

# BINOMIAL EULERIAN POLYNOMIALS FOR COLORED PERMUTATIONS

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ABSTRACT. Binomial Eulerian polynomials first appeared in work of Postnikov, Reiner and Williams on the face enumeration of generalized permutohedra. They are  $\gamma$ -positive (in particular, palindromic and unimodal) polynomials which can be interpreted as  $h$ -polynomials of certain flag simplicial polytopes and which admit interesting Schur  $\gamma$ -positive symmetric function generalizations. This paper introduces analogues of these polynomials for  $r$ -colored permutations with similar properties and uncovers some new instances of equivariant  $\gamma$ -positivity in geometric combinatorics.

## 1. INTRODUCTION

The Eulerian polynomial

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\text{des}(w)} = \sum_{w \in \mathfrak{S}_n} t^{\text{exc}(w)},$$

attached to the symmetric group  $\mathfrak{S}_n$  of permutations of the set  $\{1, 2, \dots, n\}$ , is a prototypical example of a palindromic and unimodal polynomial in combinatorics (for undefined notation and terminology, see Sections 2, 4 and 6). Its palindromicity and unimodality can be demonstrated combinatorially, for instance via an expansion of the form

$$A_n(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i} t^i (1+t)^{n-1-2i}$$

for some nonnegative integers  $\gamma_{n,i}$ , which establishes its  $\gamma$ -positivity, and geometrically (as well as by other approaches), since  $A_n(t)$  can be interpreted as the  $h$ -polynomial of the boundary complex of a simplicial polytope. For an overview of these and other important properties of Eulerian polynomials, see [23] [34, Section 1.4].

The  $n$ th *binomial Eulerian polynomial*, defined by the formula

$$(1) \quad \tilde{A}_n(t) := 1 + t \sum_{m=1}^n \binom{n}{m} A_m(t),$$

first appeared in work of Postnikov, Reiner and Williams [24, Section 10.4] on the face enumeration of generalized permutohedra and was further studied in [29] (where the name binomial Eulerian polynomial was adopted) and in [21]. The following statement, which

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can be derived from a more general result [24, Theorem 11.6] on the  $h$ -polynomials of chordal nestohedra, shows that  $\tilde{A}_n(t)$  shares some of the main properties of the Eulerian polynomial  $A_n(t)$ .

**Theorem 1.1.** (Postnikov–Reiner–Williams [24]) *The polynomial  $\tilde{A}_n(t)$  is equal to the  $h$ -polynomial of the boundary complex of an  $n$ -dimensional flag simplicial polytope for every positive integer  $n$ . Moreover,*

$$(2) \quad \tilde{A}_n(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,i} t^i (1+t)^{n-2i},$$

where  $\tilde{\gamma}_{n,i}$  is equal to the number of permutations  $w \in \mathfrak{S}_{n+1}$  which have exactly  $i$  descents, no two consecutive, and  $w(1) < w(2) < \dots < w(m) = n+1$  for some  $m$ .

In particular,  $\tilde{A}_n(t)$  is  $\gamma$ -positive.

This paper introduces a generalization  $\tilde{A}_{n,r}(t)$  of  $\tilde{A}_n(t)$  to the wreath product group  $\mathbb{Z}_r \wr \mathfrak{S}_n$  and studies further symmetric function generalizations. This section gives only a rough outline of the paper; for background, precise statements of most of the results and technicalities the reader is referred to other sections. The polynomial  $\tilde{A}_{n,r}(t)$  is defined by the formula

$$(3) \quad \tilde{A}_{n,r}(t) := \sum_{m=0}^n \binom{n}{m} t^{n-m} A_{m,r}(t),$$

where

$$A_{n,r}(t) := \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} t^{\text{des}(w)} = \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} t^{\text{exc}(w)}$$

is the Eulerian polynomial for  $\mathbb{Z}_r \wr \mathfrak{S}_n$  introduced by Steingrímsson [37, 38] and  $A_{0,r}(t) := 1$ . Since, by the palindromicity of  $A_n(t)$  and  $\tilde{A}_n(t)$ , Equation (1) may be rewritten as

$$(4) \quad \tilde{A}_n(t) = \sum_{m=0}^n \binom{n}{m} t^{n-m} A_m(t),$$

where  $A_0(t) := 1$ , the polynomial  $\tilde{A}_{n,r}(t)$  reduces to  $\tilde{A}_n(t)$  for  $r = 1$ . For  $r = 2$ , following standard notation, we prefer to write  $\tilde{B}_n(t)$ , instead of  $\tilde{A}_{n,2}(t)$ . For the first few values of  $n$ , we have

$$\tilde{B}_n(t) = \begin{cases} 1 + 2t, & \text{if } n = 1 \\ 1 + 8t + 4t^2, & \text{if } n = 2 \\ 1 + 26t + 44t^2 + 8t^3, & \text{if } n = 3 \\ 1 + 80t + 328t^2 + 208t^3 + 16t^4, & \text{if } n = 4 \\ 1 + 242t + 2072t^2 + 3072t^3 + 912t^4 + 32t^5, & \text{if } n = 5. \end{cases}$$

The following statement shows that  $\tilde{A}_{n,r}(t)$  is alternatingly increasing, in the sense of [26, Definition 2.9], and nonsymmetric  $\gamma$ -positive, in the sense of [4, Section 5.1], and

hence unimodal, for all  $n \geq 1$  and  $r \geq 2$  and thus adds another family of polynomials in the literature known to have these nice properties.

**Theorem 1.2.** *For all positive integers  $n, r$  with  $r \geq 2$  we have*

$$(5) \quad \tilde{A}_{n,r}(t) = \tilde{A}_{n,r}^+(t) + \tilde{A}_{n,r}^-(t),$$

where

- $\tilde{A}_{n,r}^+(t)$  is a  $\gamma$ -positive polynomial with center  $n/2$ , and
- $\tilde{A}_{n,r}^-(t)$  is a  $\gamma$ -positive polynomial with center  $(n+1)/2$  and zero constant term.

Moreover,  $\tilde{A}_{n,r}^+(t)$  equals the  $h$ -polynomial of the boundary complex of an  $n$ -dimensional flag simplicial polytope.

For instance, for  $r = 2$ , writing  $\tilde{B}_n^+(t)$  and  $\tilde{B}_n^-(t)$  instead of  $\tilde{A}_{n,2}^+(t)$  and  $\tilde{A}_{n,2}^-(t)$ , we have

$$\tilde{B}_n^+(t) = \begin{cases} 1+t, & \text{if } n=1 \\ 1+5t+t^2, & \text{if } n=2 \\ 1+19t+19t^2+t^3, & \text{if } n=3 \\ 1+65t+185t^2+65t^3+t^4, & \text{if } n=4 \\ 1+211t+1371t^2+1371t^3+211t^4+t^5, & \text{if } n=5 \end{cases}$$

and

$$\tilde{B}_n^-(t) = \begin{cases} t, & \text{if } n=1 \\ 3t+3t^2, & \text{if } n=2 \\ 7t+25t^2+7t^3, & \text{if } n=3 \\ 15t+143t^2+143t^3+15t^4, & \text{if } n=4 \\ 31t+701t^2+1701t^3+701t^4+31t^5, & \text{if } n=5. \end{cases}$$

The Eulerian polynomial  $A_n(t)$  affords an  $\mathfrak{S}_n$ -equivariant analogue, meaning a graded  $\mathfrak{S}_n$ -representation  $\varphi_n = \bigoplus_{j=0}^{n-1} \varphi_{n,j}$  such that  $\sum_{j=0}^{n-1} \dim(\varphi_{n,j})t^j = A_n(t)$ , which arises very often in mathematics: most importantly, in the contexts of equivariant cohomology of toric varieties [25] [30, p. 529] [39], group actions on face rings of simplicial complexes [39] and the homology of posets [27], equivariant Ehrhart theory [36, Section 9], as well as in purely enumerative contexts; see [28] for an overview. The representation  $\varphi_n$  can also be determined by the generating function formula

$$(6) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^{n-1} \text{ch}(\varphi_{n,j})(\mathbf{x})t^j = \frac{(1-t)H(\mathbf{x}; z)}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)},$$

where  $\text{ch}$  stands for Frobenius characteristic and  $H(\mathbf{x}; z) = \sum_{n \geq 0} h_n(\mathbf{x}) z^n$  is the generating function for the complete homogeneous symmetric functions in  $\mathbf{x} = (x_1, x_2, \dots)$ . An  $\mathfrak{S}_n$ -equivariant analogue  $\tilde{\varphi}_n = \oplus_{j=0}^n \tilde{\varphi}_{n,j}$  of the binomial Eulerian polynomial  $\tilde{A}_n(t)$ , satisfying

$$(7) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,j})(\mathbf{x}) t^j = \frac{(1-t)H(\mathbf{x}; z)H(\mathbf{x}; tz)}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)},$$

was found by Shareshian and Wachs [29], who computed the graded  $\mathfrak{S}_n$ -representation on the cohomology of the toric variety associated to the polytope mentioned in Theorem 1.1, namely the  $n$ -dimensional simplicial stellohedron. As shown in [29, Section 3], the coefficients of  $z^n$  in the right-hand sides of Equations (6) and (7) are Schur  $\gamma$ -positive polynomials in  $t$ , meaning that

$$(8) \quad \sum_{j=0}^{n-1} \text{ch}(\varphi_{n,j})(\mathbf{x}) t^j = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i}(\mathbf{x}) t^i (1+t)^{n-1-2i}$$

$$(9) \quad \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,j})(\mathbf{x}) t^j = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,i}(\mathbf{x}) t^i (1+t)^{n-2i}$$

for some Schur-positive symmetric functions  $\gamma_{n,i}(\mathbf{x})$  and  $\tilde{\gamma}_{n,i}(\mathbf{x})$ . These results constitute two of only few instances of equivariant  $\gamma$ -positivity (and more specifically, in this case, of the equivariant Gal phenomenon introduced in [29, Section 5]) that exist in the literature. This paper contributes a few more. For instance, we provide an  $\mathfrak{S}_n$ -equivariant analogue  $\tilde{\varphi}_{n,r}^+ = \oplus_{j=0}^n \tilde{\varphi}_{n,r,j}^+$  of  $\tilde{A}_{n,r}^+(t)$ , satisfying

$$(10) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,r,j}^+)(\mathbf{x}) t^j = \frac{H(\mathbf{x}; z)H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r H(\mathbf{x}; tz)}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r},$$

interpret it as the graded  $\mathfrak{S}_n$ -representation on the cohomology of the toric variety associated to the simplicial polytope mentioned in Theorem 1.2 and prove its equivariant  $\gamma$ -positivity for  $r = 2$ .

The content and other results of this paper may be summarized as follows. Section 2 includes preliminaries on colored permutations, symmetric functions and  $\gamma$ -positivity and fixes notation. The first part of Theorem 1.2 is derived in Section 3 from the  $\gamma$ -positivity of the derangement polynomials for  $r$ -colored permutations, established in [2]. The proof yields a combinatorial interpretation for the corresponding  $\gamma$ -coefficients (Corollary 3.3) which generalizes the one provided by Theorem 1.1 in the case  $r = 1$ .

The second statement in Theorem 1.2 is proven in Section 5, after the relevant background on simplicial complexes and their face enumeration is explained in Section 4. The proof is based on a natural construction [1] of a flag triangulation  $\Delta(\Gamma)$  of the sphere, which contains as a subcomplex a given flag triangulation  $\Gamma$  of the simplex of the same dimension. This construction is applied in Section 5 to the refinement of the barycentric

subdivision of the simplex which was introduced in [2] to provide a partial geometric interpretation for the  $r$ -colored derangement polynomial. The simplicial polytope constructed for  $r = 1$ , in which case  $\Gamma$  is the barycentric subdivision of the  $(n - 1)$ -dimensional simplex itself, turns out to be combinatorially isomorphic to the  $n$ -dimensional simplicial stellohedron, considered in [24] (see Example 4.2).

The remainder of this paper focuses on equivariant analogues of polynomials discussed so far. Section 6 reviews standard ways in which equivariant analogues of polynomials of combinatorial significance arise in the literature, namely from group actions on simplicial complexes, fans, posets and lattice points in polyhedra. Sections 7 and 8 provide  $\mathfrak{S}_n$ -equivariant analogues of  $r$ -colored derangement and binomial Eulerian polynomials, partially interpret them in terms of  $\mathfrak{S}_n$ -actions on face rings of triangulations of spheres or local face modules of triangulations of simplices and partially prove their (nonsymmetric) equivariant  $\gamma$ -positivity, thus uncovering new instances of the equivariant Gal phenomenon [29, Section 5] and its local version [4, Section 5.2] [5, Section 5]. The methods employed in these sections extend those of [5, Section 5] and [40]. A combinatorial interpretation for the  $\gamma$ -coefficients of  $\tilde{B}_n^+(t)$  and  $\tilde{B}_n^-(t)$  which is different from the one given in Section 3 is deduced via specialization (Proposition 8.6). Section 9 discusses an  $\mathfrak{S}_n$ -equivariant analogue of the Eulerian polynomial  $A_{n,r}(t)$ .

## 2. PRELIMINARIES

This section provides key definitions, fixes notation and recalls a few useful facts about group representations and symmetric and quasisymmetric functions which will be used frequently. We will write  $[a, b] = \{a, a + 1, \dots, b\}$  for integers  $a \leq b$  and set  $[n] := [1, n]$ .

**Colored permutations.** The wreath product group  $\mathbb{Z}_r \wr \mathfrak{S}_n$  consists of all permutations  $w$  of  $[n] \times [0, r - 1]$  such that  $w(a \times 0) = b \times j \Rightarrow w(a \times i) = b \times (i + j)$ , where  $i + j$  is computed modulo  $r$  and the product is composition of permutations. The elements of this group are represented as  $r$ -colored permutations of  $[n]$ , meaning pairs  $\sigma \times \varepsilon$  where  $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n)) \in \mathfrak{S}_n$ ,  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in [0, r - 1]^n$  and  $\varepsilon_k$  is thought of as the color assigned to  $\sigma(k)$ .

