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GENERALIZED CATALAN NUMBERS, WEYL GROUPS AND ARRANGEMENTS OF HYPERPLANES

CHRISTOS A. ATHANASIADIS

Abstract

For an irreducible, crystallographic root system Φ in a Euclidean space V and a positive integer m, the arrangement of hyperplanes in V given by the affine equations $(\alpha, x) = k$, for $\alpha \in \Phi$ and $k = 0, 1, \ldots, m$, is denoted here by $\mathcal{A}_\Phi^m$. The characteristic polynomial of $\mathcal{A}_\Phi^m$ is related in the paper to that of the Coxeter arrangement $\mathcal{A}_\Phi$ (corresponding to $m = 0$), and the number of regions into which the fundamental chamber of $\mathcal{A}_\Phi$ is dissected by the hyperplanes of $\mathcal{A}_\Phi^m$ is deduced to be equal to the product $\prod_{i=1}^{\ell}(e_i + mh + 1)/(e_i + 1)$, where $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ and h is the Coxeter number. A similar formula for the number of bounded regions follows. Applications to the enumeration of antichains in the root poset of Φ are included.

1. *Introduction and results*

Let V be an ℓ-dimensional Euclidean space, with inner product $(\, , \,)$, and let Φ be an irreducible, crystallographic root system [**12**, Section 2.9] spanning V. For a nonnegative integer m, we denote by $\mathcal{A}_\Phi^m$ the collection of hyperplanes in V defined by the affine equations

$$(\alpha, x) = k$$

for $\alpha \in \Phi$ and $k = 0, 1, \ldots, m$ (see Figure 1). Thus $\mathcal{A}_\Phi^m$ is a deformation of the Coxeter hyperplane arrangement $\mathcal{A}_\Phi$, in the sense of [**4**, **17**], which is invariant under the action of the Weyl group W associated to Φ. It reduces to $\mathcal{A}_\Phi$ for $m = 0$.

In the case $m = 1$, $\mathcal{A}_\Phi^m$ is referred to as a *Catalan arrangement* [**4**, Section 3], denoted Cat_Φ. Much of the interest in the combinatorics of $\mathcal{A}_\Phi^m$ comes from this case, for the following reason. The regions into which the fundamental chamber of $\mathcal{A}_\Phi$ is dissected by the hyperplanes of Cat_Φ are in bijection with a number of different kind of objects of interest in representation theory, combinatorics and algebra, most notably the admissible positive sign types of Φ, introduced by Shi [**19**, **20**] in his study of left cells for the affine Weyl groups, the antichains in the root poset of Φ [**21**, **18**, **2**] and the *ad*-nilpotent ideals of the Borel subalgebra of the corresponding simple Lie algebra [**13**]. The number of these objects is the nth Catalan number for the root system A_{n-1}, which explains the terminology for Cat_Φ. The arrangements $\mathcal{A}_\Phi^m$ have also appeared in a variety of contexts within combinatorics and discrete geometry; see, for instance, [**9**, **23**, **2**] and [**4**, Section 3].

We denote by $\chi(\mathcal{A}, q)$ the characteristic polynomial (see Section 2) of an arrangement $\mathcal{A}$ of affine hyperplanes in V. The characteristic polynomial of $\mathcal{A}_\Phi$ admits the following well-known, remarkable factorization.

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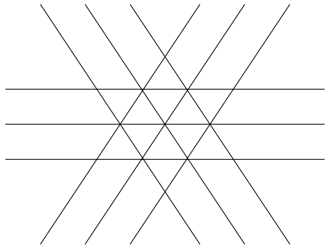

FIGURE 1. The arrangement $\mathcal{A}_\Phi^1$ for $\Phi = A_2$.

THEOREM 1.1 (Orlik and Solomon [**15**]). *We have*

$$\chi(\mathcal{A}_\Phi, q) = \prod_{i=1}^{\ell}(q - e_i),$$

where ℓ is the rank and $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ.

It is the main purpose of this paper to give an elementary, case-free proof of the following theorem (see Section 3).

