Equivariant γ-positivity

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August 29, 2018

Outline

1. Gamma-positivity
2. Equivariant gamma-positivity
3. Group actions on simplicial complexes
4. Group actions on posets

Eulerian polynomials

We let

- $\mathfrak{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$

and for $w \in \mathfrak{S}_n$

- $\mathrm{des}(w) \; := \; \# \{i \in [n-1] : \; w(i) > w(i+1)\}$
- $\mathrm{exc}(w) \; := \; \# \{i \in [n-1] : \; w(i) > i\}$

be the number of descents and excedances of w, respectively. The polynomial

$$A_n(t) \; := \; \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)} \; = \; \sum_{w \in \mathfrak{S}_n} t^{\mathrm{exc}(w)}$$

is the nth Eulerian polynomial.

Example

$$A_n(t) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + t, & \text{if } n = 2 \\ 1 + 4t + t^2, & \text{if } n = 3 \\ 1 + 11t + 11t^2 + t^3, & \text{if } n = 4 \\ 1 + 26t + 66t^2 + 26t^3 + t^4, & \text{if } n = 5 \\ 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5, & \text{if } n = 6. \end{cases}$$

Theorem

The polynomial $A_n(t)$

- *is symmetric and unimodal,*
- (Foata–Schützenberger, 1970) *can be written as*

$$A_n(t) \;=\; \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}\, t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k}$ *is the number of* $w \in \mathfrak{S}_n$ *with* $\mathrm{des}(w) = k$*, for which*

- *there is no* $i \in \{2, \ldots, n-1\}$ *such that* $w(i-1) > w(i) > w(i+1)$*,*
- $w(n-1) < w(n)$*.*

Example

$$A_n(t) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + t, & \text{if } n = 2 \\ (1+t)^2 + 2t, & \text{if } n = 3 \\ (1+t)^3 + 8t(1+t), & \text{if } n = 4 \\ (1+t)^4 + 22t(1+t)^2 + 16t^2, & \text{if } n = 5 \\ (1+t)^5 + 52t(1+t)^3 + 186t^2(1+t), & \text{if } n = 6. \end{cases}$$

Derangement polynomials

The nth derangement polynomial is defined as

$$d_n(t) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(t) = \sum_{w \in \mathcal{D}_n} t^{\mathrm{exc}(w)},$$

where $\mathcal{D}_n$ is the set of derangements in $\mathfrak{S}_n$.

Example

$$d_n(x) = \begin{cases} 0, & \text{if } n = 1 \\ x, & \text{if } n = 2 \\ x + x^2, & \text{if } n = 3 \\ x + 7x^2 + x^3, & \text{if } n = 4 \\ x + 21x^2 + 21x^3 + x^4, & \text{if } n = 5 \\ x + 51x^2 + 161x^3 + 51x^4 + x^5, & \text{if } n = 6 \\ x + 113x^2 + 813x^3 + 813x^4 + 113x^5 + x^6, & \text{if } n = 7. \end{cases}$$

Theorem (by several authors)

The polynomial $d_n(t)$

- *is symmetric and unimodal,*
- *can be written as*

$$d_n(t) \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} \xi_{n,k}\, t^k (1+t)^{n-2k},$$

where $\xi_{n,k}$ *is the number of* $w \in \mathfrak{S}_n$ *with* $\mathrm{des}(w) = k-1$*, for which*

- *there is no* $i \in \{2, \ldots, n-1\}$ *such that* $w(i-1) > w(i) > w(i+1)$*,*
- $w(1) < w(2)$ *and* $w(n-1) < w(n)$*.*

Binomial Eulerian polynomials

The nth binomial Eulerian polynomial is defined as

$$\widetilde{A}_n(t) \;:=\; 1 \;+\; t \sum_{k=1}^{n} \binom{n}{k} A_k(t).$$

Example

$$\widetilde{A}_n(t) \;=\; \begin{cases} 1+t, & \text{if} \quad n=1 \\ 1+3t+t^2, & \text{if} \quad n=2 \\ 1+7t+7t^2+t^3, & \text{if} \quad n=3 \\ 1+15t+33t^2+15t^3+t^4, & \text{if} \quad n=4 \\ 1+31t+131t^2+131t^3+31t^4+t^5, & \text{if} \quad n=5 \\ 1+63t+473t^2+883t^3+473t^4+63t^5+t^6, & \text{if} \quad n=6. \end{cases}$$

Theorem (Postnikov–Reiner–Williams, 2008; Shareshian–Wachs, 2017+)

The polynomial $\widetilde{A}_n(t)$

- *is symmetric and unimodal,*
- *can be written as*

$$\widetilde{A}_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{n,k}\, t^k (1+t)^{n-2k},$$

where $\widetilde{\gamma}_{n,k}$ is the number of $w \in \mathfrak{S}_n$ with $\mathrm{des}(w) = k$, for which there is no index $i \in \{2, \ldots, n-1\}$ such that $w(i-1) > w(i) > w(i+1)$.

