Combinatorics of uniform triangulations

Combinatorics and Geometry in Ioannina

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Combinatorics of uniform triangulations

I. Introduction

II. Definitions and Examples

III. Main results

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I. Introduction to face enumeration

We are interested in the face enumeration of simplicial complexes. Let

Δ = simplicial complex of dimension $n-1$

$f_k(\Delta)$ = # k-dimensional faces of Δ.

Definition. The f, h-polynomials of Δ are defined as

$$f(\Delta, x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k$$

$$h(\Delta, x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k (1-x)^{n-k}$$

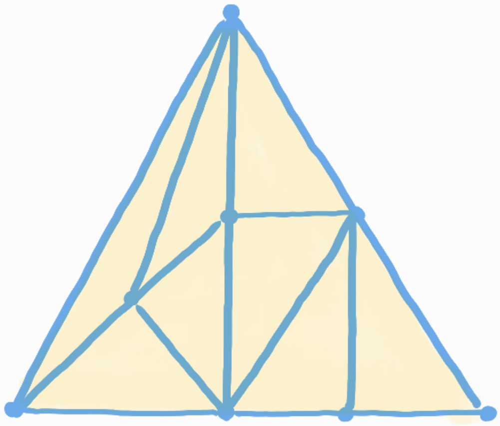
$$= (1-x)^n f\left(\Delta, \frac{x}{1-x}\right).$$

Remark.

(a) $h(\Delta, x)$ has nonnegative co-efficients if Δ is Cohen-Macaulay over some field.

(b) $h(\Delta, 1) = f_{n-1}(\Delta)$.

Example.



$n = 3$

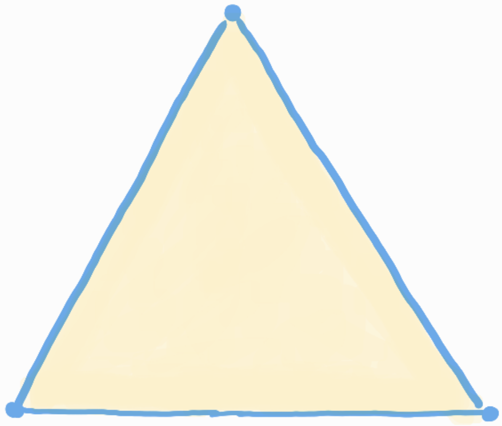
Δ

$f_0(\Delta) = 8, \quad f_1(\Delta) = 15, \quad f_2(\Delta) = 8$

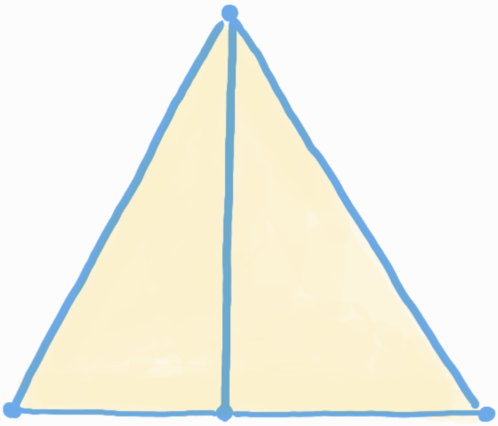
- $f(\Delta, x) = 1 + 8x + 15x^2 + 8x^3$
- $h(\Delta, x) = (1-x)^3 + 8x(1-x)^2 + 15x^2(1-x) + 8x^3$

$$= 1 + 5x + 2x^2.$$

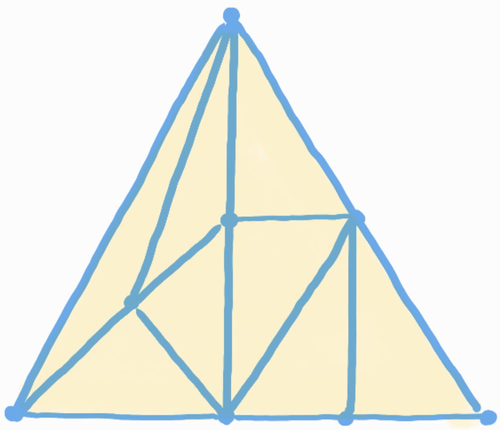
Question. How are $f(\Delta, x)$ and $h(\Delta, x)$ affected by simplicial subdivision of Δ?