THEOREM 1.2. *For any irreducible crystallographic root system Φ and positive integer m, we have*

$$\chi\big(\mathcal{A}_\Phi^m, q\big) = \chi(\mathcal{A}_\Phi, q - mh),$$

where h is the Coxeter number of Φ. Equivalently, we have

$$\chi\big(\mathcal{A}_\Phi^m, q\big) = \prod_{i=1}^{\ell}(q - mh - e_i), \tag{1}$$

where ℓ is the rank and $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ.

The following result is an immediate corollary of Theorem 1.2. In the case $m = 1$, the first statement is equivalent to the results of Shi [**21**], as we explain below.

COROLLARY 1.3. *The number of regions into which the fundamental chamber of the Coxeter arrangement $\mathcal{A}_\Phi$ is dissected by the hyperplanes of $\mathcal{A}_\Phi^m$ is equal to*

$$\prod_{i=1}^{\ell}\frac{e_i + mh + 1}{e_i + 1}, \tag{2}$$

where h is the Coxeter number, ℓ is the rank and $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ. The number of those regions that are bounded is equal to

$$\prod_{i=1}^{\ell}\frac{e_i + mh - 1}{e_i + 1}. \tag{3}$$

We remark in Section 4 that the bounded regions referred to in the previous corollary are exactly the regions of $\mathcal{A}_\Phi^m$ inside the parallelepiped defined by the inequalities $0 < (\alpha, x) < m$, where α runs through the set of simple roots of Φ.

The root poset of Φ is the set of positive roots Φ^+, partially ordered by letting $\alpha \leqslant \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots.

The following corollary will be deduced in Section 4.

COROLLARY 1.4.

(i) *The number of antichains in the root poset of* Φ *is equal to*

$$\prod_{i=1}^{\ell} \frac{e_i + h + 1}{e_i + 1} \tag{4}$$

(see [**21**, **18**, **8**]).

(ii) *The number of those antichains that do not contain any simple root is equal to*

$$\prod_{i=1}^{\ell} \frac{e_i + h - 1}{e_i + 1}. \tag{5}$$

We conclude this section with some remarks on Theorem 1.2 and its corollaries. Theorem 1.2 has previously been verified for the root systems of type A by Edelman and Reiner [**9**, Section 3], and by Postnikov and Stanley [**17**, Proposition 9.8], and for those of types A, B, C and D by the author [**1**, Theorem 5.5] (see also [**4**, Theorem 4.6]). A stronger assertion has been conjectured in [**9**, Conjecture 3.3], namely that the homogenized cone of $\mathcal{A}_\Phi^m$ is free, in the sense of Terao [**25**], with exponents $1, e_1 + mh, \dots, e_\ell + mh$. The proof of Theorem 1.2, given in Section 3, is inspired by the general method of [**1**], used to compute characteristic polynomials of hyperplane arrangements.

The numbers that appear in Corollary 1.3 may be considered as generalizations of the Catalan numbers for the Weyl groups. The product (2) is also known to count the number of orbits of the action of W on $\check{Q}/(mh+1)\check{Q}$, where $\check{Q}$ denotes the coroot lattice of Φ; see [**11**, Theorem 7.4.2]. It has also appeared in [**22**], in a more general form, as the Euler characteristic of a certain space of partial flags in an affine Lie algebra. A bijection between regions of Cat_Φ inside the fundamental chamber of $\mathcal{A}_\Phi$ and antichains in the root poset of Φ was given by Shi [**21**], and will be reviewed in Section 4. Corollary 1.4(i) was verified, case by case, by Shi [**21**], and independently by Postnikov (in unpublished work) for types A, B, C and D, although it seems that the uniform formula (4) appeared for the first time in [**18**]. A case-free proof was given recently by Cellini and Papi [**8**], via a bijection with the W-orbits of $\check{Q}/(h+1)\check{Q}$, thus answering a question raised by Krattenthaler *et al.* [**14**]. Our approach via the characteristic polynomial gives a different, case-free proof. Part (ii) of Corollary 1.4 is new. It is striking that, apart from regions of Cat_Φ, antichains in the root poset of Φ, and W-orbits of $\check{Q}/(h+1)\check{Q}$, the product (4) is also known to count noncrossing partitions associated to W (see [**5**, **18**]) and the clusters associated to Φ by Fomin and Zelevinsky [**10**] or, equivalently, vertices of their (simple) generalized associahedron; see [**5**, Proposition 5.2.1] and [**10**, Theorem 1.9], respectively. Moreover, the product (5) appears as the number of positive clusters associated to Φ; see [**10**, Proposition 3.9].