Gamma-positivity

Definition

A polynomial $f(t) \in \mathbb{R}[t]$ is called γ-positive if

$$f(t) \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k t^k (1+t)^{n-2k}$$

for some $n \in \mathbb{N}$ and nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$.

Writing $f(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$, we then have

- $p_i = p_{n-i}$ for $0 \leq i \leq n$,
- $p_0 \leq p_1 \leq \cdots \leq p_{\lfloor n/2 \rfloor}$.

Recently, γ-positivity attracted attention after the work of

- Bränden (2004, 2008) on P-Eulerian polynomials,
- Gal (2005) on flag triangulations of spheres.

Gamma-positive polynomials arise often in enumerative, algebraic and geometric contexts; see:

- A, Gamma-positivity in combinatorics and geometry, 2017.
- T.K. Petersen, Eulerian Numbers, Birkhaüser, 2015.

Gessel's identities

We let

- $\mathbf{x} = (x_1, x_2, x_3, \dots)$ be a sequence of commuting indeterminates,
- $h_n(\mathbf{x})$ be the complete homogeneous symmetric function of degree n in $\mathbf{x}$, defined by

$$H(\mathbf{x}; z) \; := \; \sum_{n \geq 0} h_n(\mathbf{x}) z^n \; = \; \prod_{i \geq 1} \frac{1}{1 - x_i z}$$

and set

- $\mathbf{x}_w = x_{w(1)} x_{w(2)} \cdots x_{w(n)}$

for $w : [n] \to \mathbb{Z}_{>0}$.

Gessel (unpublished) showed that

$$\frac{(1-t)H(\mathbf{x};z)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = 1+\sum_{n\geq 1} z^n \sum_{w} \mathbf{x}_w t^{\mathrm{des}(w)}(1+t)^{n-1-2\mathrm{des}(w)},$$

$$\frac{1-t}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = 1+\sum_{n\geq 2} z^n \sum_{w} \mathbf{x}_w t^{\mathrm{des}(w)+1}(1+t)^{n-2-2\mathrm{des}(w)},$$

where the inner sums range over all $w : [n] \to \mathbb{Z}_{>0}$ for which

- $w(n-1) \leq w(n)$ (respectively, $w(1) \leq w(2)$ and $w(n-1) \leq w(n)$),
- there is no $i \in \{2,\ldots,n-1\}$ such that $w(i-1) > w(i) > w(i+1)$.

These identities may be rewritten in the form

$$\frac{(1-t)H(\mathbf{x};z)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = 1 + \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(\mathbf{x})\, t^k (1+t)^{n-1-2k},$$

$$\frac{1-t}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = 1 + \sum_{n\geq 2} z^n \sum_{k=1}^{\lfloor n/2 \rfloor} \xi_{n,k}(\mathbf{x})\, t^k (1+t)^{n-2k},$$

where $\gamma_{n,k}(\mathbf{x})$ and $\xi_{n,k}(\mathbf{x})$ are Schur-positive symmetric functions of degree n, whose coefficients in the Schur basis refine the numbers $\gamma_{n,k}$ and $\xi_{n,k}$.

They can be considered as $\mathfrak{S}_n$-equivariant analogues of the γ-expansions of $A_n(t)$ and $d_n(t)$. For instance, write

$$\frac{(1-t)H(\mathbf{x};z)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} \;=\; \sum_{n\geq 0} \mathrm{ch}(F_n(t))\, z^n$$

for some graded $\mathfrak{S}_n$-representation $F_n(t) = \sum_{i=0}^{n} W_{n,i} t^i$, where ch denotes Frobenius characteristic. Applying a suitable exponential specialization,

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} \;=\; \frac{(1-t)e^z}{e^{tz}-te^z} \;=\; \sum_{n\geq 0}\left(\sum_{i=0}^{n} \dim(W_{n,i})\, t^i\right)\frac{z^n}{n!}$$

and hence

$$A_n(t) \;=\; \sum_{i=0}^{n} \dim(W_{n,i})\, t^i.$$

Thus, $F_n(t)$ is an $\mathfrak{S}_n$-equivariant analogue of $A_n(t)$.

