Discrete Comput Geom 31:395–403 (2004)
DOI: 10.1007/s00454-003-2869-x

Discrete & Computational
Geometry
© 2004 Springer-Verlag New York Inc.

Decompositions and Connectivity of Matching and Chessboard Complexes

Christos A. Athanasiadis

Department of Mathematics, University of Crete,
71409 Heraklion, Crete, Greece
caa@math.uoc.gr

Dedicated to Lou Billera on the occasion of his sixtieth birthday

Abstract. New lower bounds for the connectivity degree of the r-hypergraph matching and chessboard complexes are established by showing that certain skeleta of such complexes are vertex decomposable, in the sense of Provan and Billera, and hence shellable. The bounds given by Björner et al. [5] are improved for $r \geq 3$. Results on shellability of the chessboard complex due to Ziegler [16] are reproven in the case $r = 2$ and an affirmative answer to a question raised recently by Wachs for the matching complex follows. The new bounds are conjectured to be sharp.

1. Introduction and Results

We first define the main objects of study in this paper. Given integers $n \geq r \geq 2$, the *r-hypergraph matching complex* $M_n(r)$ is the simplicial complex on the vertex set $V_n(r)$ of all r-element subsets of $[n] := \{1, 2, \ldots, n\}$ with faces the subsets of $V_n(r)$ having pairwise disjoint elements. It is a pure simplicial complex of dimension $\lfloor n/r \rfloor - 1$ and is referred to simply as the matching complex M_n when $r = 2$. Given positive integers $n_1, n_2, \ldots, n_r$ with $r \geq 2$, the *$(n_1, n_2, \ldots, n_r)$-chessboard complex* $M_{n_1,n_2,\ldots,n_r}$ is the simplicial complex on the vertex set $V = [n_1] \times [n_2] \times \cdots \times [n_r]$ with faces the sets of r-tuples from V with no two having a coordinate in common. This is again a pure simplicial complex of dimension $\min\{n_i - 1 : 1 \leq i \leq r\}$. It can be described alternatively as the complex of position sets of rooks placed on a chessboard of shape $n_1 \times n_2 \times \cdots \times n_r$ so that no two of them lie in the same $(r-1)$-dimensional plane orthogonal to one of the axes of the chessboard. It is referred to simply as a chessboard complex when $r = 2$.

Matching and chessboard complexes first appeared in group theory in connection to Quillen complexes and Tits coset complexes [6], [8]. Since then they have turned out to be of importance in various mathematical contexts within algebra [11], combinatorics [14], discrete and computational geometry [17], representation theory [7] and topology [1]; see the recent survey [15] for a detailed historic account and a nice exposition of the main results and techniques in the study of matching and chessboard complexes, as well as for further references. Their hypergraph analogues were introduced and studied mainly with respect to their connectivity properties in [5]. The main result of [5] is as follows.

Theorem 1.1 [5].

(i) *For $n \geq r$ and*

$$\nu_n(r) = \left\lfloor \frac{n-2}{2r-1} \right\rfloor,$$

the complex $M_n(r)$ is $(\nu_n(r) - 1)$-connected and its $\nu_n(r)$-skeleton is homotopy Cohen–Macaulay.

(ii) *For $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ and*

$$\nu_{n_1,n_2,\ldots,n_r} = \min\left\{n_1 - 1, \left\lfloor \frac{n_1+n_2-2}{3} \right\rfloor, \ldots, \left\lfloor \frac{n_1+\cdots+n_r-r}{2r-1} \right\rfloor\right\},$$

the complex $M_{n_1,n_2,\ldots,n_r}$ is $(\nu_{n_1,n_2,\ldots,n_r} - 1)$-connected and its $\nu_{n_1,n_2,\ldots,n_r}$-skeleton is homotopy Cohen–Macaulay.

We improve the previous theorem to the following.

Theorem 1.2.

(i) *For $n \geq r$ and*

$$\mu_n(r) = \left\lfloor \frac{n-r}{r+1} \right\rfloor,$$

the complex $M_n(r)$ is $(\mu_n(r) - 1)$-connected and its $\mu_n(r)$-skeleton is shellable.