2. *Background*

In this section we introduce notation and background related to arrangements of hyperplanes and root systems. We refer the reader to the texts by Orlik and Terao [**16**] and Humphreys [**12**] for undefined terminology, and for more information on these topics.

CONVENTION. For the sake of convenience, in the rest of the paper we identify the Euclidean space V, considered in Section 1, with $\mathbb{R}^\ell$, equipped with the standard inner product $(\ ,\ )$.

We use the notation $[a,b]=\{a,a+1,\dots,b\}$ for integers $a\leqslant b$, and we denote by $\#A$ the cardinality of a finite set A.

Arrangements of hyperplanes. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^\ell$; that is, a finite collection of affine subspaces of $\mathbb{R}^\ell$ of codimension one. The *intersection poset* of $\mathcal{A}$ is the set $L_\mathcal{A}=\{\cap\mathcal{F} : \mathcal{F}\subseteq\mathcal{A}\}$, partially ordered by reverse inclusion. It has a unique minimal element $\hat{0}=\mathbb{R}^\ell$, corresponding to the empty family $\mathcal{F}$. The *characteristic polynomial* of $\mathcal{A}$ is a fundamental combinatorial and topological invariant of $\mathcal{A}$, defined by

$$\chi(\mathcal{A},q)=\sum_{x\in L_\mathcal{A}}\mu(x)\ q^{\dim x},$$

where μ stands for the Möbius function on $L_\mathcal{A}$, defined by

$$\mu(x)=\begin{cases}1, & \text{if } \ x=\hat{0},\\ -\displaystyle\sum_{y<x}\mu(y), & \text{otherwise.}\end{cases}$$

The connected components of the space obtained from $\mathbb{R}^\ell$ by removing the hyperplanes of $\mathcal{A}$ are called *regions* of $\mathcal{A}$. A region is *bounded* if it is bounded as a subset of $\mathbb{R}^\ell$ with the usual Euclidean metric. We call $\mathcal{A}$ *essential* if the normal vectors to the hyperplanes of $\mathcal{A}$ span $\mathbb{R}^\ell$. The number of regions and the number of bounded regions of $\mathcal{A}$ can be expressed in terms of the characteristic polynomial as follows.

THEOREM 2.1 (Zaslavsky [**26**]). *The number of regions into which $\mathcal{A}$ dissects $\mathbb{R}^\ell$ is equal to $(-1)^\ell\chi(\mathcal{A},-1)$. If $\mathcal{A}$ is essential, then the number of those regions of $\mathcal{A}$ that are bounded is equal to $(-1)^\ell\chi(\mathcal{A},1)$.*

Let $\mathbb{Z}_q$ denote the abelian group of integers modulo q. We call an arrangement $\mathcal{A}$ in $\mathbb{R}^\ell$ a *$\mathbb{Z}$-arrangement* if the hyperplanes of $\mathcal{A}$ are given by equations with integer coefficients. Such equations define subsets of the finite set $\mathbb{Z}_q^\ell$ if we reduce their coefficients modulo q. We denote by $V_\mathcal{A}$ the union of these subsets, suppressing q in the notation. The following result, stated as in [**3**, Theorem 2.1] and [**4**, Theorem 4.2], will be used to prove Theorem 1.2. It was discovered independently by Björner and Ekedahl [**6**]; see [**4**, Section 4] for an overview of other applications, and more references.

THEOREM 2.2 [**1**, **3**, **4**, **6**]. *Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement in $\mathbb{R}^\ell$. Then there exist positive integers r and k that depend only on $\mathcal{A}$, such that for all q relatively prime to r, with $q>k$,*

$$\chi(\mathcal{A},q)=\#\big(\mathbb{Z}_q^\ell-V_\mathcal{A}\big).$$

The next remark follows, for instance, from the proof of this theorem that is to be found in [**3**, Section 2].