Gessel's first identity implies that

$$F_n(t) \;\cong_{\mathfrak{S}_n} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M_{n,k}\, t^k (1+t)^{n-1-2k}$$

for some (non-virtual) $\mathfrak{S}_n$-representations $M_{n,k}$ and provides an $\mathfrak{S}_n$-equivariant analogue (and refinement) of the γ-positivity of $A_n(t)$. Similar remarks apply to Gessel's second identity, which may be written in the form

$$G_n(t) \;\cong_{\mathfrak{S}_n} \sum_{k=0}^{\lfloor n/2 \rfloor} N_{n,k}\, t^k (1+t)^{n-2k},$$

and $d_n(t)$.

Similarly, Shareshian and Wachs provided the $\mathfrak{S}_n$-equivariant analogue

$$\frac{(1-t)H(\mathbf{x};z)H(\mathbf{x};tz)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} \;=\; 1 + \sum_{n\geq 1} z^n \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{n,k}(\mathbf{x})\, t^k (1+t)^{n-2k}$$

of the γ-positivity of $\widetilde{A}_n(t)$, where the $\widetilde{\gamma}_{n,k}(\mathbf{x})$ are Schur-positive symmetric functions of degree n, whose coefficients in the Schur basis refine the numbers $\widetilde{\gamma}_{n,k}$.

Question: Are there more interesting equivariant analogues of γ-positivity phenomena?

Equivariant γ-positivity

We let

- G be a finite group
- $R(G)$ be the representation ring of G.

Definition

A polynomial $F(t) \in R(G)[t]$ is called γ-positive if

$$F(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} M_k t^k (1+t)^{n-2k}$$

for some $n \in \mathbb{N}$ and non-virtual G-representations $M_0, M_1, \ldots, M_{\lfloor n/2 \rfloor}$.

Writing

$$F(t) \;=\; P_0 + P_1 t + P_2 t^2 + \cdots + P_n t^n,$$

we then have

- $P_i \cong_G P_{n-i}$ for $0 \leq i \leq n$,
- $P_0 \leq_G P_1 \leq_G \cdots \leq_G P_{\lfloor n/2 \rfloor}$, where $P \leq_G Q$ means that $Q - P$ is a non-virtual G-representation.

Equivariant analogues of polynomials in combinatorics arise often from group actions on:

- simplicial complexes and their face rings
- posets and their homology
- lattice polytopes and their Ehrhart rings
- matroids

and so on.

Local face modules

We let

- $V_n = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ be the set of unit coordinate vectors in $\mathbb{R}^n$
- Σ_n be the geometric simplex with vertex set V_n
- Γ be a triangulation of Σ_n with vertex set V_Γ
- $\mathbb{C}[\Gamma]$ be the face ring of Γ over $\mathbb{C}$
- Θ be the ideal in $\mathbb{C}[\Gamma]$ generated by $\theta_1, \theta_2, \ldots, \theta_n$, where

$$\theta_i \; = \; \sum_{v \in V_\Gamma} \langle v, \varepsilon_i \rangle \, x_v$$

- $\mathbb{C}(\Gamma) = \mathbb{C}[\Gamma]/\Theta$.

Definition (Stanley, 1992)

The local face module of Γ, denoted $L_{V_n}(\Gamma)$, is defined as the image in $\mathbb{C}(\Gamma)$ of the ideal of $\mathbb{C}[\Gamma]$ generated by the square-free monomials which correspond to the faces of Γ lying in the interior of Σ_n.

The module $L_{V_n}(\Gamma)$ is a finite-dimensional, graded $\mathbb{C}$-algebra whose Hilbert polynomial

$$\ell_{V_n}(\Gamma, t) \ := \ \sum_{i=0}^{n} \dim_{\mathbb{C}}(L_{V_n}(\Gamma)_i)\, t^i$$

is the local h-polynomial of Γ (depends only on the face vectors of the restrictions of Γ to the faces of Σ).