(ii) *For $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ the complex $M_{n_1,n_2,\ldots,n_r}$ is $(\nu_{n_1,n_2} - 1)$-connected and its ν_{n_1,n_2}-skeleton is shellable.*

A few remarks on the previous theorems are in order. Theorem 1.2 was previously known in the special case $r = 2$, in which the lower bounds on the connectivity appearing in the two theorems coincide. Indeed, it was suggested in [5] that the skeleta appearing in the statement of Theorem 1.1 are always shellable, a property which is stronger than that of being homotopy Cohen–Macaulay. This was proved for the chessboard complex $M_{m,n}$ by Ziegler [16], who showed that the respective skeleton has the even stronger property of being vertex decomposable, and for the matching complex M_n by Shareshian and Wachs [12], who describe an explicit shelling order of the facets of the $\lfloor (n-2)/3 \rfloor$-skeleton of M_n. The question whether this skeleton is vertex decomposable was raised by Wachs [15, Problem 5.4].

It was further conjectured in [5] that the lower bounds on the connectivity in Theorem 1.1 are sharp in part (i) and in the special case $r = 2$. In the latter case this follows for the matching complex essentially from the work of Bouc [6] and was proved for the chessboard complex by Shareshian and Wachs [12], see Theorem 5.2 of [15]. From the result of [12] one can easily deduce that the bounds on the connectivity in Theorem 1.2(ii) are sharp for all $r \geq 2$ (see Section 5). However, the conjecture of [5] was disproved for the complexes $M_n(r)$ when $r \geq 3$ is a prime by Ksontini [9], who improved the quantity $\nu_n(r)$ in Theorem 1.1(i) to $\nu'_n(r) = \lfloor (n + r - 4)/(2r - 1) \rfloor$ for $n \geq 3r + 2$. The smallest examples of matching and chessboard complexes for which Theorem 1.2 improves the existing lower bounds on the connectivity are $M_{15}(3)$ and $M_{4,4,4}$, respectively.

This paper is organized as follows. We begin with the necessary background on simplicial complexes in Section 2. In Section 3 we prove Theorem 1.2. More specifically, we adopt the approach taken by Ziegler [16] and prove that the skeleta appearing in the statement of Theorem 1.2 are always vertex decomposable. In the special case $r = 2$ this reproves the result of [16] on $M_{m,n}$ and answers in the affirmative the question of Wachs on the $\lfloor (n - 2)/3 \rfloor$-skeleton of M_n, mentioned earlier. In Section 4 we give a generalization of the $r = 2$ case to the matching complex of any finite graph. We conclude in Section 5 with a few remarks and open problems. This includes a generalization of Theorem 1.2 to the packing complexes, introduced by Björner and Eriksson [4].

2. Preliminaries

Let E be a finite set. An (abstract) *simplicial complex* on the ground set E is a collection Δ of subsets of E such that $F \subset F' \in \Delta$ implies $F \in \Delta$. The set $V = \{v \in E : \{v\} \in \Delta\}$ is the set of *vertices* of Δ. The elements of Δ are called *faces* and those maximal with respect to inclusion are called *facets*. The dimension of a face F is defined as one less than the cardinality of F and the dimension of Δ as the maximum dimension of a face. We call Δ *pure* if all its facets have the same dimension. The k-skeleton $\Delta^{\leq k}$ of Δ is the simplicial complex formed by the faces of Δ of dimension at most k. The *cone* of Δ over a new vertex v, denoted $\Delta * v$, is the simplicial complex on $E \cup \{v\}$ whose facets are the sets $F \cup \{v\}$, where F is a facet of Δ.

For $A \subseteq E$ define the *deletion* of A from Δ as $\Delta \backslash A = \{F \in \Delta : F \cap A = \emptyset\}$ and for $A \in \Delta$ define the *link* of A as $\Delta/A = \{F - A : A \subseteq F \in \Delta\}$. Note that these two operations commute. We write $\Delta \backslash v$ for $\Delta \backslash \{v\}$ and Δ/v for $\Delta/\{v\}$. The *restriction* of Δ on $A \subseteq E$ is defined as $\Delta(A) = \{F \in \Delta : F \subseteq A\}$.