REMARK 2.3. The integer r that appears in the theorem can be specifically chosen as follows. Suppose that the hyperplanes of $\mathcal{A}$ are given by the equations $(\alpha_i, x) = b_i$ for $1 \leqslant i \leqslant n$, where $\alpha_i \in \mathbb{Z}^\ell$ and $b_i \in \mathbb{Z}$. It suffices that r be the product of the orders of the torsion subgroups of $\mathbb{Z}^\ell / \mathbb{Z}E$ for $E \subseteq \{\alpha_1, \dots, \alpha_n\}$, where $\mathbb{Z}E$ denotes the $\mathbb{Z}$-span of E in $\mathbb{Z}^\ell$.

Root systems and Weyl groups. Let Φ be an irreducible, crystallographic root system spanning $\mathbb{R}^\ell$, endowed with the standard inner product $(\, , \,)$. We fix a positive system $\Phi^+ \subseteq \Phi$ and the corresponding (ordered) set of simple roots $\Delta = (\sigma_1, \dots, \sigma_\ell)$. Thus Δ is a basis of $\mathbb{R}^\ell$, and any root $\alpha \in \Phi$ can be expressed as an integer linear combination

$$\alpha = \sum_{i=1}^{\ell} c_i(\alpha)\, \sigma_i,$$

where the coefficients $c_i(\alpha)$ are all nonnegative if $\alpha \in \Phi^+$, and all nonpositive otherwise. We let:

(i) $(\varpi_1^\vee, \varpi_2^\vee, \dots, \varpi_\ell^\vee)$ be the dual basis to Δ, with respect to the inner product $(\, , \,)$,

(ii) $e_1, e_2, \dots, e_\ell$ be the exponents of Φ,

(iii) h be its Coxeter number, and

(iv) $\tilde{\alpha}$ be the *highest root*, characterized by the condition that $c_i(\tilde{\alpha}) \geqslant c_i(\alpha)$ for all $\alpha \in \Phi$ and $1 \leqslant i \leqslant \ell$.

The following lemma can be checked directly, for instance, from the tables given in [**12**, Sections 3.18 and 4.9].

LEMMA 2.4.

(i) *We have* $\sum_{i=1}^{\ell} c_i\,(\tilde{\alpha}) = h - 1$.

(ii) *If a prime p divides $c_i\,(\tilde{\alpha})$ for some $1 \leqslant i \leqslant \ell$, then p divides h.*

We denote by $\mathcal{A}_\Phi$ the *Coxeter arrangement* associated to Φ (that is, the collection of linear hyperplanes in $\mathbb{R}^\ell$ that are orthogonal to the roots), and by W the corresponding *Weyl group*, generated by the reflections in these hyperplanes. Thus W is finite, leaves Φ invariant, and acts simply transitively on the set of regions of $\mathcal{A}_\Phi$, also called *chambers*. The *fundamental chamber* is the region defined by the inequalities $(\sigma_i, x) > 0$ for $1 \leqslant i \leqslant \ell$. The set $Z(\Phi)$ of vectors $x \in \mathbb{R}^\ell$ satisfying $(\alpha, x) \in \mathbb{Z}$ for all $\alpha \in \Phi$ is the *coweight lattice* associated to Φ. The *coroot lattice* $\check{Q}(\Phi)$ is the $\mathbb{Z}$-span of the set of coroots

$$\Phi^\vee = \left\{ \frac{2\alpha}{(\alpha, \alpha)} : \alpha \in \Phi \right\}.$$

Since Φ is crystallographic, we have $\check{Q}(\Phi) \subseteq Z(\Phi)$. The index of $\check{Q}(\Phi)$ as a subgroup of $Z(\Phi)$ is denoted by f.

For any real k and $\alpha \in \Phi$, we let $H_{\alpha,k}$ be the hyperplane in $\mathbb{R}^\ell$ defined by the equation $(\alpha, x) = k$. We denote by $\tilde{\mathcal{A}}_\Phi$ the *affine Coxeter arrangement*, which is the infinite hyperplane arrangement in $\mathbb{R}^\ell$ consisting of the hyperplanes $H_{\alpha,k}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, and by W_a the *affine Weyl group*, generated by the reflections in the hyperplanes of $\tilde{\mathcal{A}}_\Phi$. The group W_a is the semidirect product of W and the translation group in $\mathbb{R}^\ell$ corresponding to the coroot lattice $\check{Q}(\Phi)$. Its action on $\tilde{\mathcal{A}}_\Phi$ is determined by the following elementary lemma.