Theorem (Stanley, 1992)

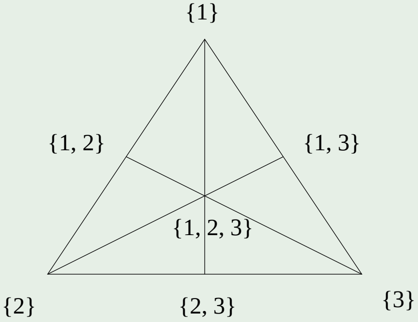
The polynomial $\ell_{V_n}(\Gamma, t)$ is:

- *symmetric,*
- *unimodal for every regular triangulation Γ of Σ_n.*

Note: The polynomial $\ell_{V_n}(\Gamma, t)$ has been shown to be γ-positive for several classes of flag triangulations of Σ_n.

Example

For the barycentric subdivision Γ_n of Σ_n



Stanley showed that $\ell_{V_n}(\Gamma_n, t) = d_n(t)$.

Now let

- G be a subgroup of the automorphism group $\mathfrak{S}_n$ of Σ_n which acts simplicially on Γ.

Then G acts on $L_{V_n}(\Gamma)$ as well, which becomes a G-equivariant analogue of $\ell_{V_n}(\Gamma, t)$.

Proposition (Stanley, 1992)

For the $\mathfrak{S}_n$-action on the barycentric subdivision Γ_n of Σ_n, we have

$$\sum_{n \geq 0} z^n \sum_{i=0}^{n} \mathrm{ch}\left(L_{V_n}(\Gamma_n)_i\right) t^i \;=\; \frac{1-t}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)}.$$

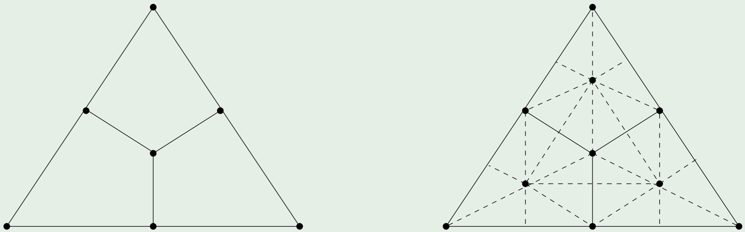
Thus, $\sum_{i=0}^{n} L_{V_n}(\Gamma_n)_i \, t^i$ is γ-positive by Gessel's second identity.

Question: Is there a $\mathcal{B}_n$-analogue of this result? Is there an analogue for the colored permutation groups $\mathbb{Z}_r \wr \mathfrak{S}_n$?

We let

- K_n be the barycentric subdivision of the standard subdivision of Σ_n into n cubes.

Example



$n = 3$

Theorem (A, 2018+)

For the $\mathfrak{S}_n$-action on K_n we have

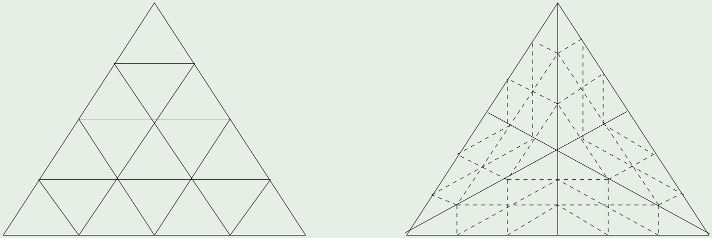
$$\sum_{n\geq 0} z^n \sum_{i=0}^{n} \mathrm{ch}\,(L_{V_n}(K_n)_i)\, t^i \;=\; \frac{H(\mathbf{x};tz) - tH(\mathbf{x};z)}{H(\mathbf{x};tz)^2 - tH(\mathbf{x};z)^2}.$$

Moreover, $\sum_{i=0}^{n} L_{V_n}(K_n)_i\, t^i$ is γ-positive for every n.

More generally, let

- $\Gamma_{n,r}$ be the r-fold edgewise subdivision of the barycentric subdivision Γ_n of Σ_n.

Example



$n = r = 3$ (on the right)

Theorem (A, 2018+)

For the $\mathfrak{S}_n$-action on $\Gamma_{n,r}$ we have

$$\sum_{n\geq 0} z^n \sum_{i=0}^{n} \mathrm{ch}\,(L_{V_n}(\Gamma_{n,r})_i)\, t^i \;=\; \frac{H(\mathbf{x};tz)^{r-1} - tH(\mathbf{x};z)^{r-1}}{H(\mathbf{x};tz)^{r} - tH(\mathbf{x};z)^{r}}.$$

Moreover, $\sum_{i=0}^{n} L_{V_n}(\Gamma_{n,r})_i\, t^i$ *is* γ*-positive for all* n, r.