Definition 2.1 [2], [10]. A simplicial complex Δ is *vertex decomposable* if it is pure and it is either empty or it has a vertex v such that $\Delta \backslash v$ and Δ/v are vertex decomposable.

For instance, any zero-dimensional complex is vertex decomposable and a one-dimensional complex is vertex decomposable if and only if it is connected. We use the following consequence of the definition.

Lemma 2.2. *Let Δ be a simplicial complex of dimension d and let $v_1, v_2, \ldots, v_t$ be distinct vertices of Δ. If $\Delta \backslash \{v_1, \ldots, v_t\}$ is vertex decomposable of dimension d and*

$(\Delta/v_i)\setminus\{v_1, \ldots, v_{i-1}\}$ is vertex decomposable of dimension $d-1$ for each $1 \le i \le t$, then Δ is vertex decomposable.

Proof. This follows by induction on t and the fact that a d-dimensional simplicial complex Δ is pure if for some vertex v, $\Delta\setminus v$ is pure of dimension d and Δ/v is pure of dimension $d-1$. □

Induction on the number of vertices also gives the next lemma.

Lemma 2.3.

(i) *If Δ is vertex decomposable, then so is its k-skeleton $\Delta^{\le k}$ for any k.*
(ii) [16, Lemma 1.2] *If the k-skeleton of a simplicial complex Δ is vertex decomposable, then the $(k+1)$-skeleton of the cone $\Delta * v$ is vertex decomposable as well.*

When we talk about topological properties of an abstract simplicial complex Δ we refer to those of its geometric realizaton $\|\Delta\|$ [3], which is unique up to linear homeomorphism. In particular, Δ is *k-connected* if the homotopy groups $\pi_i(\|\Delta\|, x)$ vanish for all $0 \le i \le k$ and $x \in \|\Delta\|$. The *connectivity degree* of Δ is the largest integer k such that Δ is *k-connected.*

3. Hypergraph Matching and Chessboard Complexes

In this section we prove Theorem 1.2. Note that the statement on the connectivity follows from that on shellability, since a simplicial complex is $(k-1)$-connected if and only if so is its k-skeleton and any shellable k-dimensional complex is $(k-1)$-connected; see, e.g. [3]. Hence it suffices to prove that the skeleta which appear in the theorem are shellable. As in [16], we will show that they are in fact vertex decomposable. We begin with the complexes $M_n(r)$.

Theorem 3.1. *For $n \ge r$ and*

$$\mu_n(r) = \left\lfloor \frac{n-r}{r+1} \right\rfloor,$$

the $\mu_n(r)$-skeleton of $M_n(r)$ is vertex decomposable.

Proof. We will prove a more general statement. For $1 \le k \le r$, let $M_{n,k}(r)$ be the restriction of $M_n(r)$ on the set of vertices u such that either $\{n, n-1, \ldots, n-k+1\} \subseteq u$ or u and $\{n, n-1, \ldots, n-k+1\}$ are disjoint. Note that $M_{n,k}(r)$ reduces to $M_n(r)$ for $k=1$. We will prove that the $\mu_n(r)$-skeleton of $M_{n,k}(r)$ is vertex decomposable of dimension $\mu_n(r)$ for all $n \ge r$.