LEMMA 2.5 [**12**, Section 4.1]. *For $w \in W$, $\alpha \in \Phi$, $\lambda \in \check{Q}(\Phi)$ and $k \in \mathbb{R}$, we have:*
(i) $wH_{\alpha,k} = H_{w\alpha,k}$;
(ii) $H_{\alpha,k} + \lambda = H_{\alpha,k+(\alpha,\lambda)}$.

The group W_a acts simply transitively on the set of regions of $\tilde{\mathcal{A}}_\Phi$, also called *alcoves*. The *fundamental alcove* of $\tilde{\mathcal{A}}_\Phi$ can be defined as

$$A_\circ = \{x \in \mathbb{R}^\ell : 0 < (\alpha, x) \text{ for all } \alpha \in \Delta \text{ and } (\tilde{\alpha}, x) < 1\}.$$

For any subset $\Phi' \subseteq \Phi$, let $L(\Phi')$ denote the $\mathbb{Z}$-span of Φ', so that $L(\Phi)$ is the *root lattice* of Φ. The following result appears in [**24**, Section 1].

LEMMA 2.6. *A prime p divides the order of the torsion subgroup of $L(\Phi)/L(\Phi')$ for some $\Phi' \subseteq \Phi$ if and only if p divides a coefficient $c_i(\tilde{\alpha})$ for $1 \leqslant i \leqslant \ell$.*

3. *Proof of Theorem* 1.2

To prove Theorem 1.2, we will show under assumptions on the positive integer t, that $\chi(\mathcal{A}_\Phi^m, t)$ counts, up to a factor, the number of elements of the coweight lattice $Z(\Phi)$ inside a dilation of the fundamental alcove of $\tilde{\mathcal{A}}_\Phi$. For the Coxeter arrangement $\mathcal{A}_\Phi$, such interpretations have appeared many times in the literature; see the papers by Blass and Sagan [**7**], Haiman [**11**, Section 7.4] and Sommers [**22**]. The next result directly generalizes [**7**, Theorem 4.1]. Our proof generalizes the proof of this theorem given in [**1**, Section 2].

THEOREM 3.1. *Let Φ and a nonnegative integer m be given. If t is an integer relatively prime to $c_i(\tilde{\alpha})$ for all $1 \leqslant i \leqslant \ell$ and $t > mh$, then*

$$\chi(\mathcal{A}_\Phi^m, t) = \frac{\#W}{f} \, \#\,((t - mh)A_\circ \cap Z(\Phi)). \tag{6}$$

Before we prove Theorem 3.1, we establish some more notation. For any positive real t, we let

$$Z_t(\Phi) = \frac{1}{t} Z(\Phi).$$

For a positive integer t and a nonnegative integer m, we denote by $V_{\Phi,t}^m$ the union of the hyperplanes $H_{\alpha,k+s/t}$, where $\alpha \in \Phi$ and s and k are integers with $|s| \leqslant m$. It follows from Lemma 2.5 that the set $V_{\Phi,t}^m$ is invariant under the action of the affine Weyl group W_a on $\mathbb{R}^\ell$.

Proof of Theorem 3.1. For $x \in \mathbb{R}^\ell$, let $x^* = (x_1^*, x_2^*, \ldots, x_\ell^*)$, where $x_i^* = (x, \sigma_i)$. In other words, x^* is the ℓ-tuple of coordinates of x in the dual basis $(\varpi_1^\vee, \varpi_2^\vee, \ldots, \varpi_\ell^\vee)$ to Δ. Note that $x \in Z(\Phi)$ if and only if $x^* \in \mathbb{Z}^\ell$, and hence the map $x \mapsto x^*$ defines a linear isomorphism of $\mathbb{R}^\ell$ under which the lattice $Z(\Phi)$ corresponds to $\mathbb{Z}^\ell$. Since

$$(\alpha, x) = \left(\sum_{i=1}^{\ell} c_i(\alpha)\, \sigma_i, x \right) = \sum_{i=1}^{\ell} c_i(\alpha) x_i^*,$$

the arrangement $\mathcal{A}_\Phi^m$ corresponds, under the isomorphism, to a $\mathbb{Z}$-arrangement in $\mathbb{R}^\ell$.