Question: Is there an analogous result for the r-fold edgewise subdivision of Σ_n?

Face rings of triangulations of spheres

We let

- Δ be the complex associated to a complete simplicial fan $\mathcal{F}$ in $\mathbb{R}^n$
- $f_i(\Delta)$ be the number of i-dimensional faces of Δ
- $\mathbb{C}[\Delta]$ be the face ring of Δ over $\mathbb{C}$
- $\mathbb{C}(\Delta) = \mathbb{C}[\Delta]/\Theta$.

The ring $\mathbb{C}(\Delta)$ is a finite-dimensional, graded $\mathbb{C}$-algebra whose Hilbert polynomial

$$h(\Delta, t) \ := \ \sum_{i=0}^{n} \dim_{\mathbb{C}}(\mathbb{C}(\Delta)_i)\, t^i \ = \ \sum_{i=0}^{n} f_{i-1}(\Delta)\, t^i (1-t)^{n-i}$$

is the h-polynomial of Δ.

Theorem

The polynomial $h(\Delta, t)$ is:

- (Klee, 1965) *symmetric,*
- (Stanley, 1980) *unimodal if Δ is the boundary complex of a simplicial polytope.*

Note: The polynomial $h(\Delta, t)$ has been shown to be γ-positive for several classes of flag triangulations of the sphere.

Note: When the fan $\mathcal{F}$ is rational, there is an associated complex projective toric variety $\mathcal{T}_\Delta$. Danilov (1978) showed that

$$H^{2i}(\mathcal{T}_\Delta; \mathbb{C}) \cong \mathbb{C}(\Delta)_i.$$

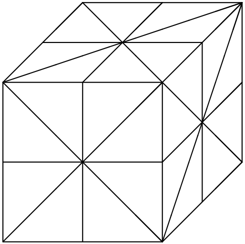
Now let

- G be a group of orthogonal transformations in $\mathbb{R}^n$ which acts simplicially on $\mathcal{F}$.

Then G acts on $\mathbb{C}(\Delta)$ as well, which becomes a G-equivariant analogue of $h(\Delta, t)$.

Prototypical example: We let

- W be a finite crystallographic Coxeter group of rank n
- Δ_W be the associated Coxeter complex.



Coxeter complex of type B_3

Because of its interpretation in terms of the toric variety $\mathcal{T}_{\Delta_W}$, the graded W-representation $\mathbb{C}(\Delta_W)$ was studied by:

- Procesi (1990)
- Stanley (1989)
- Dolgachev–Lunts (1994)
- Stembridge (1994)
- Lehrer (2008)

Question: Is $\sum_{i=0}^{n} \mathbb{C}(\Delta_W)_i \, t^i$ γ-positive?

Theorem (Procesi, Stanley)

For the $\mathfrak{S}_n$-action on the Coxeter complex $\Delta_{\mathfrak{S}_n}$, we have

$$\sum_{n \geq 0} z^n \sum_{i=0}^{n-1} \mathrm{ch}\left(\mathbb{C}(\Delta_{\mathfrak{S}_n})_i\right) t^i \;=\; \frac{(1-t)H(\mathbf{x}; z)}{H(\mathbf{x}; tz) - tH(\mathbf{x}; z)}.$$

Thus $\sum_{i=0}^{n-1} \mathbb{C}(\Delta_{\mathfrak{S}_n})_i \, t^i$ is γ-positive by Gessel's first identity.

Theorem (Dolgachev–Lunts, Stembridge, 1994)

For the $\mathcal{B}_n$-action on the Coxeter complex $\Delta_{\mathcal{B}_n}$, we have

$$\sum_{n \geq 0} z^n \sum_{i=0}^{n} \mathrm{ch}_{\mathcal{B}}\left(\mathbb{C}(\Delta_{\mathcal{B}_n})_i\right) t^i \;=\; \frac{(1-t)H(\mathbf{x};z)H(\mathbf{x};tz)}{H(\mathbf{x};tz)H(\mathbf{y};tz) - tH(\mathbf{x};z)H(\mathbf{y};z)}.$$

Proposition (A, 2018+)

We have

$$\frac{(1-t)H(\mathbf{x};z)H(\mathbf{x};tz)}{H(\mathbf{x};tz)H(\mathbf{y};tz)-tH(\mathbf{x};z)H(\mathbf{y};z)} = \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma^B_{n,k}(\mathbf{x},\mathbf{y})\, t^k (1+t)^{n-2k}$$

