Combinatorics of hyperplane arrangements: Open problems and recent progress

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Outline

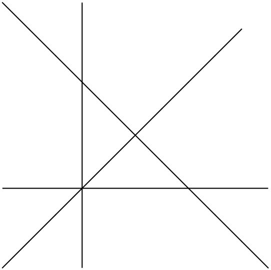
1. Basic definitions and examples
2. Deformations of Coxeter arrangements
3. Deformations of rational arrangements

Face enumeration

Given an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^n$ we let

$$\begin{aligned} f_k(\mathcal{A}) &= \# \text{ of } k\text{-dimensional faces of } \mathcal{A}, \\ r(\mathcal{A}) &= \# \text{ of regions of } \mathcal{A} \ = \ f_n(\mathcal{A}). \end{aligned}$$

Example: For

$\mathcal{A} \ =$ 

we have

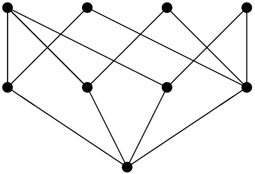
$$f_0(\mathcal{A}) = 4, \quad f_1(\mathcal{A}) = 13 \quad \text{and} \quad r(\mathcal{A}) = f_2(\mathcal{A}) = 10.$$

Combinatorial invariants

The intersection poset $\mathcal{L}_\mathcal{A}$ consists of all nonempty intersections of hyperplanes of $\mathcal{A}$, partially ordered by reverse inclusion: $x \leq y \Leftrightarrow x \supseteq y$; the element $\hat{0} = \mathbb{R}^n$ is the minimum of $\mathcal{L}_\mathcal{A}$.

Example: With $\mathcal{A}$ as before,

$\mathcal{L}_\mathcal{A} \quad =$



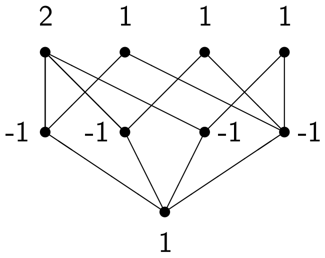
Combinatorial invariants

The Möbius function of $\mathcal{L}_\mathcal{A}$ is defined by

$$\mu(x,y) \;=\; \begin{cases} 1, & \text{if } x = y \\ -\displaystyle\sum_{x \le z < y} \mu(x,z), & \text{otherwise} \end{cases}$$

for $x, y \in \mathcal{L}_\mathcal{A}$ with $x \le y$.

Example: With $\mathcal{A}$ as before, the values $\mu(\hat{0}, x)$ are



Combinatorial invariants

Theorem (Las Vergnas, Zaslavsky, 1975)

The number of k-dimensional faces of $\mathcal{A}$ is given by

$$f_k(\mathcal{A}) = \sum (-1)^{\dim(x)-\dim(y)} \mu(x,y)$$
$$= \sum |\mu(x,y)|,$$

where the sums range over all elements $x \in \mathcal{L}_{\mathcal{A}}$ of dimension k and all elements $y \in \mathcal{L}_{\mathcal{A}}$ with $x \leq y$. In particular,

$$r(\mathcal{A}) = \sum_{x \in \mathcal{L}_{\mathcal{A}}} |\mu(\hat{0}, x)|.$$

The characteristic polynomial

The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$, defined as the generating function

$$\chi(\mathcal{A}, q) \;=\; \sum_{x \in \mathcal{L}_{\mathcal{A}}} \mu(\hat{0}, x) q^{\dim(x)}$$

of $\mu(\hat{0}, x)$ over $\mathcal{L}_{\mathcal{A}}$, is a fundamental enumerative invariant of $\mathcal{A}$.