Given an  $r$ -colored permutation  $w = \sigma \times \varepsilon$ , as above, we say that an index  $k \in [n]$  is a *descent* of  $w$  if either  $\varepsilon_k > \varepsilon_{k+1}$ , or  $\varepsilon_k = \varepsilon_{k+1}$  and  $\sigma(k) > \sigma(k + 1)$ , where  $\sigma(n + 1) := n + 1$  and  $\varepsilon_{n+1} := 0$  (in particular,  $n$  is a descent of  $w$  if and only if  $\sigma(n)$  has nonzero color). We denote by  $\text{Des}(w)$  the set of descents of  $w$  and define the set of *ascents*  $\text{Asc}(w)$  as the complement of  $\text{Des}(w)$  in  $[n]$ . An *excedance* of  $w$  is an index  $k \in [n]$  such that either  $\sigma(k) > k$ , or  $\sigma(k) = k$  and  $\varepsilon_k$  is nonzero. The number of ascents, descents and excedances of  $w$ , respectively, will be denoted by  $\text{asc}(w)$ ,  $\text{des}(w)$  and  $\text{exc}(w)$ . The *flag excedance* number is defined as  $\text{fexc}(w) = r \cdot \text{exc}_A(w) + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$ , where  $\text{exc}_A(w)$  is the number of indices  $k \in [n]$  such that  $\sigma(k) > k$  and  $\varepsilon_k = 0$ . More information, for instance about the generating functions of these statistics over  $\mathbb{Z} \wr \mathfrak{S}_n$ , and references can be found in [2, Section 2].

**Symmetric functions.** Our notation generally follows that in [22] [35, Chapter 7]. We denote by  $\Lambda(x)$  the  $\mathbb{C}$ -algebra of symmetric functions in a sequence  $\mathbf{x} = (x_1, x_2, \dots)$  of commuting independent indeterminates. We set

$$\begin{aligned} E(\mathbf{x}; z) &:= \sum_{n \geq 0} e_n(\mathbf{x}) z^n = \prod_{i \geq 1} (1 + x_i t), \\ H(\mathbf{x}; z) &:= \sum_{n \geq 0} h_n(\mathbf{x}) z^n = \prod_{i \geq 1} \frac{1}{1 - x_i t}, \end{aligned}$$

where  $e_n(\mathbf{x})$  and  $h_n(\mathbf{x})$  is the elementary and complete homogeneous, respectively, symmetric function of degree  $n$  in  $\mathbf{x}$ . As usual, we denote by  $s_\lambda(\mathbf{x})$  the Schur function associated to a partition  $\lambda$  and call a symmetric function *Schur-positive*, if it can be written as a nonnegative linear combination of Schur functions.

Given a (complex, finite-dimensional)  $\mathfrak{S}_n$ -representation  $\varphi$  with character  $\chi$ , the *Frobenius characteristic* of  $\varphi$  is defined by the formula

$$\text{ch}(\varphi)(\mathbf{x}) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \chi(w) p_w(\mathbf{x}),$$

where  $p_w(\mathbf{x}) = p_\mu(\mathbf{x})$  for every permutation  $w \in \mathfrak{S}_n$  of cycle type  $\mu \vdash n$  and  $p_\mu(\mathbf{x})$  is a power sum symmetric function. We refer to [35, Section 7.18] for a detailed discussion of the important properties of this map and mention that if  $\varphi$  is non-virtual, then  $\text{ch}(\varphi)(\mathbf{x})$  is a Schur-positive homogeneous symmetric function of degree  $n$  which determines  $\varphi$  up to isomorphism. The map  $\text{ch}$  has a natural generalization to the wreath product group  $\mathbb{Z}_r \wr \mathfrak{S}_n$  (see [22, Section I.B.6]), which we denote here by  $\text{ch}_r$ . The image under  $\text{ch}_r$  of any non-virtual  $(\mathbb{Z}_r \wr \mathfrak{S}_n)$ -representation  $\varphi$  is Schur-positive in  $\Lambda(\mathbf{x}^{(0)}) \otimes \Lambda(\mathbf{x}^{(1)}) \otimes \dots \otimes \Lambda(\mathbf{x}^{(r-1)})$ , meaning a nonnegative linear combination of products of Schur functions in the sequences of indeterminates  $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r-1)}$  of total degree  $n$ , which determines  $\varphi$  up to isomorphism.

We recall that the fundamental quasisymmetric function in  $\mathbf{x}$  of degree  $n$ , associated to  $S \subseteq [n-1]$ , is defined as

$$F_{n,S}(\mathbf{x}) := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

For a  $\mathbb{C}$ -linear combination  $f(\mathbf{x})$  of such functions, we will denote by  $\text{ex}^*(f)$  the evaluation of  $f(1-q, (1-q)q, (1-q)q^2, \dots)$  at  $q = 1$  (this is a specialization of the  $q$ -analogue of the exponential specialization, discussed in [35, Section 7.8]). We then have  $\text{ex}^*(F_{n,S}(\mathbf{x})) = 1/n!$  for every  $S \subseteq [n-1]$ . This implies that

- $\text{ex}^*(e_n(\mathbf{x})) = \text{ex}^*(h_n(\mathbf{x})) = 1/n!$  for every  $n \in \mathbb{N}$ , and
- $\text{ex}^*(s_\lambda(\mathbf{x})) = f^\lambda/n!$  for every  $\lambda \vdash n$ , where  $f^\lambda$  is the number of standard Young tableaux of shape  $\lambda$ .

As a result,  $\text{ex}^*(\text{ch}(\varphi)(\mathbf{x})) = \dim(\varphi)/n!$  for every  $\mathfrak{S}_n$ -representation  $\varphi$  and, by extending  $\text{ex}^*$  to formal power series with quasisymmetric functions as coefficients,  $\text{ex}^*(E(\mathbf{x}; z)) = \text{ex}^*(H(\mathbf{x}; z)) = e^z$ .

**Gamma-positivity.** We will generally use the term ‘palindromicity’, instead of ‘symmetry’, for polynomials to avoid confusion with the notion of symmetric function. Thus, a polynomial  $f(t) = \sum_j a_j t^j \in \mathbb{N}[t]$  is called

- *palindromic*, with center (of symmetry)  $n/2$ , if  $a_j = a_{n-j}$  for all  $j \in \mathbb{Z}$ ,
- *unimodal*, if  $a_0 \leq a_1 \leq \dots \leq a_k \geq a_{k+1} \geq \dots$  for some  $k \in \mathbb{N}$ ,
- *alternatingly increasing*, if  $a_0 \leq a_n \leq a_1 \leq a_{n-1} \leq \dots \leq a_{\lfloor (n+1)/2 \rfloor}$  for some  $n \in \mathbb{N}$  and  $a_k = 0$  for  $k > n$ ,
- $\gamma$ -*positive*, if

$$(11) \quad f(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i t^i (1+t)^{n-2i}$$

for some  $n \in \mathbb{N}$  and nonnegative integers  $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor}$ .

Gamma-positivity implies palindromicity and unimodality and, in fact, has developed into a powerful method to prove these properties; see [4] [23, Chapter 4]. There are versions of this notion for nonpalindromic polynomials [4, Section 5.1]. We say that a nonzero polynomial  $f(t) \in \mathbb{N}[t]$  has center  $(a+b)/2$ , where  $a$  (respectively,  $b$ ) is the smallest (respectively, largest) integer  $k$  for which the coefficient of  $t^k$  in  $f(t)$  is nonzero. We call  $f(t)$  *right  $\gamma$ -positive* (respectively, *left  $\gamma$ -positive*) if it can be written as a sum of two  $\gamma$ -positive polynomials with centers of symmetry  $n/2$  and  $(n+1)/2$  (respectively,  $n/2$  and  $(n-1)/2$ ), where  $n/2$  is the center of  $f(t)$ . When they exist, such expressions are unique and imply that  $f(t)$  (respectively,  $tf(t)$ ) is alternatingly increasing and, in particular, that  $f(t)$  is unimodal. For instance, the colored binomial Eulerian and colored derangement polynomials which appear in Theorems 1.2 and 3.2 are right  $\gamma$ -positive and left  $\gamma$ -positive, respectively.

The notions just introduced carry over to polynomials  $f(t) = \sum_j \varphi_j t^j$  with coefficients in the representation ring  $R(G)$  of a finite group  $G$ , when the usual total order on  $\mathbb{Z}$  is replaced by the partial order  $\leq_G$  on  $R(G)$  defined by setting  $\varphi \leq_G \psi$  if  $\psi - \varphi$  is equal to a non-virtual  $G$ -representation in  $R(G)$ . For instance,  $f(t)$  is  $\gamma$ -positive if Equation (11) holds for some  $n \in \mathbb{N}$  and non-virtual  $G$ -representations  $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor}$ . For  $G = \mathbb{Z}_r \wr \mathfrak{S}_n$ , this happens if and only if the polynomial  $\sum_j \text{ch}_r(\varphi_j) t^j$  is Schur  $\gamma$ -positive, meaning that

$$\sum_j \text{ch}_r(\varphi_j) t^j = \sum_{i=0}^{\lfloor n/2 \rfloor} g_i t^i (1+t)^{n-2i}$$

for some  $n \in \mathbb{N}$  and Schur-positive functions  $g_i \in \Lambda(\mathbf{x}^{(0)}) \otimes \Lambda(\mathbf{x}^{(1)}) \otimes \dots \otimes \Lambda(\mathbf{x}^{(r-1)})$ .

### 3. GAMMA-POSITIVITY OF BINOMIAL EULERIAN POLYNOMIALS

This section derives the  $\gamma$ -positivity statement of Theorem 1.2 from a similar property of the derangement polynomials for  $r$ -colored permutations, proven in [2], and provides an interpretation for the corresponding  $\gamma$ -coefficients. The following lemma will be crucial.

**Lemma 3.1.** *For  $n \in \mathbb{N}$ , let  $f_n(t), g_n(t), h_n(t) \in \mathbb{N}(t)$  be polynomials such that*

$$(12) \quad h_n(t) = \sum_{m=0}^n \binom{n}{m} t^{n-m} g_m(t)$$

$$(13) \quad g_m(t) = \sum_{k=0}^m \binom{m}{k} f_k(t)$$

for all  $m, n \in \mathbb{N}$ . If

$$(14) \quad f_n(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,i} t^i (1+t)^{n-2i}$$

for every  $n \in \mathbb{N}$ , then

$$(15) \quad h_n(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \left( \sum_{k=2i}^n \binom{n}{k} \xi_{k,i} \right) t^i (1+t)^{n-2i}$$

for every  $n \in \mathbb{N}$ .

In particular, if  $f_n(t)$  is palindromic (or  $\gamma$ -positive), with center  $n/2$ , for every  $n \in \mathbb{N}$ , then  $h_n(t)$  has the same property for every  $n \in \mathbb{N}$ .

*Proof.* Combining Equations (12) and (13) and changing the order of summation, we get

$$\begin{aligned} h_n(t) &= \sum_{m=0}^n \binom{n}{m} t^{n-m} \sum_{k=0}^m \binom{m}{k} f_k(t) = \sum_{k=0}^n f_k(t) \sum_{m=k}^n \binom{n}{m} \binom{m}{k} t^{n-m} \\ &= \sum_{k=0}^n f_k(t) \sum_{m=k}^n \binom{n}{k} \binom{n-k}{m-k} t^{n-m} = \sum_{k=0}^n \binom{n}{k} f_k(t) (1+t)^{n-k}. \end{aligned}$$

Replacing  $f_k(t)$  by the expression provided by Equation (14) and changing the order of summation again yields (15) and the proof follows.  $\square$

The derangement polynomial for the group  $\mathbb{Z}_r \wr \mathfrak{S}_n$ , introduced and studied on this level of generality by Chow and Mansour [13, Section 3], can be defined by either one of the formulas

$$\begin{aligned} d_{n,r}(t) &= \sum_{w \in \mathcal{D}_{n,r}} t^{\text{exc}(w)} \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} A_{k,r}(t), \end{aligned}$$



where  $\mathcal{D}_{n,r}$  is the set of derangements (colored permutations with no fixed point of zero color) in  $\mathbb{Z}_r \wr \mathfrak{S}_n$ . By the principle of inclusion-exclusion, we have

$$(16) \quad A_{n,r}(t) = \sum_{k=0}^n \binom{n}{k} d_{k,r}(t)$$

for all  $n \in \mathbb{N}$ , where  $d_{0,r}(t) := 1$ . For later use (see Section 7), we also recall the formula ([13, Theorem 5 (iv)])

$$(17) \quad \sum_{n \geq 0} d_{n,r}(t) \frac{z^n}{n!} = \frac{(1-t)e^{(r-1)tz}}{e^{rtz} - te^{rz}}.$$

The following statement is one of the main results of [2]; it reduces to [4, Theorem 2.13] for  $r = 1$ .

**Theorem 3.2.** ([2, Theorem 1.3]) *For all positive integers  $n, r$  we have*

$$(18) \quad d_{n,r}(t) = d_{n,r}^+(t) + d_{n,r}^-(t),$$

where

$$\begin{aligned} d_{n,r}^+(t) &= \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+ t^i (1+t)^{n-2i} \\ d_{n,r}^-(t) &= \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \xi_{n,r,i}^- t^i (1+t)^{n+1-2i} \end{aligned}$$

and

- $\xi_{n,r,i}^+$  is equal to the number of colored permutations  $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$  for which  $\text{Asc}(w) \subseteq [2, n]$  has exactly  $i$  elements, no two consecutive, and contains  $n$ ,
- $\xi_{n,r,i}^-$  is equal to the number of colored permutations  $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$  for which  $\text{Asc}(w) \subseteq [2, n-1]$  has exactly  $i-1$  elements, no two consecutive.

Setting

$$(19) \quad A_{n,r}^+(t) := \sum_{k=0}^n \binom{n}{k} d_{k,r}^+(t)$$

$$(20) \quad A_{n,r}^-(t) := \sum_{k=0}^n \binom{n}{k} d_{k,r}^-(t),$$

where  $d_{0,r}^+(t) := 1$  and  $d_{0,r}^-(t) := 0$ , and in view of Equations (16) and (18), we have

$$(21) \quad A_{n,r}(t) = A_{n,r}^+(t) + A_{n,r}^-(t).$$

Then (5) holds, with

$$(22) \quad \tilde{A}_{n,r}^+(t) := \sum_{m=0}^n \binom{n}{m} t^{n-m} A_{m,r}^+(t),$$

$$(23) \quad \tilde{A}_{n,r}^-(t) := \sum_{m=0}^n \binom{n}{m} t^{n-m} A_{m,r}^-(t).$$

As a result, the following corollary proves the first statement of Theorem 1.2 (note also that it generalizes the interpretation for  $\tilde{\gamma}_{n,i}$  given in Theorem 1.1).