The statement is true for $r \le n \le 2r$, since then $\mu_n(r) = 0$ and $M_{n,k}(r)$ has at least one vertex. Suppose $n \ge 2r+1$ and proceed by double induction on n and $r-k$. Note that

$M_{n,r}(r) = M_{n-r}(r) * v$, where $v = \{n, n-1, \ldots, n-r+1\}$. Since $\mu_n(r) - 1 \le \mu_{n-r}(r)$ and $n - r \ge r$, induction and Lemma 2.3 imply that the $\mu_n(r)$-skeleton of $M_{n,r}(r)$ is vertex decomposable of dimension $\mu_n(r)$, which is the statement for $k = r$. Assume now that $1 \le k < r$ and let S be the set of vertices of $M_{n,k}(r)$ not in $M_{n,k+1}(r)$, i.e. the set of r-subsets u of $[n]$ such that either $\max(u) = n - k$ or $\max([n] \setminus u) = n - k$. Let $S = \{v_1, v_2, \ldots, v_t\}$ be linearly ordered so that $i < j$ whenever $\max(v_i) = n - k$ and $\max([n] \setminus v_j) = n-k$. Let $\Delta_0 = M_{n,k}(r)$ and $\Delta_i = \Delta_{i-1} \backslash v_i$ for $1 \le i \le t$. Observe that the $\mu_n(r)$-skeleton of $\Delta_t = M_{n,k+1}(r)$ is vertex decomposable of dimension $\mu_n(r)$ by induction. Moreover, the link Δ_{i-1}/v_i is isomorphic to either $M_{n-r,k}(r)$ or $M_{n-r-1}(r)$, depending on whether $\max(v_i) = n - k$ or $\max([n] \setminus v_i) = n - k$, respectively. Since $n - r - 1 \ge r$ and $\mu_n(r) - 1 = \mu_{n-r-1}(r) \le \mu_{n-r}(r)$, induction and Lemma 2.3(i) imply that the $(\mu_n(r) - 1)$-skeleton of Δ_{i-1}/v_i is vertex decomposable of dimension $\mu_n(r) - 1$. It follows from Lemma 2.2 that the $\mu_n(r)$-skeleton of $\Delta_0 = M_{n,k}(r)$ is vertex decomposable of dimension $\mu_n(r)$ and this completes the induction. $\square$

We now shift attention to the chessboard complexes $M_{n_1,n_2,\ldots,n_r}$.

Theorem 3.2. *For* $1 \le n_1 \le n_2 \le \cdots \le n_r$ *and*

$$\nu_{n_1,n_2} = \min\left\{n_1 - 1, \left\lfloor \frac{n_1 + n_2 - 2}{3} \right\rfloor\right\},$$

the ν_{n_1,n_2}*-skeleton of* $M_{n_1,n_2,\ldots,n_r}$ *is vertex decomposable.*

Proof. Let $\gamma = (n_1, n_2, \ldots, n_r)$ and $V = [n_1] \times [n_2] \times \cdots \times [n_r]$ be the vertex set of $M_\gamma = M_{n_1,n_2,\ldots,n_r}$. Fix the vertex $v = (n_1, n_2, \ldots, n_r) \in V$. For $1 \le k \le r$, let $M_{\gamma,k}$ be the restriction of M_γ on the set of vertices $(b_1, b_2, \ldots, b_r) \in V$ such that either $b_i = n_i$ for all $1 \le i \le k$ or $b_i \ne n_i$ for all $1 \le i \le k$ and note that $M_{\gamma,1} = M_\gamma$. We will prove that the ν_{n_1,n_2}-skeleton of $M_{\gamma,k}$ is vertex decomposable of dimension ν_{n_1,n_2} for all $1 \le k \le r$.