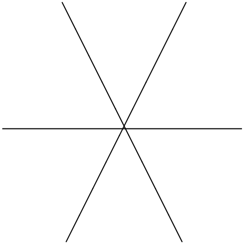
Note: By the Las Vergnas - Zaslavsky theorem, $(-1)^n \chi(\mathcal{A}, -q)$ is a monic polynomial in q of degree n with nonnegative coefficients and

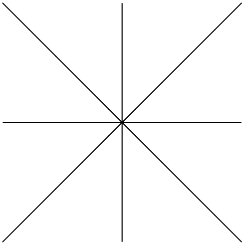
$$r(\mathcal{A}) \;=\; (-1)^n \chi(\mathcal{A}, -1).$$

Coxeter arrangements

Let

- W be an irreducible finite reflection group,
- $\mathcal{A}_W$ be the corresponding Coxeter arrangement,
- ℓ be the rank of W,
- $e_1, e_2, \ldots, e_\ell$ be the exponents of W.





Coxeter arrangements

Theorem (Brieskorn, 1971, Orlik–Solomon, 1980)

The characteristic polynomial of $\mathcal{A}_W$ factors as

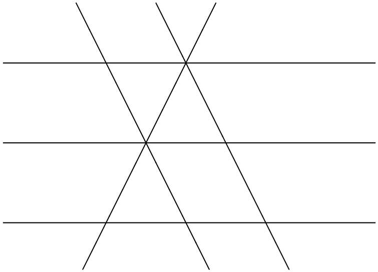
$$\chi(\mathcal{A}_W, q) \;=\; \prod_{i=1}^{\ell} (q - e_i).$$

In particular,

$$r(\mathcal{A}_W) \;=\; \prod_{i=1}^{\ell} (e_i + 1).$$

Deformations of $\mathcal{A}_W$

An arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is called a deformation of $\mathcal{A}_W$ if each hyperplane of $\mathcal{A}$ is parallel to some hyperplane of $\mathcal{A}_W$. Their combinatorial study was initiated by Richard Stanley (1996).



Deformations of $\mathcal{A}_W$

Notable and well-studied examples include:

- the Catalan arrangements,
- the Linial arrangements,
- the Shi arrangements

and their generalizations. Their combinatorics relates to

- interval orders,
- trees,
- parking functions,
- rook placements

and so on.

m-Catalan and m-Shi arrangements

Assume W is crystallographic and let

- Φ be a corresponding root system in $V = \mathbb{R}^\ell$,
- Φ^+ be a positive subsystem,
- h be the Coxeter number of W.

The m-Catalan arrangement $\mathcal{A}_\Phi^m$ consists of the hyperplanes

$$(\alpha, x) \;=\; -m, -m+1, \ldots, m-1, m, \qquad \alpha \in \Phi^+$$

in V. The m-Shi arrangement $\mathcal{S}_\Phi^m$ consists of the hyperplanes

$$(\alpha, x) \;=\; -m+1, \ldots, m-1, m, \qquad \alpha \in \Phi^+$$

in V.

m-Catalan and m-Shi arrangements

Theorem (A, 2004)

The characteristic polynomial of $\mathcal{A}_\Phi^m$ is given by

$$\chi(\mathcal{A}_\Phi^m, q) \;=\; \chi(\mathcal{A}_W, q - mh) \;=\; \prod_{i=1}^{\ell} (q - mh - e_i).$$

In particular,

$$r(\mathcal{A}_\Phi^m) \;=\; \prod_{i=1}^{\ell} (e_i + mh + 1).$$

Note: The proof is uniform and uses the finite field method (A, 1996).

This method has been pushed further by Yoshinaga who gave a uniform proof, among other results, of the following:

Theorem (Yoshinaga, 201x)

The characteristic polynomial of $\mathcal{S}_\Phi^m$ factors as

$$\chi(\mathcal{S}_\Phi^m, q) \;=\; (q - mh)^\ell.$$

In particular,

$$r(\mathcal{S}_\Phi^m) \;=\; (mh + 1)^\ell.$$

The Möbius function

Proposition

For the Möbius function μ of $\mathcal{L}_{\mathcal{A}_W}$ we have

$$\mu(\hat{0}, x) \;=\; (-1)^{\mathrm{codim}(x)} \, \# \{ w \in W : \mathrm{Fix}(w) = x \}$$

for $x \in \mathcal{L}_{\mathcal{A}_W}$, where

$$\mathrm{Fix}(w) \;=\; \{ v \in V : \; w(v) = v \}$$

is the fixed space of $w \in W$.