**Corollary 3.3.** *For all positive integers  $n, r$  with  $r \geq 2$ , we have*

$$\begin{aligned} \tilde{A}_{n,r}^+(t) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,r,i}^+ t^i (1+t)^{n-2i} \\ \tilde{A}_{n,r}^-(t) &= \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,r,i}^- t^i (1+t)^{n+1-2i}, \end{aligned}$$

where

$$(24) \quad \tilde{\gamma}_{n,r,i}^+ = \sum_{k=2i}^n \binom{n}{k} \xi_{k,r,i}^+$$

$$(25) \quad \tilde{\gamma}_{n,r,i}^- = \sum_{k=2i}^n \binom{n}{k} \xi_{k,r,i}^-.$$

As a result:

- $\tilde{\gamma}_{n,r,i}^+$  is equal to the number of  $w \in \mathbb{Z}_r \wr \mathfrak{S}_{n+1}$  for which  $\text{Asc}(w)$  has exactly  $i+1$  elements, no two consecutive, and contains  $n$  and  $w(1) > w(2) > \dots > w(m) = 1$  have all zero color, for some  $m$ ,
- $\tilde{\gamma}_{n,r,i}^-$  is equal to the number of  $w \in \mathbb{Z}_r \wr \mathfrak{S}_{n+1}$  for which  $\text{Asc}(w) \subseteq [n-1]$  has exactly  $i$  elements, no two consecutive, and  $w(1) > w(2) > \dots > w(m) = 1$  have all zero color, for some  $m$ .

*Proof.* The first part follows from Lemma 3.1, where the role of  $f_n(t)$  is played by  $d_{n,r}^+(t)$  and  $d_{n,r}^-(t)$ , respectively, and Theorem 3.2. The second follows by rewriting Equations (24) and (25) as

$$\begin{aligned} \tilde{\gamma}_{n,r,i}^+ &= \sum_{k=0}^{n-2i} \binom{n}{k} \xi_{n-k,r,i}^+ \\ \tilde{\gamma}_{n,r,i}^- &= \sum_{k=0}^{n-2i} \binom{n}{k} \xi_{n-k,r,i}^- \end{aligned}$$

and using the interpretations for  $\xi_{n,r,i}^+$  and  $\xi_{n,r,i}^-$  provided by Theorem 3.2.  $\square$

## 4. TRIANGULATIONS

This section recalls some definitions and constructions about simplicial complexes and their triangulations which are needed to complete the proof of Theorem 1.2. Familiarity with basic notions, such as the correspondence between abstract and geometric simplicial complexes, will be assumed (detailed expositions can be found in [8, 15, 33]). All simplicial complexes considered here will be finite. We will denote by  $\|K\|$  the polyhedron (union of all simplices) of a geometric simplicial complex  $K$  and by  $|V|$  and  $2^V$  the cardinality and power set, respectively, of a finite set  $V$ .

Consider two geometric simplicial complexes  $K'$  and  $K$  in some Euclidean space  $\mathbb{R}^N$ , with corresponding abstract simplicial complexes  $\Delta'$  and  $\Delta$ . Then,  $K'$  is a triangulation of  $K$ , and  $\Delta'$  is a triangulation of  $\Delta$ , if (a) every simplex of  $K'$  is contained in some simplex of  $K$ ; and (b)  $\|K'\| = \|K\|$ . Given a simplex  $L \in K$  with corresponding face  $F \in \Delta$ , the triangulation  $K'$  naturally restricts to a triangulation  $K'_L$  of  $L$ . The subcomplex  $\Delta'_F$  of  $\Delta'$  corresponding to  $K'_L$  is a triangulation of the abstract simplex  $2^F$ , called the *restriction* of  $\Delta'$  to  $F$ . A triangulation of a sphere (respectively, simplex) is called *regular* if it is combinatorially isomorphic to the boundary complex (respectively, to the complex of lower faces) of a simplicial polytope of one dimension higher.

A simplicial complex  $K$  is called *pure* if all its facets (faces which are maximal with respect to inclusion) have the same dimension and *flag* if it contains every simplex whose one-dimensional skeleton is a subcomplex of  $K$ . A pure simplicial complex  $K$  is called *shellable* if its facets can be linearly ordered, so that the intersection of any facet  $L$ , other than the first, with the union of all preceding ones is equal to the union of (one or more) codimension one faces of  $L$ .

The enumerative invariants of simplicial complexes which will be of importance here are the  $h$ -polynomial of a triangulation of a sphere [33, Chapter II] and the local  $h$ -polynomial of a triangulation of a simplex [32] [33, Section III.10]. The  $h$ -polynomial of an  $(n-1)$ -dimensional abstract simplicial complex  $\Delta$  is defined as

$$(26) \quad h(\Delta, t) = \sum_{i=0}^n f_{i-1}(\Delta) t^i (1-t)^{n-i},$$

where  $f_i(\Delta)$  is the number of  $i$ -dimensional faces of  $\Delta$ . Given a triangulation  $\Gamma$  of an  $(n-1)$ -dimensional simplex  $2^V$ , the *local  $h$ -polynomial* of  $\Gamma$  (with respect to  $V$ ) is defined [32, Definition 2.1] by the formula

$$(27) \quad \ell_V(\Gamma, t) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, t),$$

where  $\Gamma_F$  is the restriction of  $\Gamma$  to the face  $F \in 2^V$ . By the principle of inclusion-exclusion, we have

$$(28) \quad h(\Gamma, t) = \sum_{F \subseteq V} \ell_F(\Gamma_F, t).$$

The polynomials (26) and (27) have especially attractive properties when  $\Gamma$  and  $\Delta$  triangulate an  $(n - 1)$ -dimensional simplex or sphere, respectively [32] [33, Chapter III]. For instance, they both have nonnegative and palindromic coefficients, with center  $n/2$ . Moreover, they are unimodal if  $\Gamma$  and  $\Delta$ , respectively, are regular triangulations and are conjectured to be  $\gamma$ -positive [1, Conjecture 5.4] [17, Conjecture 2.1.7] (see also [4, Section 3]) when the triangulations are assumed to be flag.

**The complex  $\Delta(\Gamma)$ .** Every triangulation of a simplex can be extended to a triangulation of a sphere of the same dimension in a way which preserves important properties, such as flagness and regularity. This construction, which we now recall, was exploited in [1].

Let  $V = \{v_1, v_2, \dots, v_n\}$  be an  $n$ -element set and  $\Gamma$  be a triangulation of the simplex  $2^V$ . Pick an  $n$ -element set  $U = \{u_1, u_2, \dots, u_n\}$  which is disjoint from the vertex set of  $\Gamma$  and denote by  $\Delta(\Gamma)$  the collection of sets of the form  $E \cup G$ , where  $E = \{u_i : i \in I\}$  is a face of the simplex  $2^U$  for some  $I \subseteq [n]$  and  $G$  is a face of the restriction  $\Gamma_F$  of  $\Gamma$  to the face  $F = \{v_i : i \in [n] \setminus I\}$  of the simplex  $2^V$  which is complementary to  $E$ . Clearly,  $\Delta(\Gamma)$  is a simplicial complex which contains  $2^U$  and  $\Gamma$  as subcomplexes. When  $\Gamma = 2^V$  is the trivial triangulation, the complex  $\Delta(\Gamma)$  is combinatorially isomorphic to the boundary complex of the  $n$ -dimensional cross-polytope, defined as the convex hull of the set of unit coordinate vectors in  $\mathbb{R}^n$  and their negatives. Part of the following statement appeared in a more general setting in [1, Section 4].

**Proposition 4.1.** *The simplicial complex  $\Delta(\Gamma)$  triangulates an  $(n-1)$ -dimensional sphere for every triangulation  $\Gamma$  of the  $(n-1)$ -dimensional simplex  $2^V$ . Moreover,  $\Delta(\Gamma)$ :*

- (a) *is a flag complex, if so is  $\Gamma$ ,*
- (b) *is a regular triangulation, if so is  $\Gamma$ ,*
- (c) *is shellable, if so are all restrictions of  $\Gamma$  to the faces of  $2^V$ , and*
- (d) *satisfies*

$$(29) \quad h(\Delta(\Gamma), t) = \sum_{F \subseteq V} t^{n-|F|} h(\Gamma_F, t).$$

*Proof.* The first statement holds because  $\Delta(\Gamma)$  naturally triangulates  $\Delta(2^V)$ . Part (a) follows from [1, Proposition 4.6] (iii). For part (b), consider a geometric realization, say  $\Delta(K)$ , of  $\Delta(\Gamma)$  in which the elements of  $V$  are realized by the unit coordinate vectors in  $\mathbb{R}^n$  and those of  $U$  by their negatives, and let  $K$  be the subcomplex of  $\Delta(K)$  which realizes  $\Gamma$ . Since  $\Gamma$  is regular, Lemma 4.3.5 in [15] guarantees that it can be extended to a regular triangulation of  $\Delta(2^V)$ , which is realized by the boundary complex of the  $n$ -dimensional cross-polytope. On the other hand, since the boundary of this polytope does not contain any line segment joining a unit coordinate vector to its negative,  $\Delta(K)$  is the only triangulation of its boundary complex extending  $K$  and the proof follows.

To prove part (c), assume that all restrictions of  $\Gamma$  to the  $2^n$  faces of  $2^V$  are shellable. Then, the same holds for the restrictions of  $\Delta(\Gamma)$  to the  $2^n$  facets of  $\Delta(2^V)$ , since every such restriction is the simplicial join of the restriction of  $\Gamma$  to a face  $F \in 2^V$  with the subsimplex of  $2^U$  which is complementary to  $F$ . We partition the set of facets of  $\Delta(\Gamma)$

into  $2^n$  blocks, according to the facet of  $\Delta(2^V)$  in which each facet of  $\Delta(\Gamma)$  lies, and linearly order the  $2^n$  facets of  $\Delta(2^V)$  in any way which extends the inclusion order on their intersections with  $U$ . This gives a linear order on the  $2^n$  blocks, which we may extend to a total order of all facets of  $\Delta(\Gamma)$  by choosing a linear order for the facets within each block which is a shelling order for the corresponding restriction of  $\Delta(\Gamma)$ . We leave to the reader to verify that this is indeed a shelling order for  $\Delta(\Gamma)$ .

To prove part (d), we apply [1, Equation (4-2)] to the triangulation  $\Delta(\Gamma)$  of  $\Delta(2^V)$ . Taking into account the fact that cones of triangulations of simplices have zero local  $h$ -polynomials [32, p. 821], we get

$$h(\Delta(\Gamma), t) = \sum_{F \subseteq V} \ell_F(\Gamma_F, t) (1+t)^{n-|F|}.$$

Using the defining Equation (27), we conclude that

$$\begin{aligned} h(\Delta(\Gamma), t) &= \sum_{F \subseteq V} \left( \sum_{G \subseteq F} (-1)^{|F \setminus G|} h(\Gamma_G, t) \right) (1+t)^{n-|F|} \\ &= \sum_{G \subseteq V} h(\Gamma_G, t) \left( \sum_{G \subseteq F \subseteq V} (-1)^{|F \setminus G|} (1+t)^{n-|F|} \right) \\ &= \sum_{G \subseteq V} t^{n-|G|} h(\Gamma_G, t) \end{aligned}$$

and the proof follows.  $\square$

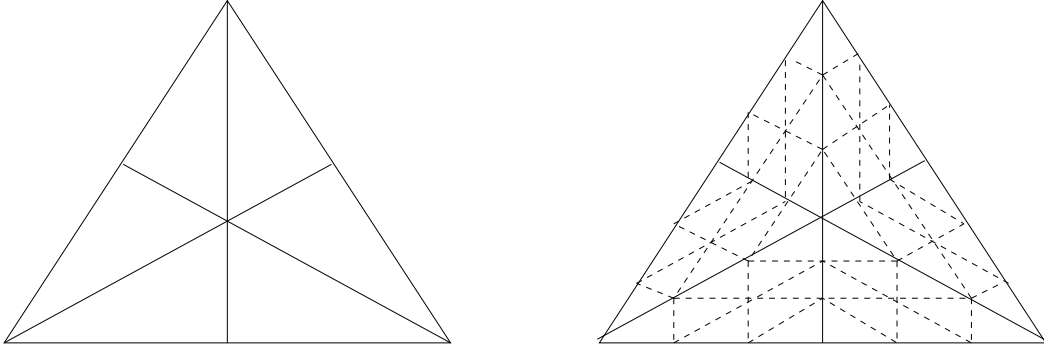
**Example 4.2.** Let  $\Gamma_n$  be the (first) barycentric subdivision of  $2^V$ , consisting of all chains of nonempty subsets of  $V$ . Then,  $h(\Gamma_n, t) = A_n(t)$  (see, for instance, [23, Theorem 9.1]) and  $\Gamma_n$  restricts to the barycentric subdivision of  $2^F$  for every face  $F \in 2^V$ . Therefore, from Equations (29) and (4) we get

$$h(\Delta(\Gamma_n), t) = \sum_{k=0}^n \binom{n}{k} t^{n-k} A_k(t) = \tilde{A}_n(t).$$

It was shown in [24, Section 10.4] that  $\tilde{A}_n(t) = h(\tilde{\Delta}_n, t)$ , where  $\tilde{\Delta}_n$  is the boundary complex of the  $n$ -dimensional simplicial stellohedron. This polytope can be constructed by successively stellarly subdividing the boundary faces of an  $n$ -dimensional simplex which contain a fixed vertex, in any order of decreasing dimension of these faces. Although this may not be obvious, the complex  $\Delta(\Gamma_n)$  is combinatorially isomorphic to  $\tilde{\Delta}_n$  (the proof is omitted). The construction of  $\Delta(\Gamma_n)$  will allow us in the following section to provide the right generalization of  $\tilde{\Delta}_n$  to the setting of  $r$ -colored permutations.

## 5. EDGEWISE SUBDIVISIONS AND THE PROOF OF THEOREM 1.2

This section employs the construction of  $\Delta(\Gamma)$  when  $\Gamma$  is the  $r$ -fold edgewise subdivision of the barycentric subdivision of a simplex to complete the proof of Theorem 1.2.