We proceed by double induction on $n_1 + n_2 + \cdots + n_r$ and $r - k$. We may assume that $n_1 \ge 2$ and $n_2 \ge 3$, since otherwise $\nu_{n_1,n_2} = 0$ and $M_{\gamma,k}$ has at least one vertex. Note first that $M_{\gamma,r} = M_{n_1-1,\ldots,n_r-1} * v$. Since $\nu_{n_1,n_2} - 1 \le \nu_{n_1-1,n_2-1}$, induction and Lemma 2.3 imply that the ν_{n_1,n_2}-skeleton of $M_{\gamma,r}$ is vertex decomposable of dimension ν_{n_1,n_2}, which is the statement for $k = r$. Assume that $1 \le k < r$ and let S be the set of vertices of $M_{\gamma,k}$ not in $M_{\gamma,k+1}$, i.e. those of the form $(b_1, b_2, \ldots, b_r)$ with either $b_i \ne n_i$ for $1 \le i \le k$ and $b_{k+1} = n_{k+1}$ or $b_i = n_i$ for $1 \le i \le k$ and $b_{k+1} \ne n_{k+1}$. Let $S = \{v_1, v_2, \ldots, v_t\}$ be linearly ordered so that those vertices with $b_{k+1} = n_{k+1}$ come first and set $\Delta_0 = M_{\gamma,k}$ and $\Delta_i = \Delta_{i-1} \backslash v_i$ for $1 \le i \le t$. Observe that the ν_{n_1,n_2}-skeleton of $\Delta_t = M_{\gamma,k+1}$ is vertex decomposable of dimension ν_{n_1,n_2} by induction. Moreover, the link Δ_{i-1}/v_i is isomorphic to either $M_{\bar{\gamma},k}$ or $M_{\gamma'}$, where $\bar{\gamma} = (n_1 - 1, n_2 - 1, \ldots, n_r - 1)$ and $\gamma' = (m_1, m_2, \ldots, m_r)$ with

$$m_j = \begin{cases} n_j - 2, & \text{if } j = k + 1, \\ n_j - 1, & \text{otherwise.} \end{cases}$$

Using Lemma 2.2 once again, to complete the induction we need to check that the $(\nu_{n_1,n_2} - 1)$-skeleton of each of these two complexes is vertex decomposable of dimension

$\nu_{n_1,n_2} - 1$. This is clear for $M_{\tilde{\gamma},k}$ by induction and Lemma 2.3(i), since $\nu_{n_1,n_2} - 1 \le \nu_{n_1-1,n_2-1}$. For $M_{\gamma'}$ we need to verify that $\nu_{n_1,n_2} - 1 \le \nu_{m'_1,m'_2}$, where $(m'_1, m'_2, \ldots, m'_r)$ is the increasing rearrangement of γ'. This is clear if $m'_1 = n_1 - 1$ since then $m'_2 \ge n_2 - 2$. Otherwise we must have $n_1 = n_2 = \ldots = n_{k+1} = q$ and $m'_1 = q - 2$, $m'_2 = q - 1$ for some $q \ge 3$ and one can check that $\nu_{n_1,n_2} - 1 = \nu_{m'_1,m'_2} = \lfloor (2q-5)/3 \rfloor$. □

Theorem 1.2 implies that the skeleta of the complexes which appear there are Cohen–Macaulay over any field **k**. As in [5], we can draw the following corollary on their Stanley–Reisner rings; see [13] for relevant background.

Corollary 3.3. *For any field* **k** *and* $2 \le r \le n$, $1 \le n_1 \le n_2 \le \cdots \le n_r$ *we have*

$$\text{depth}\,(\mathbf{k}[M_n(r)]) \ge \left\lfloor \frac{n+1}{r+1} \right\rfloor$$

and

$$\text{depth}\,(\mathbf{k}[M_{n_1,n_2,\ldots,n_r}]) \ge \min\left\{ n_1, \left\lfloor \frac{n_1+n_2+1}{3} \right\rfloor \right\}.$$

In particular, $\mathbf{k}[M_{n_1,n_2,\ldots,n_r}]$ *is Cohen–Macaulay if* $n_2 \ge 2n_1 - 1$.

4. Matching Complexes of Graphs

Let G be a finite graph, meaning a pair (N, E) of a finite node set N and a set E of 2-element subsets of N, called edges. The *matching complex* of G, denoted by M_G, is the simplicial complex on the ground set E defined as follows. A set F of edges of G is a face of M_G if each node of G is contained in at most one edge in F. In graph-theoretic terminology, the faces of M_G are the sets of edges forming a partial matching of G. If G is the complete graph K_n on the node set $[n]$ or the complete bipartite graph $K_{m,n}$ on the node set $[m] \cup [n']$, where $[n'] := \{1', 2', \ldots, n'\}$, then M_G is the matching complex M_n or the chessboard complex $M_{m,n}$, respectively. The proof of Theorem 3.1 can be adapted in this situation to give the following theorem.

