n-k)a_{n-1,k-1}^- + 2(n-1)a_{n-2,k-1}^- + d_{n-1,k-1}^B.$$

Proof. Equation (7.6) follows from the formula of Proposition 7.8 by comparing the coefficients of x^k . Since $a_{n,k}^- = d_{n,k}^B - a_{n,k}^+$, equation (7.7) follows from (7.6) and the recurrence relation for $d_{n,k}^B$ given in [15, Corollary 4.3]. $\square$

Third proof of Theorem 1.3. As in the second proof, we denote by $\ell_n^+(x)$ and $\ell_n^-(x)$ the right-hand sides of (1.15) and (1.16), respectively, and note that $\ell_n^+(x) + \ell_n^-(x) = d_n^B(x)$ and that $\ell_n^+(x)$ is equal to the local h -polynomial of K_n . In particular, we have $\ell_n^+(x) = x^n \ell_n^+(1/x)$ by Theorem 2.2 (b). The proofs of Proposition 7.8 and Corollary 7.9 show that the coefficients of $\ell_n^+(x)$ and $\ell_n^-(x)$ satisfy (7.6) and (7.7), respectively. Since $a_{n,k}^- = d_{n,k}^B - a_{n,k}^+$, we may rewrite (7.6) as

$$a_{n,k}^+ = 2ka_{n-1,k}^+ + 2(n-k)a_{n-1,k-1}^+ + 2(n-1)a_{n-2,k-1}^+ + a_{n-1,k}^-.$$

Switching k to $n-k$ in this equality and using the symmetry $a_{n,k}^+ = a_{n,n-k}^+$ shows that

$$a_{n-1,k}^- = a_{n-1,n-k}^-.$$

Equivalently, we have $a_{n,k}^- = a_{n,n-k+1}^-$ for all n and k and hence $\ell_n^-(x) = x^{n+1} \ell_n^-(1/x)$ for every $n \in \mathbb{N}$. The uniqueness of the defining properties of $f_n^+(x)$ and $f_n^-(x)$ shows that $\ell_n^+(x) = f_n^+(x)$ and $\ell_n^-(x) = f_n^-(x)$ for every $n \in \mathbb{N}$. $\square$

We have verified that $f_n^+(x)$ and $f_n^-(x)$ are real-rooted for $2 \leq n \leq 10$. Thus, it is natural to conjecture the following statement.

Conjecture 7.10. *The polynomials $f_n^+(x)$ and $f_n^-(x)$ are real-rooted for every $n \geq 2$.*

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DEPARTMENT OF MATHEMATICS (DIVISION OF ALGEBRA-GEOMETRY), UNIVERSITY OF ATHENS,
PANEPISTIMIOUPOLIS, ATHENS 15784, HELLAS (GREECE)

E-mail address: caath@math.uoa.gr, savvtina@math.uoa.gr