FIGURE 1. The triangulations  $\Gamma_3$  and  $\Gamma_{3,3}$ 

We first recall the definition of edgewise subdivision. Let  $\Delta$  be an abstract simplicial complex whose vertex set  $V(\Delta) = \{v_1, v_2, \dots, v_m\}$  is equipped with a fixed linear order. Given a positive integer  $r$ , denote by  $V_r(\Delta)$  the set of maps  $f : V(\Delta) \rightarrow \mathbb{N}$  such that  $\text{supp}(f) \in \Delta$  and  $f(v_1) + f(v_2) + \dots + f(v_m) = r$ , where  $\text{supp}(f)$  is the set of all  $v \in V(\Delta)$  for which  $f(v) \neq 0$ . For  $f \in V_r(\Delta)$ , let  $\iota(f) : V(\Delta) \rightarrow \mathbb{N}$  be the map defined by setting  $\iota(f)(v_j) = f(v_1) + f(v_2) + \dots + f(v_j)$  for  $j \in [m]$ . The  $r$ -fold edgewise subdivision of  $\Delta$  is the abstract simplicial complex  $\text{esd}_r(\Delta)$  on the vertex set  $V_r(\Delta)$  of which a set  $E \subseteq V_r(\Delta)$  is a face if the following two conditions are satisfied:

- $\bigcup_{f \in E} \text{supp}(f) \in \Delta$  and
- $\iota(f) - \iota(g) \in \{0, 1\}^{V(\Delta)}$ , or  $\iota(g) - \iota(f) \in \{0, 1\}^{V(\Delta)}$ , for all  $f, g \in E$ .

Clearly,  $\text{esd}_r(\Delta)$  is combinatorially isomorphic to  $\Delta$  for  $r = 1$ .

The simplicial complex  $\text{esd}_r(\Delta)$  can be realized as a triangulation of  $\Delta$ ; see, for instance, [4, Section 3.3.1] [12, Section 6] [16] and references therein, where its importance and long history in mathematics is also discussed, and [19, Section 4], where it is studied under the name *canonical triangulation*. More precisely, if  $K$  is a geometric realization of  $\Delta$  with corresponding ordered vertex set  $V(K)$ , then  $\text{esd}_r(\Delta)$  can be realized by a triangulation of  $K$  in which the vertex  $f \in V_r(\Delta)$  is represented by the point in  $\|K\|$  with barycentric coordinates  $f(v_i)/r$  with respect to  $V(K)$ . The restriction of  $\text{esd}_r(\Delta)$  to  $F \in \Delta$  coincides with the triangulation  $\text{esd}_r(2^F)$  of the simplex  $2^F$  (where  $F$  is considered with the induced linear order).

For the remainder of this section we set  $\Gamma_{n,r} = \text{esd}_r(\Gamma_n)$ , where  $\Gamma_n$  denotes the barycentric subdivision of the  $(n-1)$ -dimensional simplex (with its vertex set linearly ordered arbitrarily); see Figure 1. This triangulation was introduced in [2] in order to extend to the context of  $r$ -colored permutations Stanley's interpretation [32, Proposition 2.4] of the derangement polynomial as a local  $h$ -polynomial. Since edgewise subdivision preserves flagness and regularity (see, for instance, [19, Theorem 4.11]),  $\Gamma_{n,r}$  is in fact a regular flag triangulation of the  $(n-1)$ -dimensional simplex. The relevance of  $\Gamma_{n,r}$  to the study of Eulerian polynomials is explained by the following statement (recall that  $A_{n,r}^+(t)$  was

defined by Equation (19) and the flag excedance number  $\text{fexc}(w)$  for  $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$  was defined in Section 2); part (c) is included for completeness.

**Proposition 5.1.** *For all positive integers  $n, r$ :*

- (a)  $A_{n,r}^+(t) = h(\Gamma_{n,r}, t)$ .
- (b)

$$\frac{A_{n,r}^+(t)}{(1-t)^n} = \sum_{k \geq 0} ((rk+1)^n - (rk)^n) t^k.$$

- (c)

$$A_{n,r}^+(t) = \sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_n)^+} t^{\text{des}(w)} = \sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_n)^b} t^{\text{fexc}(w)/r},$$

where  $(\mathbb{Z}_r \wr \mathfrak{S}_n)^+$  is the set of colored permutations  $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$  with first coordinate of zero color and  $(\mathbb{Z}_r \wr \mathfrak{S}_n)^b$  is the set of  $w \in \mathbb{Z}_r \wr \mathfrak{S}_n$  such that the sum of the colors of the coordinates of  $w$  is divisible by  $r$ .

*Proof.* It was shown in [2, Section 5] that  $\ell_V(\Gamma_{n,r}, t) = d_{n,r}^+(t)$ . Thus, part (a) follows from this fact and Equations (19) and (28). It was also shown in [2, Section 5] that

$$h(\Gamma_{n,r}, t) = E_r((1+t+t^2+\cdots+t^{r-1})^n A_n(t)) = \sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_n)^b} t^{\text{fexc}(w)/r},$$

where  $E_r : \mathbb{R}[t] \rightarrow \mathbb{R}[t]$  is the linear operator defined by setting  $E_r(t^k) = t^{k/r}$ , if  $k$  is divisible by  $r$ , and  $E_r(t^k) = 0$  otherwise. Furthermore, the equality

$$\sum_{w \in (\mathbb{Z}_r \wr \mathfrak{S}_n)^+} x^{\text{des}(w)} = E_r((1+t+t^2+\cdots+t^{r-1})^n A_n(t))$$

follows by setting  $q = 1$  in the Carlitz identity [6, Theorem A.1] for  $\mathbb{Z}_r \wr \mathfrak{S}_n$ . Thus, part (c) follows from part (a) and the previous remarks. For part (b), we use the previous formulas and Worpitzky's identity  $A_n(t)/(1-t)^{n+1} = \sum_{k \geq 1} k^n t^{k-1}$  to conclude that

$$\begin{aligned} A_{n,r}^+(t) &= E_r((1+t+t^2+\cdots+t^{r-1})^n A_n(t)) = E_r\left(\frac{(1-t^r)^n}{(1-t)^n} A_n(t)\right) \\ &= (1-t)^n E_r\left(\frac{A_n(t)}{(1-t)^n}\right) = (1-t)^n E_r\left(\sum_{k \geq 1} k^n (t^{k-1} - t^k)\right) \\ &= (1-t)^n \sum_{k \geq 0} ((rk+1)^n - (rk)^n) t^k \end{aligned}$$

and the proof follows. □

The following statement completes the proof of Theorem 1.2.

**Corollary 5.2.** *We have  $\tilde{A}_{n,r}^+(t) = h(\Delta(\Gamma_{n,r}), t)$  for all positive integers  $n, r$ . In particular,  $\tilde{A}_{n,r}^+(t)$  is equal to the  $h$ -polynomial of the boundary complex of an  $n$ -dimensional flag simplicial polytope.*

*Proof.* The first statement is an immediate consequence of Proposition 5.1 (a) and Equations (22) and (29). In view of Proposition 4.1, the second statement follows from the first and the fact that  $\Gamma_{n,r}$  is a regular flag triangulation.  $\square$

Interpretations for the polynomials  $d_{n,r}^+(t)$  and  $d_{n,r}^-(t)$ , which appear in Theorem 3.2, as the local  $h^*$ -polynomials of certain lecture hall simplices were found by Gustafsson and Solus [18]. This raises the following question.

**Question 5.3.** *Is there an Ehrhart-theoretic interpretation of  $\tilde{A}_{n,r}^-(t)$ , possibly similar to the ones provided for  $d_{n,r}^+(t)$  and  $d_{n,r}^-(t)$  in [18]?*

## 6. GROUP ACTIONS

This section provides background from [5] [32, Section 4] [40] which will be necessary to study  $\mathfrak{S}_n$ -equivariant analogues of colored Eulerian, derangement and binomial Eulerian polynomials in the following sections. We assume familiarity with Stanley–Reisner theory, Ehrhart theory and the homology of posets and refer the reader to the sources [7, 31, 33, 41] for detailed expositions.

**6.1. Fans and simplicial complexes.** Our discussion here follows closely [40, Section 1] and [32, Section 4]. A *simplicial fan* in Euclidean space  $\mathbb{R}^n$  is defined as a finite collection  $\mathcal{F}$  of pointed simplicial cones in  $\mathbb{R}^n$  with apex at the origin, such that (a) every face of every cone in  $\mathcal{F}$  belongs to  $\mathcal{F}$ ; and (b) the intersection of any two cones in  $\mathcal{F}$  is a face of both. To such a fan  $\mathcal{F}$  one can associate an abstract simplicial complex  $\Delta$  on the vertex set of one-dimensional cones (rays) of  $\mathcal{F}$  in the obvious way. We denote by  $\|\mathcal{F}\|$  the union of the cones of  $\mathcal{F}$  and assume that  $\Delta$  is homeomorphic to the  $(n-1)$ -dimensional sphere (this happens if  $\|\mathcal{F}\| = \mathbb{R}^n$ , in which case  $\mathcal{F}$  is called *complete*), or to the  $(n-1)$ -dimensional ball; in particular,  $\Delta$  is Cohen–Macaulay over  $\mathbb{C}$  (and any other field).

Suppose that  $G$  is a finite group of orthogonal transformations of  $\mathbb{R}^n$  which acts on  $\mathcal{F}$  and denote by  $V(\Delta)$  the vertex set of  $\Delta$ . Then,  $G$  acts simplicially on  $\Delta$  and linearly on the polynomial ring  $S = \mathbb{C}[x_v : v \in V(\Delta)]$  in commuting indeterminates which are in one-to-one correspondence with the vertices of  $\Delta$ . This action preserves the ideal  $I_\Delta$  of  $S$  which is generated by the square-free monomials which correspond to the non-faces of  $\Delta$  and hence  $G$  acts linearly on the *face ring*  $\mathbb{C}[\Delta] = S/I_\Delta$  as well. As explained on [40, p. 250], the group  $G$  leaves invariant the  $\mathbb{C}$ -linear span of the linear forms

$$(30) \quad \theta_i = \sum_{v \in V(\Delta)} \langle v, e_i \rangle x_v$$

for  $i \in [n]$ , where  $\langle \cdot, \cdot \rangle$  is the standard inner product on  $\mathbb{R}^n$  and  $(e_1, e_2, \dots, e_n)$  is the basis of unit coordinate vectors (any other basis works, but this one will be convenient in the



sequel). As a result,  $G$  acts linearly on each homogeneous component of the (standard) graded ring

$$\mathbb{C}(\Delta) := \mathbb{C}[\Delta]/\Theta = \bigoplus_{j=0}^n \mathbb{C}(\Delta)_j,$$

where  $\Theta$  is the ideal of  $\mathbb{C}[\Delta]$  generated by  $\theta_1, \theta_2, \dots, \theta_n$ . Since (by [33, Lemma III.2.4]) this sequence is a linear system of parameters for  $\mathbb{C}[\Delta]$ , which is Cohen–Macaulay over  $\mathbb{C}$ , we have  $\sum_{j=0}^n \dim(\mathbb{C}(\Delta)_j)t^j = h(\Delta, t)$  and hence  $\mathbb{C}(\Delta) = \bigoplus_{j=0}^n \mathbb{C}(\Delta)_j$  is a  $G$ -equivariant analogue of  $h(\Delta, t)$ . By a theorem of Danilov [14], if  $\mathcal{F}$  is complete and its cones are generated by elements of some lattice, then the space  $\mathbb{C}(\Delta)_j$  is isomorphic to the cohomology of degree  $2j$  over  $\mathbb{C}$  of the complex projective toric variety associated to  $\mathcal{F}$  (although this is not the point of view adopted here). The pair  $(\Delta, G)$  is said to satisfy the *equivariant Gal phenomenon* [29, Section 5] if the polynomial  $\sum_{j=0}^n \mathbb{C}(\Delta)_j t^j$  is  $\gamma$ -positive.

Following [40, Section 1], we say that the action of  $G$  on  $\Delta$  is *proper* (and that  $\mathcal{F}$  carries a proper  $G$ -action) if every  $w \in G$  fixes pointwise the vertices of every face  $F \in \Delta$  which is fixed by  $w$ . Under this assumption, the set  $\Delta^w$  of faces of  $\Delta$  which are fixed by  $w$  forms an induced subcomplex of  $\Delta$ , for every  $w \in G$ . Assuming that  $G = \mathfrak{S}_n$  acts on  $\mathbb{R}^n$  by permuting coordinates and combining Theorem 1.4 with the considerations in Section 6 of [40], we conclude that the formula

$$(31) \quad \sum_{j=0}^n \text{ch}(\mathbb{C}(\Delta)_j)(\mathbf{x})t^j = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \frac{h(\Delta^w, t)}{(1-t)^{1+\dim(\Delta^w)}} \prod_{i \geq 1} (1-t^{\lambda_i(w)}) p_{\lambda_i(w)}(\mathbf{x})$$

holds, provided  $\mathfrak{S}_n$  acts properly on  $\Delta$ , where  $\lambda_1(w) \geq \lambda_2(w) \geq \dots$  are the lengths of the cycles of  $w \in \mathfrak{S}_n$  and  $p_k(\mathbf{x})$  is a power sum symmetric function. Stembridge [40, Section 6] used this formula to give a new proof of Equation (6), essentially due to Procesi [25], where  $\varphi_n$  is the graded  $\mathfrak{S}_n$ -representation  $\mathbb{C}(\Delta_n) = \bigoplus_{j=0}^n \mathbb{C}(\Delta_n)_j$  obtained from the  $\mathfrak{S}_n$ -action on the Coxeter fan associated to  $\mathfrak{S}_n$  (we note that, although [40, Theorem 1.4] is stated for complete fans, its proof exploits the Cohen–Macaulayness of  $\Delta$  and hence applies in our situation).