Theorem 4.1. *Let $\mathcal{G}$ be the class of pairs (G, k), where G is a graph and k is a nonnegative integer, such that*

(i) *the k-skeleton of M_G is pure of dimension k and, for $k \ge 1$,*
(ii) *there exist nodes a, b of G connected by an edge, such that if H is obtained from G by deleting a and any node connected by an edge to it, or a, b and any node connected by an edge to b, then $(H, k-1) \in \mathcal{G}$.*

Then for any $(G, k) \in \mathcal{G}$, the k-skeleton of M_G is vertex decomposable.

Proof. Let $(G, k) \in \mathcal{G}$ and proceed by induction on the number n of nodes of G, the result being trivial for $n \le 3$. We may assume that $k \ge 1$.

Let e be the edge $\{a, b\}$ in assumption (ii) of the theorem and let H be the graph obtained from G by deleting nodes a and b. Let $v_1, v_2, \ldots, v_s$ be the edges of G other than e incident to a, listed in any order, and let $v_{s+1}, \ldots, v_t$ be the edges of G other than e incident to b, listed in any order. Let $\Delta_0 = M_G$ and $\Delta_i = \Delta_{i-1} \backslash v_i$ for $1 \leq i \leq t$. By assumption we have $(H, k-1) \in \mathcal{G}$ and hence the $(k-1)$-skeleton of M_H is vertex decomposable of dimension $k-1$. Clearly, Δ_t is the cone $M_H * e$ and hence its k-skeleton is vertex decomposable of dimension k, by Lemma 2.3(ii). Moreover, the link Δ_{i-1}/v_i is isomorphic to the matching complex of the graph obtained from G by deleting a and the endpoints of v_i and hence its $(k-1)$-skeleton is vertex decomposable of dimension $k-1$ by assumption (ii) and induction. Lemma 2.2 implies that the k-skeleton of $\Delta_0 = M_G$ is vertex decomposable and completes the proof. □

The following corollary gives a simple solution to Problem 5.4 of [15] and another proof to the result of [16] on $M_{m,n}$. Let $\nu_n = \nu_n(2) = \lfloor (n-2)/3 \rfloor$ for $n \geq 2$ and

$$\nu_{m,n} = \min\left\{m-1, n-1, \left\lfloor \frac{m+n-2}{3} \right\rfloor\right\}$$

for $m, n \geq 1$.

Corollary 4.2.

(i) *The ν_n-skeleton of the matching complex M_n is vertex decomposable for all $n \geq 2$.*
(ii) [16] *The $\nu_{m,n}$-skeleton of the chessboard complex $M_{m,n}$ is vertex decomposable for all $m, n \geq 1$.*

Proof. Let $\mathcal{G}$ be as in Theorem 4.1. It suffices to check that $(G, k) \in \mathcal{G}$ if G is the complete graph K_n on the node set $[n]$ and $k \leq \nu_n$ or G is the complete bipartite graph $K_{m,n}$ on the node set $[m] \cup [n']$ and $k \leq \nu_{m,n}$, respectively. For this we proceed by induction on the size of G. For (i) we may assume that $n \geq 5$, since otherwise $\nu_n = 0$. Assumption (i) of Theorem 4.1 is clear. To verify assumption (ii) choose the nodes a, b arbitrarily and note that the graphs H considered there are isomorphic to either K_{n-2} or K_{n-3}. Since $k-1 \leq \nu_n - 1 = \nu_{n-3} \leq \nu_{n-2}$, we have $(H, k-1) \in \mathcal{G}$ by induction. Similarly for (ii) we may assume that $n \geq m \geq 2$ and $n \geq 3$. Assumption (i) of Theorem 4.1 is satisfied since $M_{m,n}$ is pure of dimension $m-1$. To verify assumption (ii) let a be any node in $[n']$, let b be any node in $[m]$ and note that any of the graphs H considered there is isomorphic to either $K_{m-1,n-1}$ or $K_{m-1,n-2}$. Since $k-1 \leq \nu_{m,n}-1 \leq \nu_{m-1,n-2} \leq \nu_{m-1,n-1}$, where the middle inequality has been verified in the proof of Theorem 3.2, we have $(H, k-1) \in \mathcal{G}$ by induction. It follows in both cases that $(G, k) \in \mathcal{G}$, as desired. □