In view of Remark 2.3 and Lemma 2.6, Theorem 2.2 implies that there exists a positive integer k such that for any integer $t > k$ with t relatively prime to the coefficients $c_i(\tilde{\alpha})$,

$$\begin{aligned}\chi\big(\mathcal{A}_\Phi^m, t\big) &= \#\{x^* \in [0, t-1]^\ell : \ (\alpha, x) \neq 0, 1, \ldots, m \mod t \text{ for all } \alpha \in \Phi\} \\ &= \#\left\{x^* \in [0, t-1]^\ell : \ \frac{x}{t} \text{ is not in } V_{\Phi,t}^m\right\} \\ &= \#\big(P_t - V_{\Phi,t}^m\big),\end{aligned}$$

where $P_t = P \cap Z_t(\Phi)$ and P is the parallelepiped

$$\left\{\sum_{i=1}^{n} y_i \varpi_i^\vee : \ 0 \leqslant y_i \leqslant 1\right\}.$$

Let $V_{\tilde{\mathcal{A}}_\Phi}$ be the union of the hyperplanes of $\tilde{\mathcal{A}}_\Phi$. It is known that $P - V_{\tilde{\mathcal{A}}_\Phi}$ has $\#W/f$ connected components [**12**, p. 99], and that W_a acts transitively on them. Clearly, W_a preserves the points of $Z_t(\Phi)$ and, as we noted earlier, it preserves $V_{\Phi,t}^m$ as well. It follows that each connected component of $P - V_{\tilde{\mathcal{A}}_\Phi}$ has the same number of points belonging to $P_t - V_{\Phi,t}^m$. Hence

$$\chi\big(\mathcal{A}_\Phi^m, t\big) = \frac{\#W}{f} \ \#\left(A_\circ \cap Z_t(\Phi) - V_{\Phi,t}^m\right)$$

or, equivalently,

$$\chi\big(\mathcal{A}_\Phi^m, t\big) = \frac{\#W}{f} \ \#\left(t\, A_\circ \cap Z(\Phi) - t\, V_{\Phi,t}^m\right).$$

Note that $t\, A_\circ \cap Z(\Phi) - t\, V_{\Phi,t}^m$ is the set of points of $Z(\Phi)$ in the open simplex defined by the inequalities $m < (\sigma_i, x)$ for all $1 \leqslant i \leqslant \ell$ and $(\tilde{\alpha}, x) < t - m$, while $(t - mh)\, A_\circ$ is the open simplex defined by $0 < (\sigma_i, x)$ for all $1 \leqslant i \leqslant \ell$ and $(\tilde{\alpha}, x) < t - mh$. Lemma 2.4(i) implies that the translation by the negative of $m(\varpi_1^\vee + \varpi_2^\vee + \ldots + \varpi_\ell^\vee) \in Z(\Phi)$ maps the set $tA_\circ \cap Z(\Phi) + tV_{\Phi,t}^m$ bijectively onto $(t + mh)A_\circ \cap Z(\Phi)$, and the proposed expression for $\chi(\mathcal{A}_\Phi^m, t)$ follows.

Finally, observe that $\#(tA_\circ \cap Z(\Phi))$ is a polynomial in t for all t positive and relatively prime to the $c_i(\tilde{\alpha})$. Indeed, it is certainly a quasi-polynomial for such t, namely the Ehrhart quasi-polynomial of the open simplex in $\mathbb{R}^\ell$ bounded by the hyperplanes $x_i^* = 0$ and $\sum_{i=1}^{\ell} c_i(\tilde{\alpha}) x_i^* = 1$, and agrees with the polynomial $\chi(\mathcal{A}_\Phi, t)$ for large t by what we have already shown for $m = 0$. By Lemma 2.4(ii), if t is relatively prime to the $c_i(\tilde{\alpha})$, then so is $t - mh$. Consequently, the right-hand side of (6) is a polynomial in t for all t, as in the statement of the theorem, and hence agrees with $\chi(\mathcal{A}_\Phi^m, t)$ for all such t. □

Proof of Theorem 1.2 *and Corollary* 1.3. Theorem 3.1 implies that

$$\chi\big(\mathcal{A}_\Phi^m, q\big) = \chi(\mathcal{A}_\Phi, q - mh)$$

for infinitely many positive integers q. Since both sides are polynomials in q, this equation has to hold identically.