Suppose now that  $\|\mathcal{F}\|$  is the positive orthant in  $\mathbb{R}^n$ , generated by the unit coordinate vectors  $e_1, e_2, \dots, e_n$ , which is thus simplicially subdivided by  $\mathcal{F}$ . We will denote by  $\Gamma$  the associated simplicial complex, which can be considered as a triangulation of the geometric simplex  $\Sigma_n$  on the vertex set  $V_n = \{e_1, e_2, \dots, e_n\}$ . Recall that  $G$  acts on  $\mathcal{F}$  by orthogonal transformations and note that, in this case,  $G$  must be a subgroup of the automorphism group  $\mathfrak{S}_n$  of  $\Sigma_n$ . The *local face module*

$$L_{V_n}(\Gamma) = \bigoplus_{j=0}^n L_{V_n}(\Gamma)_j$$

is a graded  $\mathbb{C}$ -vector space (and  $S$ -module) defined [32, Definition 4.5] as the image in  $\mathbb{C}(\Gamma)$  of the ideal of  $\mathbb{C}[\Gamma]$  generated by the square-free monomials which correspond to the faces of  $\Gamma$  lying in the relative interior of  $\Sigma_n$ . Since the coefficient of  $x_v$  in the right-hand side of Equation (30) is equal to zero for every  $v \in V(\Gamma)$  lying on the facet of  $\Sigma_n$  opposite to  $e_i$ ,

the sequence  $\theta_1, \theta_2, \dots, \theta_n$  is a special linear system of parameters for  $\mathbb{C}[\Gamma]$ , in the sense of [32, Definition 4.2]. By [32, Theorem 4.6], we have  $\sum_{j=0}^n \dim(L_{V_n}(\Gamma))_j t^j = \ell_{V_n}(\Gamma, t)$  and, as explained in [32, p. 823], the group  $G$  acts on each homogeneous component of  $L_{V_n}(\Gamma)$ , which becomes a graded  $G$ -representation. The pair  $(\Gamma, G)$  satisfies the *local equivariant Gal phenomenon* [4, Section 5] [5, Section 5] if the polynomial  $\sum_{j=0}^n L_{V_n}(\Gamma)_j t^j$ , which is a  $G$ -equivariant analogue of  $\ell_{V_n}(\Gamma, t)$ , is  $\gamma$ -positive.

Our prototypical example is the standard action of  $\mathfrak{S}_n$ , viewed as the group of symmetries of the simplex  $\Sigma_n$ , on the barycentric subdivision of  $\Sigma_n$  (this action is easily verified to be proper). The following result combines [32, Proposition 4.20] with an identity due to Gessel [4, Equation (87)] [29, Equation (6.3)] (a proof of which is given in [5, Section 4]).

**Proposition 6.1.** *For the  $\mathfrak{S}_n$ -action on the barycentric subdivision  $\Gamma_n$  of the simplex  $\Sigma_n$  and the corresponding graded  $\mathfrak{S}_n$ -representation  $\psi_n = \bigoplus_{j=0}^n \psi_{n,j}$  on the local face module  $L_{V_n}(\Gamma_n)$ , we have*

$$1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\psi_{n,j})(\mathbf{x}) t^j = \frac{1-t}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)}.$$

Moreover, the polynomial  $\sum_{j=0}^n \psi_{n,j} t^j$  is  $\gamma$ -positive for every  $n \geq 1$ .

A careful examination of the proof of [32, Proposition 4.20] shows that it actually yields the following more general statement. By the standard embedding  $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ , we may consider  $\Sigma_m$  as a face of  $\Sigma_n$  for  $m \leq n$ .

**Proposition 6.2.** *Let  $K_n$  be a triangulation of the simplex  $\Sigma_n$ , for every  $n \geq 1$ , so that  $K_m$  is the restriction of  $K_n$  to  $\Sigma_m$  for all  $m \leq n$ . Suppose that for every  $n \geq 1$ , the full automorphism group  $\mathfrak{S}_n$  of  $\Sigma_n$  acts on  $K_n$ . Then,*

$$1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\mathbb{C}(K_n)_j)(\mathbf{x}) t^j = H(\mathbf{x}; z) \left( 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(L_{V_n}(K_n)_j)(\mathbf{x}) t^j \right).$$

**6.2. Posets.** Group actions on posets induce representations on their homology which often have combinatorial significance; see [31] [41, Chapter 2]. Here we describe a special situation which will be useful in the following sections, referring to [5] for more details and to [34, Chapter 3] for any undefined poset terminology.

Given finite graded posets  $\mathcal{P}$  and  $\mathcal{Q}$  with rank functions  $\rho_{\mathcal{P}}$  and  $\rho_{\mathcal{Q}}$ , respectively, the *Rees product*  $\mathcal{P} * \mathcal{Q}$  was defined by Björner and Welker [9] as the set  $\{(p, q) \in \mathcal{P} \times \mathcal{Q} : \rho_{\mathcal{P}}(p) \geq \rho_{\mathcal{Q}}(q)\}$ , partially ordered by setting  $(p_1, q_1) \preceq (p_2, q_2)$  if the following conditions are satisfied:

- $p_1 \preceq p_2$  holds in  $\mathcal{P}$ ,
- $q_1 \preceq q_2$  holds in  $\mathcal{Q}$  and
- $\rho_{\mathcal{P}}(p_2) - \rho_{\mathcal{P}}(p_1) \geq \rho_{\mathcal{Q}}(q_2) - \rho_{\mathcal{Q}}(q_1)$ .

The poset  $\mathcal{P} * \mathcal{Q}$  is graded and if a finite group  $G$  acts on  $\mathcal{P}$  by order preserving bijections, then it does so on  $\mathcal{P} * \mathcal{Q}$  as well by acting trivially on the second coordinate of its elements. Thus, there is an induced  $G$ -representation on the homology  $\tilde{H}_*(\mathcal{P} * \mathcal{Q}; \mathbb{C})$ .

Assuming  $\mathcal{P}$  is bounded, let us denote by  $\mathcal{P}^-$ ,  $\mathcal{P}_-$  and  $\bar{\mathcal{P}}$  the poset obtained from  $\mathcal{P}$  by removing its maximum element, or minimum element, or both, respectively, and by  $T_{t,n}$  the poset whose Hasse diagram is a complete  $t$ -ary tree of height  $n$ , rooted at the minimum element. The following statement is a direct consequence of [5, Theorem 1.2], which is an equivariant analogue of [27, Corollary 3.8]; it plays a key role in the proofs of all equivariant  $\gamma$ -positivity results in the following sections.

**Theorem 6.3.** ([5]) *Let  $\mathcal{P}$  be a finite bounded poset of rank  $n+1$  which is Cohen–Macaulay over  $\mathbb{C}$  and  $G$  be a finite group which acts on  $\mathcal{P}$  by order preserving bijections. Then, there exist non-virtual  $G$ -representations  $\beta_{\mathcal{P},i}^+$ ,  $\beta_{\mathcal{P},i}^-$ ,  $\gamma_{\mathcal{P},i}^+$  and  $\gamma_{\mathcal{P},i}^-$  such that*

$$(32) \quad \tilde{H}_{n-1}(\bar{\mathcal{P}} * T_{t,n-1}; \mathbb{C}) \cong_G \sum_{i=1}^{\lfloor n/2 \rfloor} \beta_{\mathcal{P},i}^+ t^i (1+t)^{n-2i} + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \beta_{\mathcal{P},i}^- t^i (1+t)^{n-1-2i}$$

and

$$(33) \quad \tilde{H}_{n-1}((\mathcal{P}^- * T_{t,n})_-; \mathbb{C}) \cong_G \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{\mathcal{P},i}^+ t^i (1+t)^{n-2i} + \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{\mathcal{P},i}^- t^i (1+t)^{n+1-2i}$$

for every positive integer  $t$ .

**6.3. Lattice points.** Let  $P$  be an  $n$ -dimensional lattice polytope in  $\mathbb{R}^N$ . The *Ehrhart  $h$ -polynomial* (or  *$h^*$ -polynomial*) of  $P$  is defined by the equation

$$\sum_{k \geq 0} |kP \cap \mathbb{Z}^N| t^k = \frac{h^*(P, t)}{(1-t)^{n+1}},$$

where  $kP$  is the  $k$ th dilate of  $P$ . The function  $h^*(P, t)$  is indeed a polynomial in  $t$ , with nonnegative integer coefficients and degree not exceeding  $n$ , which is very well studied in algebraic and geometric combinatorics; see [7, Chapter 3] [34, Section 4.6] and references therein. The problem to investigate when  $h^*(P, t)$  is unimodal, in particular, has been of great interest; see [11] for a recent survey on this topic.

Equivariant analogues of  $h^*(P, t)$  are provided by Stapledon’s equivariant Ehrhart theory [36] as follows. Suppose  $G$  is a finite group which acts linearly on  $\mathbb{R}^N$ , preserving the lattice  $\mathbb{Z}^N$ , and leaves  $P$  invariant. Assume further that the affine span of  $P$  contains the origin and let  $\mathcal{L}$  be its intersection with  $\mathbb{Z}^N$ . Stapledon [36] defines the formal power series  $\varphi_P^*(t)$ , with coefficients in the representation ring of  $G$ , via the generating function formula

$$(34) \quad \sum_{k \geq 0} \chi_{kP} t^k = \frac{\varphi_P^*(t)}{(1-t) \det(I - \rho t)},$$

where  $\chi_{kP}$  is the permutation representation defined by the  $G$ -action on the set of lattice points of  $kP$  and  $\rho : G \rightarrow GL(\mathcal{L})$  is the induced representation.

The series  $\varphi_P^*(t)$  is a  $G$ -equivariant analogue of  $h^*(P, t)$  which, under additional assumptions (see [36, Section 7]) is a polynomial in  $t$  whose coefficients are non-virtual  $G$ -representations. For example, if  $P$  is the standard unit cube in  $\mathbb{R}^n$  on which the symmetric group  $\mathfrak{S}_n$  acts by permuting coordinates, then  $h^*(P, t) = A_n(t)$  and  $\varphi_P^*(t) = \sum_{j=0}^{n-1} \varphi_{n,j} t^j$ , in the notation of Section 1. For this particular  $\mathfrak{S}_n$ -action on an  $n$ -dimensional lattice polytope  $P$  in  $\mathbb{R}^n$ , and identifying  $G$ -representations with their characters, Equation (34) yields that

$$(35) \quad \sum_{k \geq 0} \chi_{kP}(w) t^k = \frac{\varphi_P^*(t)(w)}{(1-t) \prod_{i \geq 1} (1-t^{\lambda_i(w)})}$$

for every  $w \in \mathfrak{S}_n$ , where  $\lambda_i(w)$  are the lengths of the cycles of  $w$ .

## 7. EQUIVARIANT ANALOGUES OF COLORED DERANGEMENT POLYNOMIALS

This section studies an  $\mathfrak{S}_n$ -equivariant analogue of the colored derangement polynomial  $d_{n,r}(t)$  and confirms the local equivariant Gal phenomenon for the natural  $\mathfrak{S}_n$ -action on the triangulation  $\Gamma_{n,r}$ , considered in Section 5. These results generalize Proposition 6.1 (the case  $r = 1$ ) and are partially extended in the following section, which focuses on the colored binomial Eulerian polynomials.

We define (up to isomorphism) the graded  $\mathfrak{S}_n$ -representation  $\psi_{n,r} = \bigoplus_{j=0}^n \psi_{n,r,j}$  by the generating function formula

$$(36) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\psi_{n,r,j})(\mathbf{x}) t^j = \frac{(1-t)H(\mathbf{x}; tz)^{r-1}}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}.$$

Applying the specialization  $\text{ex}^*$  (see Section 2) on both sides gives

$$1 + \sum_{n \geq 1} \frac{z^n}{n!} \sum_{j=0}^n \dim(\psi_{n,r,j}) t^j = \frac{(1-t)e^{(r-1)tz}}{e^{rtz} - te^{rz}}$$

and hence, a comparison with Equation (17) shows that  $\psi_{n,r}$  is indeed an  $\mathfrak{S}_n$ -equivariant analogue of  $d_{n,r}(t)$ . Clearly, for  $r = 1$  it reduces to the graded  $\mathfrak{S}_n$ -representation  $\psi_n$  which appears in Proposition 6.1.

To define precisely the  $\mathfrak{S}_n$ -action on  $\Gamma_{n,r}$  that was mentioned earlier, consider an abstract simplicial complex  $\Delta$ , as in the beginning of Section 5, on a linearly ordered vertex set  $V(\Delta)$ . Suppose that  $G$  acts simplicially on  $\Delta$  and preserves the induced linear order on the vertex sets of faces of  $\Delta$ , in other words, if  $\{u, v\} \in \Delta$  and  $u$  precedes  $v$ , then  $w \cdot u$  precedes  $w \cdot v$  for every  $w \in G$  (note that such an action is automatically proper). Then,  $G$  acts on  $V_r(\Delta)$  by the rule  $(w \cdot f)(v) = f(w^{-1} \cdot v)$  for  $w \in G$ ,  $f \in V_r(\Delta)$  and  $v \in V(\Delta)$ . The proof of the following statement is fairly straightforward.

**Proposition 7.1.** *Under the stated assumptions:*

- (a) *the  $G$ -action on  $V_r(\Delta)$  induces a proper simplicial  $G$ -action on  $\text{esd}_r(\Delta)$ , and*

- (b) *the subcomplex  $\text{esd}_r(\Delta)^w$  is combinatorially isomorphic to  $\text{esd}_r(\Delta^w)$  for every  $w \in G$ , where the vertex set of  $\Delta^w$  is considered with the induced linear order.*

*Proof.* For part (a), given a face  $E \in \text{esd}_r(\Delta)$  and  $w \in G$ , we need to show that  $w(E) := \{w \cdot f : f \in E\} \in \text{esd}_r(\Delta)$ . Since, by definition of the  $G$ -action on  $V_r(\Delta)$ ,  $\text{supp}(w \cdot f) = w \cdot \text{supp}(f)$  for every  $f \in V_r(\Delta)$ , we have

$$\bigcup_{f \in E} \text{supp}(w \cdot f) = w \left( \bigcup_{f \in E} \text{supp}(f) \right).$$

Thus, the first condition in the definition of  $\text{esd}_r(\Delta)$  that the union of the supports of the elements of  $E$  is a face of  $\Delta$  transfers to  $w(E)$ . The second condition transfers as well because the  $G$ -action respects the given linear order on  $V(\Delta)$  and hence the nonzero values of  $\iota(w \cdot f) - \iota(w \cdot g)$  are exactly those of  $\iota(f) - \iota(g)$ . To verify that the  $G$ -action on  $\text{esd}_r(\Delta)$  is proper, suppose that  $w \in G$  fixes  $E \in \text{esd}_r(\Delta)$ . Then,  $w$  fixes the union, say  $F \in \Delta$ , of the supports of all elements of  $E$ . Since, as has already been commented, the  $G$ -action on  $\Delta$  is proper,  $w$  fixes  $F$  pointwise. Therefore, it fixes all elements of  $V_r(\Delta)$  whose support is contained in  $F$ ; in particular,  $w$  fixes  $E$  pointwise.