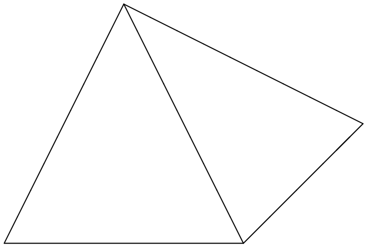
for some Schur positive functions $\gamma^B_{n,k}(\mathbf{x},\mathbf{y}) \in \Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{y})$ of total degree n. As a result, $\sum_{i=0}^{n} \mathbb{C}(\Delta_{\mathcal{B}_n})_i\, t^i$ is γ-positive for every n.

Note: No combinatorial interpretations for the coefficients of $\gamma^B_{n,k}(\mathbf{x},\mathbf{y})$ in the Schur basis are known at present.

Note: Given other evidence provided by results by Stembridge and Lehrer, it seems reasonable to conjecture that $\sum_{i=0}^{n} \mathbb{C}(\Delta_W)_i \, t^i$ is γ-positive for every finite (crystallographic) Coxeter group W.

We now let

- $\widetilde{\Delta}_n$ be the boundary complex of the n-dimensional simplicial stellohedron.



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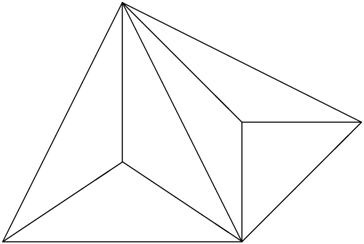

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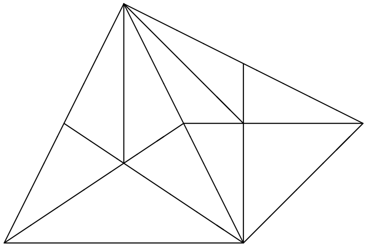


Note: Postnikov–Reiner–Williams showed that $h(\widetilde{\Delta}_n, t) = \widetilde{A}_n(t)$.

Recall the Shareshian–Wachs identity

$$\frac{(1-t)H(\mathbf{x};z)H(\mathbf{x};tz)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{n,k}(\mathbf{x})\, t^k (1+t)^{n-2k}.$$

Theorem (Shareshian–Wachs, 2017+)

For the natural $\mathfrak{S}_n$-action on $\widetilde{\Delta}_n$, we have

$$\sum_{n\geq 0} z^n \sum_{i=0}^{n} \mathrm{ch}\left(\mathbb{C}(\widetilde{\Delta}_n)_i\right) t^i = \frac{(1-t)H(\mathbf{x};z)H(\mathbf{x};tz)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)}.$$

In particular, $\sum_{i=0}^{n} \mathbb{C}(\widetilde{\Delta}_n)_i\, t^i$ is γ-positive for every n.

Question: Is there a $\mathcal{B}_n$-analogue of this result? Is there an analogue for the colored permutation groups $\mathbb{Z}_r \wr \mathfrak{S}_n$?

The triangulation $\Delta(\Gamma)$

We let

- Γ be a triangulation of the simplex Σ_n
- Γ_F be the restriction of Γ to the face F of Σ_n.

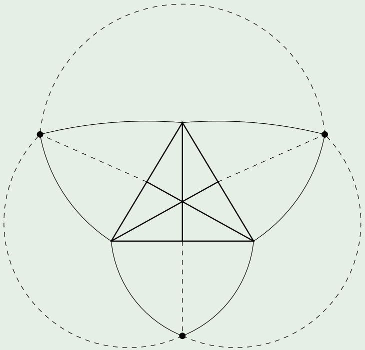
Then, there exists a triangulation $\Delta(\Gamma)$ of the standard n-dimensional cros-spolytope which restricts to Γ on one of its facets and satisfies

$$h(\Delta(\Gamma), t) \; = \; \sum_{F} t^{n-\dim(F)} \, h(\Gamma_F, t),$$

summed over all faces of Σ_n.

Example

For the barycentric subdivision Γ_n of Σ_n we get



$$h(\Delta(\Gamma_n), t) \;=\; \sum_{k=0}^{n} \binom{n}{k} t^{n-k} A_k(t) \;=\; 1 \;+\; t \sum_{k=1}^{n} \binom{n}{k} A_k(t) \;=\; \widetilde{A}_n(t).$$

More generally, we may consider $\Delta(\Gamma_{n,r})$.