From the proofs in this section one can deduce explicit vertex decompositions of the skeleta which appear in Corollary 4.2. For instance, if E is the set of vertices of M_n and F is the facet $\{\{1, 2\}, \{3, 4\}, \ldots\}$, then the linear ordering of E which first lists the elements of $E \backslash F$ in the lexicographic order and then those of F in any order induces a vertex decomposition of the ν_n-skeleton of M_n.

5. Remarks

1. Let $\tilde{H}_*(\Delta)$ denote the reduced, integral homology of Δ. The next corollary follows from Theorem 1.2(i).

Corollary 5.1. *We have $\tilde{H}_i\,(M_n(r)) = 0$ for $i < \mu_n(r)$ and $n \geq r \geq 2$.*

In the case $r = 2$ it follows essentially from the work of Bouc [6, Section 3.3] that $\tilde{H}_{\nu_n}\,(M_n) \neq 0$ for $n > 2$. In view of Theorem 1.2(i), the following conjecture would imply that the connectivity degree of $M_n(r)$ is equal to $\mu_n(r) - 1$ for all $n > r \geq 2$.

Conjecture 5.2. *For all $n > r \geq 2$ we have $\tilde{H}_{\mu_n(r)}\,(M_n(r)) \neq 0$.*

For the chessboard complex the result $\tilde{H}_{\nu_{n_1,n_2}}(M_{n_1,n_2}) \neq 0$ for $2 \leq n_1 \leq n_2$ has been announced by Shareshian and Wachs [12], [15, Theorem 5.2]. Since $\tilde{H}_*(M_{n_1,n_2})$ is a direct summand of $\tilde{H}_*(M_{n_1,n_2,\ldots,n_r})$ [5, Proposition 3.4], this implies that $\tilde{H}_{\nu_{n_1,n_2}}(M_{n_1,n_2,\ldots,n_r}) \neq 0$ and hence, in view of Theorem 1.2(i), that the connectivity degree of $M_{n_1,n_2,\ldots,n_r}$ is equal to $\nu_{n_1,n_2} - 1$ for $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$.

2. Packing complexes were introduced by Björner and Eriksson [4] as a common generalization to the hypergraph matching and chessboard complexes as follows. Let $N_1, N_2, \ldots, N_s$ be mutually disjoint sets of cardinalities $n_1, n_2, \ldots, n_s$, respectively. Fix integers $1 \leq r_i \leq n_i$ for $1 \leq i \leq s$ and let V be the set of r-element subsets of the union of the N_i which contain exactly r_i elements from N_i for each $1 \leq i \leq s$, so that $r = r_1 + \cdots + r_s$. The *packing complex* $M_{\binom{n_1}{r_1}\binom{n_2}{r_2}\cdots\binom{n_s}{r_s}}$ is the simplicial complex on the vertex set V whose faces are the subsets of V having pairwise disjoint elements. Björner and Eriksson gave a lower bound on the connectivity of the packing complex which reduces to those of Theorem 1.1 in the special cases in which $s = 1$ or $r_1 = \cdots = r_s = 1$. Theorem 1.2 can be generalized in this situation as follows.

Theorem 5.3. *Let $n_1, n_2, \ldots, n_s$ be ordered so that for some $\ell \geq 1$ we have $r_i = 1$ for $i < \ell$ and $r_i \geq 2$ otherwise. If μ denotes the minimum of the set*

$$\{n_i - 1 : i < \ell\} \cup \left\{ \left\lfloor \frac{n_i + n_j - 2}{3} \right\rfloor : i < j < \ell \right\} \cup \left\{ \left\lfloor \frac{n_i - r_i}{r_i + 1} \right\rfloor : i \geq \ell \right\},$$

then the complex $M_{\binom{n_1}{r_1}\binom{n_2}{r_2}\cdots\binom{n_s}{r_s}}$ is $(\mu - 1)$-connected and its μ-skeleton is shellable.