Part (b) follows easily from the fact that the  $G$ -actions on  $\Delta$  and  $\text{esd}_r(\Delta)$  are proper; the details are left to the reader.  $\square$

Following the notation of Section 6, we linearly order the vertex set of the barycentric subdivision  $\Gamma_n$  of the simplex  $\Sigma_n$  in any way which respects the inclusion order on the corresponding faces of  $\Sigma_n$  and define the  $r$ -fold edgewise subdivision  $\Gamma_{n,r}$  using this order. Since the  $\mathfrak{S}_n$ -action on  $\Gamma_n$  preserves the inclusion order on its faces, it induces a proper  $\mathfrak{S}_n$ -action on  $\Gamma_{n,r}$  which, as discussed in Section 5, can be realized as a triangulation of  $\Sigma_n$  which refines  $\Gamma_n$ . Thus,  $\mathfrak{S}_n$  acts linearly on the local face module  $L_{V_n}(\Gamma_{n,r})$  and the resulting graded  $\mathfrak{S}_n$ -representation is an  $\mathfrak{S}_n$ -equivariant analogue of the local  $h$ -polynomial  $\ell_{V_n}(\Gamma_{n,r}, t)$ . The latter was shown in [2] to equal the polynomial  $d_{n,r}^+(t)$  which appeared in Theorem 3.2. Therefore, the following statement may be viewed as an  $\mathfrak{S}_n$ -equivariant analogue of some of the main results of [2].

**Theorem 7.2.** *There exists an isomorphism*

$$(37) \quad \psi_{n,r} \cong_{\mathfrak{S}_n} \psi_{n,r}^+ \oplus \psi_{n,r}^-$$

*of graded  $\mathfrak{S}_n$ -representations, where  $\psi_{n,r}^+ = \bigoplus_{j=0}^n \psi_{n,r,j}^+$  and  $\psi_{n,r}^- = \bigoplus_{j=0}^n \psi_{n,r,j}^-$  are such that*

- $\sum_{j=0}^n \psi_{n,r,j}^+ t^j$  is a  $\gamma$ -positive polynomial with center  $n/2$  and zero constant term,
- $\sum_{j=0}^n \psi_{n,r,j}^- t^j$  is a  $\gamma$ -positive polynomial with center  $(n+1)/2$  and zero constant term.

Moreover,  $\psi_{n,r}^+$  is isomorphic to the graded  $\mathfrak{S}_n$ -representation on the local face module  $L_{V_n}(\Gamma_{n,r})$ , induced by the  $\mathfrak{S}_n$ -action on  $\Gamma_{n,r}$ , and

$$(38) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\psi_{n,r,j}^+)(\mathbf{x}) t^j = \frac{H(\mathbf{x}; tz)^{r-1} - tH(\mathbf{x}; z)^{r-1}}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}$$

$$(39) \quad \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\psi_{n,r,j}^-)(\mathbf{x}) t^j = \frac{t(H(\mathbf{x}; z)^{r-1} - H(\mathbf{x}; tz)^{r-1})}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}.$$

**Remark 7.3.** For  $r = 2$ , it was shown in [5, Proposition 5.2] that Equation (38) holds for the graded  $\mathfrak{S}_n$ -representation on the local face module of a triangulation of the  $(n-1)$ -dimensional simplex (sometimes referred to as the interval triangulation) which is different from  $\Gamma_{n,2}$ . This should come as no surprise, in view of the similarities in the combinatorial properties of the two triangulations; see [3, Remark 4.5].  $\square$

We first confirm the statement of Theorem 7.2 about  $L_{V_n}(\Gamma_{n,r})$  using the machinery of [40], explained in Section 6.

**Proposition 7.4.** *The graded  $\mathfrak{S}_n$ -representation on the local face module  $L_{V_n}(\Gamma_{n,r})$ , induced by the  $\mathfrak{S}_n$ -action on the triangulation  $\Gamma_{n,r}$ , satisfies Equation (38).*

*Proof.* Since, for any fixed  $r$ , Proposition 6.2 applies to the family of triangulations  $\Gamma_{n,r}$ , it suffices to show that

$$(40) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\mathbb{C}(\Gamma_{n,r})_j)(\mathbf{x}) t^j = \frac{H(\mathbf{x}; z)H(\mathbf{x}; tz)^{r-1} - tH(\mathbf{x}; z)^r}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}.$$

This can be achieved by a computation similar to those in [5, Section 5] [40, Section 6] as follows. We will apply Equation (31) to the proper  $\mathfrak{S}_n$ -action on  $\Delta := \Gamma_{n,r}$ . Since, as the reader can verify,  $\Gamma^w$  is combinatorially isomorphic to  $\Gamma_{c(w)}$ , where  $c(w)$  is the number of cycles of  $w \in \mathfrak{S}_n$ , part (b) of Proposition 7.1 shows that  $\Gamma_{n,r}^w$  is combinatorially isomorphic to  $\Gamma_{c(w),r}$ . As a result, and in view of Proposition 5.1 (a), we get

$$(41) \quad \begin{aligned} \sum_{j=0}^n \text{ch}(\mathbb{C}(\Gamma_{n,r})_j)(\mathbf{x}) t^j z^n &= \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \frac{A_{c(w),r}^+(t)}{(1-t)^{c(w)}} \prod_{i \geq 1} (1 - t^{\lambda_i(w)}) p_{\lambda_i(w)}(\mathbf{x}) z^{\lambda_i(w)} \\ &= \sum_{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n} m_\lambda^{-1} \frac{A_{\ell(\lambda),r}^+(t)}{(1-t)^{\ell(\lambda)}} \prod_{i \geq 1} (1 - t^{\lambda_i}) p_{\lambda_i}(\mathbf{x}) z^{\lambda_i}, \end{aligned}$$

where  $n!/m_\lambda$  is the cardinality of the conjugacy class of  $\mathfrak{S}_n$  which corresponds to  $\lambda \vdash n$  and  $\ell(\lambda)$  is the number of parts of  $\lambda$ . By Proposition 5.1 (b), the expression (41) may be rewritten as

$$\sum_{k \geq 0} t^k \sum_{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n} m_\lambda^{-1} ((rk+1)^{\ell(\lambda)} - (rk)^{\ell(\lambda)}) \prod_{i \geq 1} (1 - t^{\lambda_i}) p_{\lambda_i}(\mathbf{x}) z^{\lambda_i}.$$

Summing over all  $n \geq 1$  and using the identity (see the proof of [39, Proposition 3.3])

$$\sum_{\lambda=(\lambda_1, \lambda_2, \dots)} m_{\lambda}^{-1} k^{\ell(\lambda)} \prod_{i \geq 1} (1 - t^{\lambda_i}) p_{\lambda_i}(\mathbf{x}) z^{\lambda_i} = \frac{H(\mathbf{x}; z)^k}{H(\mathbf{x}; tz)^k}$$

we get

$$\begin{aligned} 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\mathbb{C}(\Gamma_{n,r})_j)(\mathbf{x}) t^j &= 1 + \sum_{k \geq 0} t^k \left( \frac{H(\mathbf{x}; z)^{rk+1}}{H(\mathbf{x}; tz)^{rk+1}} - \frac{H(\mathbf{x}; z)^{rk}}{H(\mathbf{x}; tz)^{rk}} \right) \\ &= 1 + \left( \frac{H(\mathbf{x}; z)}{H(\mathbf{x}; tz)} - 1 \right) \left( 1 - t \frac{H(\mathbf{x}; z)^r}{H(\mathbf{x}; tz)^r} \right)^{-1} \\ &= \frac{H(\mathbf{x}; z) H(\mathbf{x}; tz)^{r-1} - t H(\mathbf{x}; z)^r}{H(\mathbf{x}; tz)^r - t H(\mathbf{x}; z)^r} \end{aligned}$$

and the proof of (40) follows.  $\square$

To proceed with the proof of Theorem 7.2, just as in [2, Section 3], we need to consider the colored analogue  $\mathcal{B}_{n,r}$  of the Boolean lattice of subsets of  $[n]$ . An  $r$ -colored subset of  $[n]$  is any subset of  $[n] \times \mathbb{Z}_r$  which contains at most one pair  $(i, j)$  for each  $i \in [n]$  (where  $j$  is thought of as the color assigned to  $i$ ). The set  $\mathcal{B}_{n,r}$  consists of all  $r$ -colored subsets of  $[n]$  and is partially ordered by inclusion. The group  $\mathbb{Z}_r \wr \mathfrak{S}_n$  acts on  $r$ -colored subsets of  $[n]$  by permuting their elements and cyclically shifting their colors. This action induces an action on the poset  $\mathcal{B}_{n,r}$  by order preserving bijections. We also recall from Section 2 the notation  $\text{ch}_r$  for the Frobenius characteristic for  $\mathbb{Z}_r \wr \mathfrak{S}_n$  and that the image under this map of any non-virtual representation of  $\mathbb{Z}_r \wr \mathfrak{S}_n$  is Schur-positive in  $\Lambda(\mathbf{x}^{(0)}) \otimes \Lambda(\mathbf{x}^{(1)}) \otimes \dots \otimes \Lambda(\mathbf{x}^{(r-1)})$ , of total degree  $n$ .

*Proof of Theorem 7.2.* Our plan is to apply the first part of Theorem 6.3 to the poset  $\mathcal{P}$  obtained from  $\mathcal{B}_{n,r}$  by adding a maximum element and the group  $G = \mathbb{Z}_r \wr \mathfrak{S}_n$ , which acts on  $\mathcal{P}$  by order preserving bijections. We first claim that

$$1 + \sum_{n \geq 1} \text{ch}_r \left( \tilde{H}_{n-1}((\mathcal{B}_{n,r} \setminus \{\emptyset\}) * T_{t,n-1}; \mathbb{C}) \right) z^n = \frac{(1-t) \prod_{i=1}^{r-1} E(\mathbf{x}^{(i)}; z)}{\prod_{i=0}^{r-1} E(\mathbf{x}^{(i)}; tz) - t \prod_{i=0}^{r-1} E(\mathbf{x}^{(i)}; z)}$$

and omit the proof, which is entirely similar to that of the special case  $r = 2$ , treated in [5, Proposition 4.3], and uses properties of  $\text{ch}_r$  which are direct generalizations of those of  $\text{ch}_2$  mentioned in [5, Section 2]. We now replace the representation on the left-hand side of this formula with the right-hand side of the isomorphism (32), set all sequences  $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r-1)}$  equal to each other and apply the standard involution  $\omega$ . Since these operations on the functions  $\text{ch}_r(\beta_{\mathcal{P},i}^+)$  and  $\text{ch}_r(\beta_{\mathcal{P},i}^-)$  preserve their Schur-positivity, we

conclude that

$$\begin{aligned} \frac{(1-t)H(\mathbf{x}; z)^{r-1}}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r} &= 1 + \sum_{n \geq 1} z^n \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+(\mathbf{x}) t^i (1+t)^{n-2i} \\ &\quad + \sum_{n \geq 1} z^n \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi_{n,r,i}^-(\mathbf{x}) t^i (1+t)^{n-1-2i} \end{aligned}$$

for some Schur-positive symmetric functions  $\xi_{n,r,i}^+(\mathbf{x})$  and  $\xi_{n,r,i}^-(\mathbf{x})$ . Replacing  $z$  with  $tz$  and  $t$  with  $1/t$ , this equation may be rewritten as

$$(42) \quad \begin{aligned} \frac{(1-t)H(\mathbf{x}; tz)^{r-1}}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r} &= 1 + \sum_{n \geq 1} z^n \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+(\mathbf{x}) t^i (1+t)^{n-2i} \\ &\quad + \sum_{n \geq 1} z^n \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi_{n,r,i}^-(\mathbf{x}) t^{i+1} (1+t)^{n-1-2i}. \end{aligned}$$

We define the graded  $\mathfrak{S}_n$ -representations  $\psi_{n,r}^+$  and  $\psi_{n,r}^-$  by Equations (38) and (39) and note that (37) holds, since the right-hand side, say  $F(\mathbf{x}, t; z)$ , of (36) equals the sum of the right-hand sides, say  $F^+(\mathbf{x}, t; z)$  and  $F^-(\mathbf{x}, t; z)$ , of (38) and (39), respectively.

Finally, we observe that  $F^+(\mathbf{x}, t; z)$  is left invariant under replacing  $z$  with  $tz$  and  $t$  with  $1/t$ . This implies that the coefficient of  $z^n$  in  $F^+(\mathbf{x}, t; z)$  is a palindromic polynomial in  $t$ , centered at  $n/2$ . Similarly, we find that the coefficient of  $z^n$  in  $F^-(\mathbf{x}, t; z)$  is a palindromic polynomial in  $t$ , centered at  $(n+1)/2$ . Since the corresponding properties are clear for the coefficient of  $z^n$  in the two summands in the right-hand side of Equation (42) and because of the uniqueness of the decomposition of a polynomial  $f(t)$  as a sum of two palindromic polynomials with centers  $n/2$  and  $(n+1)/2$ , we must have

$$\begin{aligned} \sum_{j=0}^n \text{ch}(\psi_{n,r,j}^+)(\mathbf{x}) t^j &= \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+(\mathbf{x}) t^i (1+t)^{n-2i}, \\ \sum_{j=0}^n \text{ch}(\psi_{n,r,j}^-)(\mathbf{x}) t^j &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi_{n,r,i}^-(\mathbf{x}) t^{i+1} (1+t)^{n-1-2i} \end{aligned}$$

and the proof follows.  $\square$

**Problem 7.5.** Find explicit combinatorial interpretations for the coefficients of the expansions of  $\xi_{n,r,i}^+(\mathbf{x})$  and  $\xi_{n,r,i}^-(\mathbf{x})$  as linear combinations of Schur functions.

## 8. EQUIVARIANT ANALOGUES OF COLORED BINOMIAL EULERIAN POLYNOMIALS

This section studies an  $\mathfrak{S}_n$ -equivariant analogue of  $\tilde{A}_{n,r}(t)$  and confirms the equivariant Gal phenomenon for the  $\mathfrak{S}_n$ -action on the triangulated sphere  $\Delta(\Gamma_{n,r})$ , considered in Section 5, for  $r = 2$ .



Following the approach of Section 7, we define the graded  $\mathfrak{S}_n$ -representation  $\tilde{\varphi}_{n,r} = \bigoplus_{j=0}^n \tilde{\varphi}_{n,r,j}$  by the formula

$$(43) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,r,j})(\mathbf{x}) t^j = \frac{(1-t)H(\mathbf{x}; z)H(\mathbf{x}; tz)^r}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}.$$

Applying the specialization  $\text{ex}^*$  on both sides gives

$$1 + \sum_{n \geq 1} \frac{z^n}{n!} \sum_{j=0}^n \dim(\tilde{\varphi}_{n,r,j}) t^j = \frac{(1-t)e^{(rt+1)z}}{e^{rtz} - te^{rz}}$$

which, in view of Equation (52) (see the proof of Proposition 8.6), shows that  $\tilde{\varphi}_{n,r}$  is an  $\mathfrak{S}_n$ -equivariant analogue of  $\tilde{A}_{n,r}(t)$ . For  $r = 1$ , it reduces to the graded  $\mathfrak{S}_n$ -representation  $\tilde{\varphi}_n$  discussed in Section 1.