Theorem (A, 2018+)

For the $\mathfrak{S}_n$-action on $\Delta(\Gamma_{n,r})$ we have

$$\sum_{n\geq 0} z^n \sum_{i=0}^{n} \mathrm{ch}\left(\mathbb{C}(\Delta(\Gamma_{n,r}))_i\right) t^i = \frac{H(\mathbf{x};z)H(\mathbf{x};tz)\left(H(\mathbf{x};tz)^{r-1} - tH(\mathbf{x};z)^{r-1}\right)}{H(\mathbf{x};tz)^r - tH(\mathbf{x};z)^r}.$$

Moreover, $\sum_{i=0}^{n} \mathbb{C}(\Delta(\Gamma_{n,r}))_i\, t^i$ is γ-positive for $r = 2$ and every n.

Note: Gamma-positivity is open for $r \geq 3$.

Note: This construction motivates the definition

$$\widetilde{A}_{n,r}(t) \;=\; \sum_{k=0}^{n} \binom{n}{k} t^{n-k} A_{k,r}(t)$$

of the binomial Eulerian polynomial for r-coroled permutations, where

$$A_{n,r}(t) \;:=\; \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} t^{\mathrm{des}(w)} \;=\; \sum_{w \in \mathbb{Z}_r \wr \mathfrak{S}_n} t^{\mathrm{exc}(w)}$$

is the Eulerian polynomial for $\mathbb{Z}_r \wr \mathfrak{S}_n$. One can then show that

$$\widetilde{A}_{n,r}(t) \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}^{+}_{n,r,i}\, t^i (1+t)^{n-2i} \;+\; \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \widetilde{\gamma}^{-}_{n,r,i}\, t^i (1+t)^{n+1-2i}$$

for some nonnegative integers $\widetilde{\gamma}^{\pm}_{n,r,i}$ and that the first summand is equal to $h(\Delta(\Gamma_{n,r}), t)$.

Rees products of posets

We let

- P be a finite graded poset with rank function ρ_P,
- Q be a finite graded poset with rank function ρ_Q.

Definition (Björner–Welker, 2005)

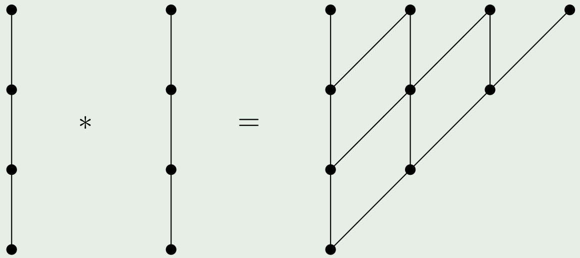
The Rees product of P and Q is defined as

$$P * Q \;=\; \{(p,q) \in P \times Q : \; \rho_P(p) \geq \rho_Q(q)\},$$

with partial order defined by setting $(p_1, q_1) \leq (p_2, q_2)$ if and only if:

- $p_1 \leq p_2$ *holds in* P,
- $q_1 \leq q_2$ *holds in* Q, *and*
- $\rho_P(p_2) - \rho_P(p_1) \geq \rho_Q(q_2) - \rho_Q(q_1)$.

Example



We let

- B_n be the Boolean lattice of subsets of $[n]$
- C_n be the n-element chain.

For a finite poset P we let

- $\mu(P) \;=\; \mu_{\hat{P}}(\hat{0}, \hat{1})$,

where $\mu_{\hat{P}}$ is the Möbius function of $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$.

Theorem (Johnsson, 2005)

The number $|\mu((B_n \setminus \varnothing) * C_n)|$ is equal to the derangement number d_n.

Note: When the group G acts on P by order-preserving bijections, it acts on $P * Q$ as well.

For positive integers n, t we let

- $C_{n,t}$ be the poset whose Hasse diagram is a complete t-ary tree of height $n-1$, with root at the bottom.

Theorem (Shareshian–Wachs, 2009)

For the $\mathfrak{S}_n$-action on $(B_n \setminus \varnothing) * C_n$ we have

$$\sum_{n \geq 0} \operatorname{ch} \tilde{H}_{n-1}((B_n \setminus \varnothing) * C_{n,t}; \mathbb{C})\, z^n \;=\; \frac{1-t}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)}.$$

For a graded poset P of rank $n+1$ with minimum element $\hat{0}$, maximum element $\hat{1}$ and rank function $\rho : P \to \{0, 1, \ldots, n+1\}$ we set

- $\bar{P} \; = \; P \smallsetminus \{\hat{0}, \hat{1}\}$.