This can be proved by generalizing the proof of Theorem 1.2 given in Section 3 in a fairly straightforward way. To be more precise, let P denote the packing complex which appears in the statement of the theorem. It suffices again to show that the μ-skeleton of P is shellable. Assume that $n_1 \leq n_2 \leq \cdots \leq n_{\ell-1}$ and fix a vertex $v = \{a_1, a_2, \ldots, a_r\}$ of P such that $a_1 \in N_1$ if $\ell \geq 2$. For $1 \leq i \leq r$, let P_i be the restriction of P on the set of vertices u such that either $\{a_1, a_2, \ldots, a_i\} \subseteq u$ or u and $\{a_1, a_2, \ldots, a_i\}$ are disjoint. One can then show by induction that the μ-skeleton of P_i is vertex decomposable of dimension μ for all $1 \leq i \leq r$ as in the proof of Theorems 3.1 and 3.2. We leave the details to the interested reader.

Acknowledgements

This research was carried out while the author was serving in the Hellenic military. Thanks go to fellow soldiers for their understanding and to Anders Björner and Michelle Wachs for useful suggestions.

References

1. E. Babson, A. Björner, S. Linusson, J. Shareshian and V. Welker, Complexes of not i-connected graphs, *Topology* **38** (1999), 271–299.
2. L.J. Billera and J.S. Provan, A decomposition property for simplicial complexes and its relation to diameters and shellings, *Ann. NY Acad. Sci.* **319** (1979), 82–85.
3. A. Björner, Topological methods, in *Handbook of Combinatorics* (R.L. Graham, M. Grötschel and L. Lovász, eds.), North-Holland, Amsterdam, 1995, pp. 1819–1872.
4. A. Björner and H. Eriksson, Connectivity of packing complexes, manuscript, 1998.
5. A. Björner, L. Lovász, S.T. Vrećica and R.T. Živaljević, Chessboard complexes and matching complexes, *J. London Math. Soc.* **49** (1994), 25–39.
6. S. Bouc, Homologie de certains ensembles de 2-sous-groups des groupes symmétriques, *J. Algebra* **150** (1992), 158–186.
7. J. Friedman and P. Hanlon, On the Betti numbers of chessboard complexes, *J. Algebraic Combin.* **8** (1998), 193–203.
8. P.F. Garst, Cohen–Macaulay complexes and group actions, Ph.D. Thesis, University of Wisconsin, Madison, WI, 1979.
9. R. Ksontini, Propriétés homotopiques du complexe de Quillen du groupe symmétrique, Thèse de Doctorate, Université de Lausanne, 2000.
10. J.S. Provan and L.J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra, *Math. Oper. Res.* **5** (1980), 576–594.
11. V. Reiner and J. Roberts, Minimal resolutions and the homology of matching and chessboard complexes, *J. Algebraic Combin.* **11** (2000), 135–154.
12. J. Shareshian and M.L. Wachs, Torsion in the homology of matching and chessboard complexes, in preparation.
13. R.P. Stanley, *Combinatorics and Commutative Algebra*, second edition, Birkhäuser, Boston, MA, 1996.
14. S. Sundaram, On the topology of two partition posets with forbidden block sizes, *J. Pure Appl. Algebra* **155** (2001), 271–304.
15. M.L. Wachs, Topology of matching, chessboard and general bounded degree graph complexes, in "Gian-Carlo Rota Memorial Issue", *Algebra Universalis* **49** (2003), 345–385.
16. G.M. Ziegler, Shellability of chessboard complexes, *Israel J. Math.* **87** (1994), 97–110.
17. R.T. Živaljević and S.T. Vrećica, The colored Tverberg problem and complexes of injective functions, *J. Combin. Theory Ser. A* **61** (1992), 309–318.

Received April 15, 2002, *and in final form May* 28, 2002, *and April* 25, 2003.
Online publication February 5, 2004.