Example: For the symmetric group $W = \mathfrak{S}_n$,

$$\begin{aligned} \mu(\hat{0}, \hat{1}) \;&=\; (-1)^{n-1} \, \# \{ \text{cyclic permutations } w \in \mathfrak{S}_n \} \\ &=\; (-1)^{n-1} \, (n-1)!. \end{aligned}$$

The Möbius function

Combined with results of Shephard–Todd (1954) and Solomon (1963), this statement implies that

$$\chi(\mathcal{A}_W, q) \;=\; \sum_{w \in W} (-1)^{\mathrm{codim}(\mathrm{Fix}(w))}\, q^{\dim(\mathrm{Fix}(w))} \;=\; \prod_{i=1}^{\ell} (q - e_i).$$

Problem

Find an analogous expression for the Möbius function of the intersection poset of $\mathcal{A}_\Phi^m$ and use it to show that

$$\chi(\mathcal{A}_\Phi^m, q) \;=\; \prod_{i=1}^{\ell} (q - mh - e_i).$$

Similarly for $\mathcal{S}_\Phi^m$.

Faces of the braid arrangement

The braid arrangement $\mathcal{A}_n$ consists of the $\binom{n}{2}$ hyperplanes $x_i = x_j$ in $\mathbb{R}^n$. The number of k-dimensional faces is given by

$$\begin{aligned} f_k(\mathcal{A}_n) &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \\ &= \#\text{ of surjective maps } \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}. \end{aligned}$$

In particular,

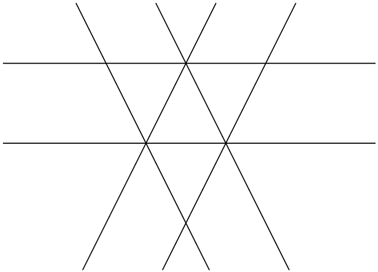
$$r(\mathcal{A}_n) = \#\text{ of permutations of } \{1, 2, \ldots, n\} = n!.$$

Faces of the Shi arrangement

The Shi arrangement $\mathcal{S}_n$ consists of the $n(n-1)$ hyperplanes

- $x_i - x_j = 0, \quad 1 \le i < j \le n,$
- $x_i - x_j = 1, \quad 1 \le i < j \le n$

in $\mathbb{R}^n$. For $n = 3$



$$f_1(\mathcal{S}_3) = 6, \quad f_2(\mathcal{S}_3) = 21 \quad \text{and} \quad r(\mathcal{S}_3) = 16.$$

Regions of the Shi arrangement

Several bijective proofs are known that

$$\begin{aligned} r(\mathcal{S}_n) &= \#\text{ of trees on the vertex set } \{1, 2, \ldots, n\} \\ &= \#\text{ of parking functions on } \{1, 2, \ldots, n\}, \\ &= (n+1)^{n-1}. \end{aligned}$$

Faces of the Shi arrangement

Theorem (A, 1996)

The number of k-dimensional faces of $\mathcal{S}_n$ is given by

$$\begin{aligned} f_k(\mathcal{S}_n) &= \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} (n-i+1)^{n-1} \\ &= \binom{n}{k} \# \{ f : [n-1] \to [n+1] \ : \ [n-k] \subseteq \mathrm{Im}\,(f) \}, \end{aligned}$$

where $[m] := \{1, 2, \dots, m\}$ and $\mathrm{Im}\,(f)$ is the image of the map f.

In particular, $f_1(\mathcal{S}_n) = n!$ and $r(\mathcal{S}_n) = (n+1)^{n-1}$.

Note: The proof uses the finite field method (A, 1996).

Faces of the Shi arrangement

Problem

Find a bijective proof of this result.

The Tits product of a face F and a region C of $\mathcal{A}$ is defined as the region FC which is closest to C among all regions of $\mathcal{A}$ whose closure contains F.

Problem

Describe the Tits product of a face and a region of $\mathcal{S}_n$ in terms of nice combinatorial objects and operations.

Regions of the Linial arrangement

The Linial arrangement $\mathcal{L}_n$ consists of the $\binom{n}{2}$ hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$ in $\mathbb{R}^n$. A tree T on the vertex $\{1, 2, \ldots, n\}$ is alternating if every vertex of T is either smaller than all its neighbors, or larger that all its neighbors.