For $S \subseteq [n]$ we set

- $b_P(S) \; = \; (-1)^{|S|-1} \mu(\bar{P}_S)$,

where

$$\bar{P}_S \; = \; \{x \in P : \; \rho(x) \in S\}$$

is a rank-selected subposet.

Theorem (Linusson–Shareshian–Wachs, 2012)

For every EL-shellable poset P of rank $n+1$ and every positive integer t

$$|\mu(\bar{P} * C_{n,t})| = \sum_{S \in \mathrm{Stab}(\{2,\ldots,n-1\})} b_P([n] \setminus S)\, t^{|S|}\, (1+t)^{n-1-2|S|} \quad + \sum_{S \in \mathrm{Stab}(\{2,\ldots,n-2\})} b_P([n-1] \setminus S)\, t^{|S|+1}\, (1+t)^{n-2-2|S|},$$

where $\mathrm{Stab}(\Theta)$ *denotes the set of all subsets of* Θ *which do not contain two consecutive integers.*

Note: These authors proved a similar formula for

$$|\mu(((P \setminus \{\hat{1}\}) * C_{n,t}) \setminus \{\hat{0}\})|.$$

Suppose now that G acts on P by order-preserving bijections and that P is Cohen–Macaulay over $\mathbb{C}$.

Theorem (A, 2017+)

$$\tilde{H}_{n-1}(\bar{P} * C_{n,t}; \mathbb{C}) \cong_G \sum_{S \in \mathrm{Stab}([2,n-1])} \beta_P([n] \setminus S)\, t^{|S|}\, (1+t)^{n-1-2|S|} + \sum_{S \in \mathrm{Stab}([2,n-2])} \beta_P([n-1] \setminus S)\, t^{|S|+1}\, (1+t)^{n-2-2|S|},$$

where $\beta_P(S)$ is the (non-virtual) G-representation on $\tilde{H}_{|S|-1}(\bar{P}_S; \mathbb{C})$.

Note: A similar result holds for $\tilde{H}_{n-1}(((P \setminus \{\hat{1}\}) * C_{n,t}) \setminus \{\hat{0}\}; \mathbb{C})$.

Note: Gessel's identities follow from the special case of the action of $\mathfrak{S}_n$ on B_n. Our other equivariant γ-positivity results follow by applying this to the natural r-colored version of B_n. For instance, letting

- $e_n(\mathbf{x})$ be the elementary symmetric function of degree n, defined by

$$E(\mathbf{x}; z) \ := \ \sum_{n \geq 0} e_n(\mathbf{x}) z^n \ = \ \prod_{i \geq 1} (1 + x_i z)$$

we have:

Corollary (A, 2017+)

We have

$$\frac{E(\mathbf{x};tz) - tE(\mathbf{x};z)}{E(\mathbf{x};tz)E(\mathbf{y};tz) - tE(\mathbf{x};z)E(\mathbf{y};z)} = \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \xi^{+}_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-2k}$$

and

$$\frac{E(\mathbf{x};z)E(\mathbf{x};tz)\left(E(\mathbf{y};tz) - tE(\mathbf{y};z)\right)}{E(\mathbf{x};tz)E(\mathbf{y};tz) - tE(\mathbf{x};z)E(\mathbf{y};z)} = \sum_{n\geq 0} z^n \sum_{k=0}^{\lfloor n/2\rfloor} \gamma^{+}_{n,k}(\mathbf{x},\mathbf{y})\, t^k(1+t)^{n-2k}$$

for some Schur-positive functions $\xi^{+}_{n,k}(\mathbf{x},\mathbf{y}), \gamma^{+}_{n,k}(\mathbf{x},\mathbf{y}) \in \Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{y})$ *of total degree* n*.*

This corollary implies:

- Gessel's second identity (set $\mathbf{x} = 0$ to the first one)
- the γ-positivity of $\sum_{i=0}^{n} L_{V_n}(K_n)_i \, t^i$ (set $\mathbf{x} = \mathbf{y}$ to the first one)
- the Shareshian–Wachs identity (set $\mathbf{y} = 0$ to the second one)
- the γ-positivity of $\sum_{i=0}^{n} \mathbb{C}(\Delta_{\mathcal{B}_n})_i \, t^i$ (combine the two).

Thank you for your attention!