Given this, equation (1) follows from Theorem 1.1. Then Corollary 1.3 follows from equation (1), Theorem 2.1, and the fact that each chamber of $\mathcal{A}_\Phi$ contains the same number of regions (or bounded regions, respectively) as $\mathcal{A}_\Phi^m$, as well as the formula $\prod_{i=1}^{\ell}(e_i + 1)$ for the number of chambers of $\mathcal{A}_\Phi$. □

4. *Antichains in the root poset*

In this section we give a characterization of the bounded regions of $\mathcal{A}^m_\Phi$, referred to in Corollary 1.3, and we prove Corollary 1.4.

Let P denote the parallelepiped spanned by the dual basis $(\varpi_1^\vee, \varpi_2^\vee, \dots, \varpi_\ell^\vee)$ to Δ, as in the proof of Theorem 1.2, so that P can be described alternatively by the inequalities $0 \leqslant (\sigma_i, x) \leqslant 1$ for $1 \leqslant i \leqslant \ell$. Let mP denote the dilation $\{mx : x \in P\}$ of P.

LEMMA 4.1. *Let R be a region of $\mathcal{A}^m_\Phi$ inside the fundamental chamber of the Coxeter arrangement $\mathcal{A}_\Phi$. Then R is bounded if and only if $R \subseteq mP$.*

Proof. If $R \subseteq mP$, then clearly R is bounded. Suppose that R is not contained in mP, and let $x \in R$, so that for some simple root σ_i we have $(\sigma_i, x) > m$. We claim that $x_t = x + t\varpi_i^\vee \in R$ for all positive real t, which implies that R is unbounded. Indeed, in view of the relations $(\sigma_j, \varpi_i^\vee) = \delta_{ij}$, for $\alpha \in \Phi^+$ we have $(\alpha, x_t) = (\alpha, x)$ if α is a nonnegative linear combination of simple roots other than σ_i, and $(\alpha, x_t) > (\alpha, x) \geqslant (\sigma_i, x) > m$ otherwise. This implies that $x_t \in R$. $\square$

Recall that the root poset of Φ is the set of positive roots Φ^+, partially ordered by letting $\alpha \leqslant \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots. An *antichain* in Φ^+ is a subset A of the root poset of Φ with no two elements comparable. The *dual order ideal* I_A, defined by A, is the set of all $\beta \in \Phi^+$ such that $\alpha \leqslant \beta$ for some $\alpha \in A$. Given an antichain A in Φ^+, let R_A be the set of points x in $\mathbb{R}^\ell$ that satisfy

$$\begin{aligned} (\beta, x) &> 1, && \text{if } \beta \in I_A; \\ 0 < (\beta, x) &< 1, && \text{if } \beta \in \Phi^+ - I_A. \end{aligned}$$

For instance, if A is the empty antichain, then R_A is the fundamental alcove of $\tilde{\mathcal{A}}_\Phi$. Let τ be the map that sends an antichain A in Φ^+ to the set R_A. The following result has been discovered independently for the root systems of types A, B, C and D by A. Postnikov (in unpublished work).

THEOREM 4.2 (Shi [**21**]). *The map τ is a bijection between the set of antichains in the root poset of Φ and the set of regions of* Cat_Φ *inside the fundamental chamber of $\mathcal{A}_\Phi$.*

We remark that the nontrivial part in the proof of the previous theorem is to show that the set R_A is actually nonempty. As explained in [**21**], this follows from the results of Shi given in [**20**].

Proof of Corollary 1.4. Part (i) follows immediately from the case $m = 1$ of Corollary 1.3, and Theorem 4.2.

For part (ii), note that an antichain A in the root poset of Φ does not contain any simple root if and only if $R_A \subseteq P$. The result then follows from Theorem 4.2 and the cases $m = 1$ of Lemma 4.1 and Corollary 1.3, respectively. $\square$

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Christos A. Athanasiadis
Department of Mathematics
University of Crete
71409 Heraklion
Crete
Greece

caa@math.uoc.gr