As in Section 7, we consider  $\Gamma_{n,r}$  as a triangulation of the simplex  $\Sigma_n$  on which  $\mathfrak{S}_n$  acts by permuting coordinates. This action extends to one on  $\Delta(\Gamma_{n,r})$  which, as explained in Section 5, can be considered as a triangulation of the  $n$ -dimensional cross-polytope (note that this extended  $\mathfrak{S}_n$ -action is not proper, since  $-\Sigma_n$  is a face of  $\Delta(\Gamma_{n,r})$  which is fixed by the action, although not pointwise). As a result, we have a linear  $\mathfrak{S}_n$ -action on  $\mathbb{C}(\Delta(\Gamma_{n,r}))$ . The following theorem, in view of our discussion in Example 4.2, partially generalizes [29, Theorem 5.2] (the case  $r = 1$ ); we have no reason to doubt that the  $\gamma$ -positivity statement holds for every  $r \geq 2$ .

**Theorem 8.1.** *There exists an isomorphism*

$$(44) \quad \tilde{\varphi}_{n,r} \cong_{\mathfrak{S}_n} \tilde{\varphi}_{n,r}^+ \oplus \tilde{\varphi}_{n,r}^-$$

of graded  $\mathfrak{S}_n$ -representations, where  $\tilde{\varphi}_{n,r}^+ = \bigoplus_{j=0}^n \tilde{\varphi}_{n,r,j}^+$  and  $\tilde{\varphi}_{n,r}^- = \bigoplus_{j=0}^n \tilde{\varphi}_{n,r,j}^-$  are such that

- $\sum_{j=0}^n \tilde{\varphi}_{n,r,j}^+ t^j$  is palindromic and unimodal, with center  $n/2$ , and
- $\sum_{j=0}^n \tilde{\varphi}_{n,r,j}^- t^j$  is palindromic with center  $(n+1)/2$  and zero constant term.

Moreover, these polynomials are  $\gamma$ -positive for  $r = 2$ , we have

$$(45) \quad 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,r,j}^+)(\mathbf{x}) t^j = \frac{H(\mathbf{x}; z)H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r H(\mathbf{x}; tz)}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}$$

$$(46) \quad \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,r,j}^-)(\mathbf{x}) t^j = \frac{tH(\mathbf{x}; z)H(\mathbf{x}; tz)(H(\mathbf{x}; z)^{r-1} - H(\mathbf{x}; tz)^{r-1})}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}$$

for  $r \geq 1$  and  $\tilde{\varphi}_{n,r}^+$  is isomorphic to the graded  $\mathfrak{S}_n$ -representation  $\mathbb{C}(\Delta(\Gamma_{n,r}))$ , induced by the  $\mathfrak{S}_n$ -action on  $\Delta(\Gamma_{n,r})$ .

The proof of Theorem 8.1 follows a similar path with that of Theorem 7.2. One essential difference is that the  $\mathfrak{S}_n$ -action on  $\Delta(\Gamma_{n,r})$  is not proper and hence one cannot hope to adapt the proof of Proposition 7.4. The following proposition, whose proof we postpone for a while, will be used instead.

**Proposition 8.2.** *Let  $(K_n)$  be a sequence of triangulations, as in Proposition 6.2. Assuming that each  $K_n$  is shellable, we have*

$$1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\mathbb{C}(\Delta(K_n))_j)(\mathbf{x}) t^j = H(\mathbf{x}; tz) \left( 1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\mathbb{C}(K_n)_j)(\mathbf{x}) t^j \right).$$

*Proof of Theorem 8.1.* We define the graded  $\mathfrak{S}_n$ -representations  $\tilde{\varphi}_{n,r}^+$  and  $\tilde{\varphi}_{n,r}^-$  by Equations (45) and (46). As in the proof of Theorem 7.2 we find that (44) holds and that the polynomials  $\sum_{j=0}^n \tilde{\varphi}_{n,r,j}^+ t^j$  and  $\sum_{j=0}^n \tilde{\varphi}_{n,r,j}^- t^j$  are symmetric, with center of symmetry  $n/2$  and  $(n+1)/2$ , respectively, for every  $n$ . The last statement of the theorem follows from Proposition 8.2, applied to the triangulation  $\Gamma_{n,r}$  (which is regular, and hence shellable), and Equation (40) and implies the unimodality of  $\sum_{j=0}^n \tilde{\varphi}_{n,r,j}^+ t^j$  via an application of the hard Lefschetz theorem, as pointed out by Stanley on [30, p. 528].

To prove the  $\gamma$ -positivity statement for  $r = 2$ , we apply the second part of Theorem 6.3 to the  $(\mathbb{Z}_2 \wr \mathfrak{S}_n)$ -action on the poset  $\mathcal{P}$  obtained from  $\mathcal{B}_{n,2}$  by adding a maximum element. The left-hand side of Equation (33) in this case has already been computed in [5, Proposition 4.6]; its image under  $\text{ch}_2$  was shown to equal the coefficient of  $z^n$  in

$$(47) \quad \frac{(1-t)E(\mathbf{y}; z)E(\mathbf{x}; tz)E(\mathbf{y}; tz)}{E(\mathbf{x}; tz)E(\mathbf{y}; tz) - tE(\mathbf{x}; z)E(\mathbf{y}; z)},$$

where  $\mathbf{x} := \mathbf{x}^{(0)}$  and  $\mathbf{y} := \mathbf{x}^{(1)}$ . Setting  $\mathbf{x} = \mathbf{y}$ , applying the involution  $\omega$  and comparing with Equation (43), we get

$$\sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,2,j})(\mathbf{x}) t^j = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,2,i}^+(\mathbf{x}) t^i (1+t)^{n-2i} + \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,2,i}^-(\mathbf{x}) t^i (1+t)^{n+1-2i}$$

for some Schur-positive symmetric functions  $\tilde{\gamma}_{n,2,i}^+(\mathbf{x})$  and  $\tilde{\gamma}_{n,2,i}^-(\mathbf{x})$ . Using the uniqueness of the symmetric decomposition (44), as in the proof of Theorem 7.2 we conclude that

$$\begin{aligned} \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,2,j}^+)(\mathbf{x}) t^j &= \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,2,i}^+(\mathbf{x}) t^i (1+t)^{n-2i}, \\ \sum_{j=0}^n \text{ch}(\tilde{\varphi}_{n,2,j}^-)(\mathbf{x}) t^j &= \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,2,i}^-(\mathbf{x}) t^i (1+t)^{n+1-2i} \end{aligned}$$

and the proof follows.  $\square$

**Problem 8.3.** *Find explicit combinatorial interpretations for the coefficients of the expansions of  $\tilde{\gamma}_{n,2,i}^+(\mathbf{x})$  and  $\tilde{\gamma}_{n,2,i}^-(\mathbf{x})$  as linear combinations of Schur functions.*

**Remark 8.4.** One may try to extend this proof to any  $r \geq 2$  by considering the  $(\mathbb{Z}_r \wr \mathfrak{S}_n)$ -action on  $\mathcal{B}_{n,r}$ , just as in the proof of Theorem 7.2. However, the image of the left-hand side of Equation (33) under  $\text{ch}_r$  turns out to be the coefficient of  $z^n$  in

$$\frac{(1-t) \prod_{i=1}^{r-1} E(\mathbf{x}^{(i)}; z) \prod_{i=0}^{r-1} E(\mathbf{x}^{(i)}; tz)}{\prod_{i=0}^{r-1} E(\mathbf{x}^{(i)}; tz) - t \prod_{i=0}^{r-1} E(\mathbf{x}^{(i)}; z)}.$$

After specializing and applying  $\omega$  this expression becomes

$$\frac{(1-t)H(\mathbf{x}; z)^{r-1}H(\mathbf{x}; tz)^r}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r},$$

which equals the right-hand side of Equation (43) only for  $r = 2$ .  $\square$

For the proof of Proposition 8.2, which is somewhat technical, we assume familiarity with the notion of shellability and its connections to Stanley–Reisner theory, as explained in [33, Section III.2]. The assumption that each  $K_n$  is shellable simplifies the proof, but we make no claim that it is necessary for the validity of the proposition.

*Proof of Proposition 8.2.* As was the case with  $\Delta(\Gamma_{n,r})$ , we consider  $\Delta(K_n)$  as a triangulation of the boundary complex  $\Delta(\Sigma_n)$  of the  $n$ -dimensional cross-polytope, defined as the convex hull of the set of unit coordinate vectors in  $\mathbb{R}^n$  and their negatives.

Part (c) of Proposition 4.1 applies to  $K_n$  and hence, we may choose a shelling order of the set of facets of  $\Delta(K_n)$  as in the proof given there. As described in [33, Theorem 2.5], this shelling order gives rise to a basis  $\mathcal{B}$  of the  $\mathbb{C}$ -vector space  $\mathbb{C}(\Delta(K_n))$  which consists of monomials, one for every facet of  $\Delta(K_n)$ . We recall that the faces of  $\Sigma_n$  are in one-to-one correspondence with the facets of  $\Delta(\Sigma_n)$ . For a face  $F$  of  $\Sigma_n$ , we will denote by  $\mathcal{B}_F$  the set of those basis elements which are associated to the facets of the restriction of  $\Delta(K_n)$  to the facet of  $\Delta(\Sigma_n)$  corresponding to  $F$  and by  $W_F$  the  $\mathbb{C}$ -linear span of  $\mathcal{B}_F$  in  $\mathbb{C}(\Delta(K_n))$ . Then, each  $W_F$  is a (standard) graded  $\mathbb{C}$ -vector space and we have a decomposition

$$(48) \quad \mathbb{C}(\Delta(K_n)) = \bigoplus_F W_F = \bigoplus_{m=0}^n W_{n,m}$$

of graded  $\mathbb{C}$ -vector spaces, where  $F$  ranges over all faces of  $\Sigma_n$  and  $W_{n,m}$  is the direct sum of those spaces  $W_F$  for which  $F$  has codimension  $m$ .

We claim that  $W_F$  is isomorphic to  $\mathbb{C}((K_n)_F)$  as a graded  $\mathbb{C}$ -vector space, provided the grading of the latter is shifted by  $m$ . To verify the claim, and for the ease of notation, we let  $F = \Sigma_n$  (the general case being similar). Consider the linear system of parameters (30) for  $\mathbb{C}[\Delta(K_n)]$ . Since setting all variables  $x_v$  associated to vertices  $v$  not in  $K_n$  equal to zero produces the corresponding system of parameters for  $\mathbb{C}[K_n]$ , the natural projection map  $\mathbb{C}[\Delta(K_n)] \rightarrow \mathbb{C}[K_n]$  induces a well defined linear surjection  $\pi : \mathbb{C}(\Delta(K_n)) \rightarrow \mathbb{C}(K_n)$ . Since, by [33, Theorem 2.5], the images of the elements of the basis  $\mathcal{B}_{\Sigma_n}$  form a basis of  $\mathbb{C}(K_n)$ , the map  $\pi$  restricts to an isomorphism  $W_{\Sigma_n} \rightarrow \mathbb{C}(K_n)$  and the proof of the claim follows. Next, we observe that  $\mathfrak{S}_n$  acts (transitively) on the set of faces of  $\Sigma_n$  of given codimension  $m$  and, by our assumptions, it acts on the set of restrictions of  $K_n$  to

these faces and hence on the set of restrictions of  $\Delta(K_n)$  on the corresponding facets of  $\Delta(\Sigma_n)$ , mapping shelling orders to shelling orders and hence bases  $\mathcal{B}_F$  to other bases (not necessarily the ones we have chosen) of the spaces  $W_F$ . As a result, each subspace  $W_{n,m}$  of  $\mathbb{C}(\Delta(K_n))$  is  $\mathfrak{S}_n$ -invariant and

$$W_{n,m} \cong_{\mathfrak{S}_n} \bigoplus_F \mathbb{C}((K_n)_F)$$

where, essentially by the definition of induction, the direct sum on the right is isomorphic to the  $\mathfrak{S}_n$ -representation which is induced from the  $\mathfrak{S}_m$ -representation  $\mathbb{C}(K_m)$ . By standard properties of Frobenius characteristic, we infer that

$$\text{ch}((W_{n,m})_j)(\mathbf{x}) = h_{m-n}(\mathbf{x}) \cdot \text{ch}(\mathbb{C}(K_m)_j)(\mathbf{x})$$

for  $0 \leq j \leq m$ . By Equation (48) and our previous discussion, we also have

$$\text{ch}(\mathbb{C}(\Delta(K_n))_j)(\mathbf{x}) = \sum_{m=n-j}^n \text{ch}((W_{n,m})_{j-n+m})(\mathbf{x})$$

and hence

$$\text{ch}(\mathbb{C}(\Delta(K_n))_j)(\mathbf{x}) = \sum_{m=n-j}^n h_{m-n}(\mathbf{x}) \cdot \text{ch}(\mathbb{C}(K_m)_{j-n+m})(\mathbf{x})$$

for  $0 \leq j \leq n$ , which is equivalent to the proposed formula.  $\square$

**Example 8.5.** Let  $K_n$  be the trivial triangulation of  $\Sigma_n$ . Then,  $\Delta(\Sigma_n)$  is the boundary complex of the  $n$ -dimensional cross-polytope and Proposition 8.2 implies that

$$1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\sigma_{n,j})(\mathbf{x}) t^j = H(\mathbf{x}; z) H(\mathbf{x}; tz),$$

where  $\sigma_n = \bigoplus_{j=0}^n \sigma_{n,j}$  is the graded  $\mathfrak{S}_n$ -representation on  $\mathbb{C}(\Delta(\Sigma_n))$  induced by the  $\mathfrak{S}_n$ -action on  $\Delta(\Sigma_n)$ . Equivalently,  $\sigma_{n,j}$  is isomorphic to the permutation  $\mathfrak{S}_n$ -representation defined by the standard  $\mathfrak{S}_n$ -action on the set of  $j$ -element subsets of  $[n]$  (a fact which can be shown directly too). The polynomial  $\sum_{j=0}^n \sigma_{n,j} t^j$  is an  $\mathfrak{S}_n$ -equivariant analogue of  $h(\Delta(\Sigma_n), t) = (1+t)^n$  and clearly cannot be  $\gamma$ -positive for  $n \geq 2$ . Hence, the equivariant Gal phenomenon fails for this action. A similar example was given in [29, Section 5].  $\square$

We conclude this section with a combinatorial interpretation for the coefficients in the  $\gamma$ -expansion of  $\tilde{B}_n^+(t)$  and  $\tilde{B}_n^-(t)$  which is similar to the one provided for  $\tilde{A}_n(t)$  in [29] and to others known for  $A_n(t)$  (see, for instance, [4, Theorem 2.1]).