Theorem (A, Postnikov-Stanley, 1996)

The number of regions of $\mathcal{L}_n$ is given by

$$r(\mathcal{L}_n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$$

$$= \# \text{ of alternating trees on } \{1, 2, \ldots, n+1\}.$$

Regions of the Linial arrangement

Problem

Find a bijection from the set of regions of $\mathcal{L}_n$ to that of alternating trees on $\{1, 2, \ldots, n+1\}$.

Combinatorial interpretations for the number of relatively bounded regions of $\mathcal{L}_n$ in terms of Postnikov's local binary search trees have been found by David Forge and Vasu Tewari.

Problem

Find analogues of this result for other Coxeter types.

Deformations of rational arrangements

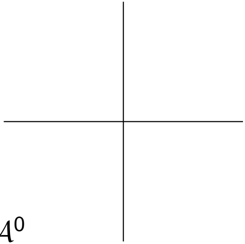
Consider n linear hyperplanes

$$H_i \;=\; \{x \in \mathbb{R}^d :\; \alpha_i(x) = 0\}, \qquad 1 \le i \le n$$

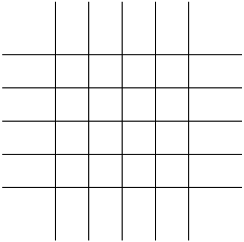
in $\mathbb{R}^d$, defined by rational linear forms α_i spanning the dual space $(\mathbb{R}^d)^*$, and denote by $\mathcal{A}^m$ the arrangement of affine hyperplanes

$$\alpha_i(x) \;=\; -m, -m+1, \ldots, m, \qquad 1 \le i \le n$$

in $\mathbb{R}^d$.



$\mathcal{A}^0$



$\mathcal{A}^m$

Coordinate hyperplanes

Example: Suppose $H_i = \{x \in \mathbb{R}^d : \; x_i = 0\}$ for $1 \leq i \leq d$ are the coordinate hyperplanes in $\mathbb{R}^d$. Then

$$\chi(\mathcal{A}^m, q) \; = \; (q - 2m - 1)^d.$$

Moreover, the arrangement $\mathcal{A}^m$ has $r_{\mathcal{A}}(m) = (2m+2)^d$ regions of which $b_{\mathcal{A}}(m) = (2m)^d$ are bounded and

$$(-1)^d r_{\mathcal{A}}(-m) \; = \; b_{\mathcal{A}}(m-1).$$

Deformations of rational arrangements

Theorem (A, 2010)

The characteristic polynomial $\chi_{\mathcal{A}}(m,q) := \chi(\mathcal{A}^m, q)$ is a quasi-polynomial in m which satisfies the reciprocity law

$$\chi_{\mathcal{A}}(-m,q) \;=\; (-1)^d\, \chi_{\mathcal{A}}(m-1,-q).$$

In particular, the number $r_{\mathcal{A}}(m)$ of regions of $\mathcal{A}^m$ and the number $b_{\mathcal{A}}(m)$ of bounded regions are quasi-polynomials in m related by

$$(-1)^d r_{\mathcal{A}}(-m) \;=\; b_{\mathcal{A}}(m-1).$$

Note: For the Coxeter arrangement $\mathcal{A} = \mathcal{A}_W$ the reciprocity law reduces to the known fact that $\{h-e_1, h-e_2, \ldots, h-e_d\} = \{e_1, e_2, \ldots, e_d\}$.

Deformations of rational arrangements

Problem

Under what conditions is $\chi(\mathcal{A}^m, q)$ a polynomial in m and q?

Recall that, given a graph G on the vertex set $\{1, 2, \ldots, d\}$, the graphical arrangement $\mathcal{A}_G$ consists of the hyperplanes

$$x_i - x_j \; = \; 0, \qquad \{i, j\} \in E_G$$

in $\mathbb{R}^d$. The function $\chi(\mathcal{A}_G^m, q)$ reduces to the chromatic polynomial of G for $m = 0$.

Problem

Study the function $\chi(\mathcal{A}_G^m, q)$.