For a signed permutation  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$ , we denote by  $\text{Des}_B(w)$  the set of indices  $i \in [n]$  for which either  $w(i)$  is positive and  $w(i) > w(i+1)$ , or else both  $w(i)$  and  $w(i+1)$  are negative and  $|w(i)| > |w(i+1)|$ , where  $w(n+1) := 0$ . The set  $\text{Asc}_B(w)$  is defined as the complement of  $\text{Des}_B(w)$  in  $[n]$ .

**Proposition 8.6.** *For every positive integer  $n$ ,*

$$(49) \quad \tilde{B}_n^+(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,2,i}^+ t^i (1+t)^{n-2i}$$

$$(50) \quad \tilde{B}_n^-(t) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,2,i}^- t^i (1+t)^{n+1-2i},$$

where

- $\tilde{\gamma}_{n,2,i}^+$  is equal to the number of signed permutations  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$  for which  $\text{Des}_B(w)$  has  $i$  elements, no two consecutive, and does not contain  $n$ ,
- $\tilde{\gamma}_{n,2,i}^-$  is equal to the number of signed permutations  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$  for which  $\text{Des}_B(w)$  has  $i$  elements, no two consecutive, and contains  $n$ .

For the first few values of  $n$ , the numbers  $\tilde{\gamma}_{n,2,i}^\pm$  appear in the expansions

$$\tilde{B}_n^+(t) = \begin{cases} 1+t, & \text{if } n=1 \\ (1+t)^2 + 3t, & \text{if } n=2 \\ (1+t)^3 + 16t(1+t), & \text{if } n=3 \\ (1+t)^4 + 61t(1+t)^2 + 57t^2, & \text{if } n=4 \\ (1+t)^5 + 206t(1+t)^3 + 743t^2(1+t), & \text{if } n=5 \end{cases}$$

and

$$\tilde{B}_n^-(t) = \begin{cases} t, & \text{if } n=1 \\ 3t(1+t), & \text{if } n=2 \\ 7t(1+t)^2 + 11t^2, & \text{if } n=3 \\ 15t(1+t)^3 + 98t^2(1+t), & \text{if } n=4 \\ 31t(1+t)^4 + 577t^2(1+t)^2 + 361t^3, & \text{if } n=5. \end{cases}$$

For the proof, we need some explicit combinatorial formula demonstrating the (non-symmetric)  $\gamma$ -positivity of the coefficient of  $z^n$  in (47). Conveniently, such a formula has been provided in [5, Corollary 4.7]. We set  $\text{Des}^*(w) = \text{Des}_B(w) \setminus \{n\}$  for  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$ . By adding the two equations considered in [5, Corollary 4.7], setting  $\mathbf{x} = \mathbf{y}$  and noting that the signed quasisymmetric function  $F_{\text{Des}(w)}(\mathbf{x}, \mathbf{y})$  specializes to  $F_{\text{Des}^*(w)}(\mathbf{x})$  for  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$ , we get

$$(51) \quad \frac{(1-t)E(\mathbf{x}; z)E(\mathbf{x}; tz)^2}{E(\mathbf{x}; tz)^2 - tE(\mathbf{x}; z)^2} = 1 + \sum_{n \geq 1} z^n \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,2,i}^+(\mathbf{x}) t^i (1+t)^{n-2i} \\ + \sum_{n \geq 1} z^n \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,2,i}^-(\mathbf{x}) t^i (1+t)^{n+1-2i},$$

where

$$\begin{aligned}\tilde{\gamma}_{n,2,i}^+(\mathbf{x}) &= \sum_w F_{\text{Des}^*(w)}(\mathbf{x}), \\ \tilde{\gamma}_{n,2,i}^-(\mathbf{x}) &= \sum_w F_{\text{Des}^*(w)}(\mathbf{x})\end{aligned}$$

and  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$  runs through all signed permutations for which  $\text{Asc}_B(w^{-1})$  has  $i$  elements, no two consecutive, and does not contain (respectively, contains)  $n$ .

*Proof of Proposition 8.6.* The defining equation (3) can be rewritten as

$$\sum_{n \geq 0} \tilde{A}_{n,r}(t) \frac{z^n}{n!} = \left( \sum_{n \geq 0} A_{n,r}(t) \frac{z^n}{n!} \right) \left( \sum_{n \geq 0} t^n \frac{z^n}{n!} \right) = \left( \sum_{n \geq 0} A_{n,r}(t) \frac{z^n}{n!} \right) e^{tz}.$$

From these equalities and [38, Theorem 20] we get

$$(52) \quad \sum_{n \geq 0} \tilde{A}_{n,r}(t) \frac{z^n}{n!} = \frac{(1-t)e^{(rt+1)z}}{e^{rtz} - te^{rz}}$$

and, in particular,

$$(53) \quad \sum_{n \geq 0} \tilde{B}_n(t) \frac{z^n}{n!} = \frac{(1-t)e^{(2t+1)z}}{e^{2tz} - te^{2z}}.$$

On the other hand, applying  $\text{ex}^*$  to (51) gives

$$\begin{aligned}\frac{(1-t)e^{(2t+1)z}}{e^{2tz} - te^{2z}} &= 1 + \sum_{n \geq 1} \left( \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,2,i}^+ t^i (1+t)^{n-2i} \right) \frac{z^n}{n!} \\ &\quad + \sum_{n \geq 1} \left( \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,2,i}^- t^i (1+t)^{n+1-2i} \right) \frac{z^n}{n!},\end{aligned}$$

where  $\tilde{\gamma}_{n,2,i}^+$  (respectively,  $\tilde{\gamma}_{n,2,i}^-$ ) stands for the number of signed permutations  $w \in \mathbb{Z}_2 \wr \mathfrak{S}_n$  for which  $\text{Asc}_B(w)$  has  $i$  elements, no two consecutive, and does not contain (respectively, contains)  $n$ . Equating the coefficients of  $z^n/n!$  on both sides, in view of (53), we get

$$\tilde{B}_n(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,2,i}^+ t^i (1+t)^{n-2i} + \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \tilde{\gamma}_{n,2,i}^- t^i (1+t)^{n+1-2i}.$$

Since there is an obvious involution on  $\mathbb{Z}_2 \wr \mathfrak{S}_n$  which exchanges the set-valued statistics  $\text{Asc}_B$  and  $\text{Des}_B$ , the expression just obtained for  $\tilde{B}_n(t)$  is equivalent to the statement of the proposition.  $\square$

**Problem 8.7.** Find interpretations for  $\tilde{\gamma}_{n,r,i}^\pm$  for all  $r \geq 2$ , similar to those provided by Proposition 8.6 in the case  $r = 2$ .

## 9. EQUIVARIANT ANALOGUES OF COLORED EULERIAN POLYNOMIALS

Given the results of the previous two sections, it is natural to inquire about the existence of a well behaved  $\mathfrak{S}_n$ -equivariant analogue of  $A_{n,r}(t)$ . Although one can use the machinery of Section 6.1 to obtain such an analogue, it is perhaps more straightforward to employ Stapledon's equivariant Ehrhart theory [36] instead.

Consider the  $r$ th dilate  $P = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq r\}$  of the standard unit cube in  $\mathbb{R}^n$ . The Ehrhart  $h$ -polynomial of  $P$  satisfies

$$\sum_{k \geq 0} (rk + 1)^n t^k = \frac{h^*(P, t)}{(1 - t)^{n+1}}$$

and hence, in view of [38, Theorem 17], we have  $h^*(P, t) = A_{n,r}(t)$ . The group  $\mathfrak{S}_n$  acts on  $\mathbb{R}^n$  by permuting coordinates, as usual, and leaves  $P$  invariant. As explained in Section 6.3, denoting by  $\varphi_{n,r,j}$  the coefficient of  $t^j$  in  $\varphi_P^*(t)$  we have an  $\mathfrak{S}_n$ -equivariant analogue  $\varphi_{n,r} = \bigoplus_{j \geq 0} \varphi_{n,r,j}$  of the Ehrhart  $h$ -polynomial  $A_{n,r}(t)$  of  $P$ . Noting that  $\chi_{kP}(w) = (rk + 1)^{c(w)}$ , where  $c(w)$  is the number of cycles of  $w$ , Equation (35) and the definition of Frobenius characteristic imply that

$$\sum_{j \geq 0} \text{ch}(\varphi_{n,r,j})(\mathbf{x}) t^j = (1 - t) \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \sum_{k \geq 0} (rk + 1)^{c(w)} t^k \prod_{i \geq 1} (1 - t^{\lambda_i(w)}) p_{\lambda_i(w)}(\mathbf{x}).$$

An computation analogous to the one in the proof of Proposition 7.4 results in the formula

$$(54) \quad 1 + \sum_{n \geq 1} z^n \sum_{j \geq 0} \text{ch}(\varphi_{n,r,j})(\mathbf{x}) t^j = \frac{(1 - t)H(\mathbf{x}; z)H(\mathbf{x}; tz)^{r-1}}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}.$$

For  $r = 2$ , the right-hand side is left invariant after replacing  $z$  with  $tz$  and  $t$  with  $1/t$  and hence  $\sum_{j \geq 0} \varphi_{n,2,j} t^j = \sum_{j=0}^n \varphi_{n,2,j} t^j$  is palindromic, with center of symmetry  $n/2$ . The following stronger statement can be derived from results of previous sections.

**Proposition 9.1.** *The polynomial  $\sum_{j=0}^n \varphi_{n,2,j} t^j$  is  $\gamma$ -positive for every positive integer  $n$ .*

*Proof.* Using Equation (9), due to Shareshian and Wachs [29], and Theorem 7.2 for  $r = 2$ , we find that

$$\begin{aligned} \frac{(1 - t)H(\mathbf{x}; z)H(\mathbf{x}; tz)}{H(\mathbf{x}; tz)^2 - tH(\mathbf{x}; z)^2} &= \frac{(1 - t)H(\mathbf{x}; z)H(\mathbf{x}; tz)}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)} \cdot \frac{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)}{H(\mathbf{x}; tz)^2 - tH(\mathbf{x}; z)^2} \\ &= \left( 1 + \sum_{n \geq 1} z^n \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,i}(\mathbf{x}) t^i (1 + t)^{n-2i} \right) \\ &\quad \left( 1 + \sum_{n \geq 1} z^n \sum_{i=1}^{\lfloor n/2 \rfloor} \xi_{n,2,i}^+(\mathbf{x}) t^i (1 + t)^{n-2i} \right) \end{aligned}$$

for some Schur-positive symmetric functions  $\tilde{\gamma}_{n,i}(\mathbf{x})$  and  $\xi_{n,2,i}^+(\mathbf{x})$ . Computing the coefficient of  $z^n$  on the right-hand side shows that

$$\sum_{j=0}^n \text{ch}(\varphi_{n,2,j})(\mathbf{x})t^j = \sum_{k+\ell=n} \sum_{i,j} \tilde{\gamma}_{k,i}(\mathbf{x}) \xi_{\ell,2,j}^+(\mathbf{x}) t^{i+j} (1+t)^{n-2i-2j},$$

where  $\tilde{\gamma}_{n,i}(\mathbf{x}) = \xi_{n,2,j}^+(\mathbf{x}) := 1$  for  $n = i = j = 0$ . Given that sums and products of Schur-positive symmetric functions are Schur-positive, this formula implies that the left-hand side is Schur  $\gamma$ -positive for every  $n$  and the proof follows.  $\square$

For  $r \geq 3$ , proceeding as with  $\psi_{n,r}$  and  $\tilde{\varphi}_{n,r}$ , we find that there is an isomorphism

$$\varphi_{n,r} \cong_{\mathfrak{S}_n} \varphi_{n,r}^+ \oplus \varphi_{n,r}^-$$

of graded  $\mathfrak{S}_n$ -representations, where  $\varphi_{n,r}^+ = \bigoplus_{j \geq 0} \varphi_{n,r,j}^+$  and  $\varphi_{n,r}^- = \bigoplus_{j \geq 0} \varphi_{n,r,j}^-$  are such that

- $\sum_{j \geq 0} \varphi_{n,r,j}^+ t^j$  is a palindromic polynomial, with center of symmetry  $n/2$ , and
- $\sum_{j \geq 0} \varphi_{n,r,j}^- t^j$  is a palindromic polynomial, with center of symmetry  $(n+1)/2$  and zero constant term

and

$$1 + \sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\varphi_{n,r,j}^+)(\mathbf{x})t^j = \frac{H(\mathbf{x}; z)H(\mathbf{x}; tz) (H(\mathbf{x}; tz)^{r-2} - tH(\mathbf{x}; z)^{r-2})}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}$$

$$\sum_{n \geq 1} z^n \sum_{j=0}^n \text{ch}(\varphi_{n,r,j}^-)(\mathbf{x})t^j = \frac{tH(\mathbf{x}; z)H(\mathbf{x}; tz) (H(\mathbf{x}; z)^{r-2} - H(\mathbf{x}; tz)^{r-2})}{H(\mathbf{x}; tz)^r - tH(\mathbf{x}; z)^r}.$$

A recent result of Brändén and Solus [10, Section 3] implies that the graded dimensions  $\sum_{j=0}^n \dim(\varphi_{n,r,j}^+)t^j$  and  $\sum_{j=0}^n \dim(\varphi_{n,r,j}^-)t^j$  of  $\varphi_{n,r}^+$  and  $\varphi_{n,r}^-$  are real-rooted, hence  $\gamma$ -positive polynomials for every  $n \geq 1$ . Thus, we expect that the following question has an affirmative answer.

**Question 9.2.** *Are  $\sum_{j=0}^n \varphi_{n,r,j}^+ t^j$  and  $\sum_{j=0}^n \varphi_{n,r,j}^- t^j$   $\gamma$ -positive for all  $n \geq 1$  and  $r \geq 3$ ?*

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