$h(\cdot, x) = 1$



$h(\cdot, x) = 1 + x$



$h(\cdot, x) = 1 + 5x + 2x^2$

Let

V = n-element set

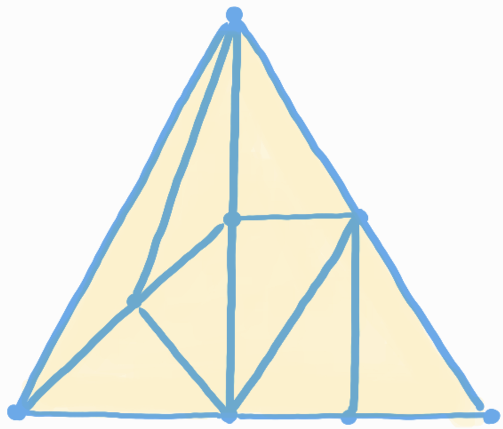
Γ = triangulation of 2^V

Γ_F = restriction of Γ on $F \in 2^V$.

Definition (Stanley 1992). The local h-polynomial of Γ (with respect to V) is defined as

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x).$$

Example.



Γ

- $\ell(\Gamma_V, x) = (1+5x+2x^2) - (1+2x) - (1+x) - 1 + 1 + 1 + 1 - 1$
  $= 2x + 2x^2.$

Theorem (Stanley 1992). For every triangulation Δ' of a pure simplicial complex Δ,

$$h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x)\, h(\mathrm{link}_\Delta(F), x)$$

where Δ'_F is the restriction of Δ' to $F \in \Delta$.

Theorem (Stanley 1992). The polynomial $\ell_V(\Gamma, x)$

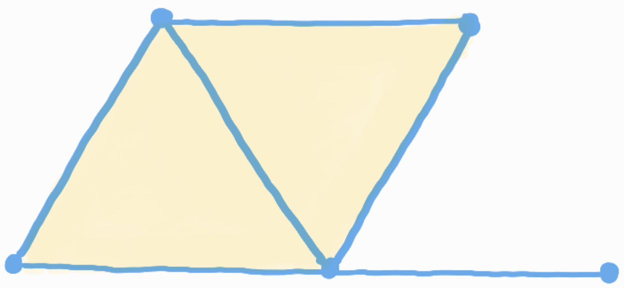
- is symmetric, with center of symmetry $n/2$, for every triangulation Γ of the simplex 2^V
- has nonnegative coefficients for every triangulation Γ of the simplex 2^V
- is unimodal for every regular triangulation Γ of the simplex 2^V.

II. Uniform triangulations: Definitions and examples

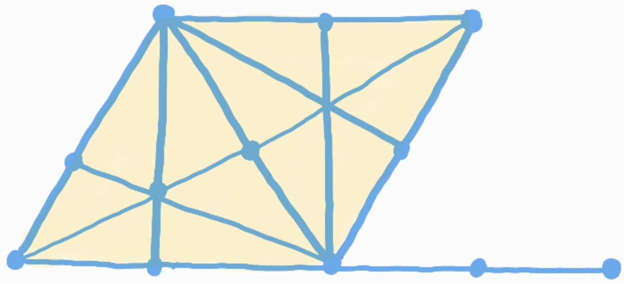
Barycentric subdivision. Let

Δ = simplicial complex of dimension $n-1$

$sd(\Delta)$ = barycentric subdivision of Δ.



Δ



$sd(\Delta)$

Theorem (Brenti–Welker, 2008)

(a) There exist $p_{n,k,j} \in \mathbb{N}$ such that

$$h_j(sd(\Delta), x) = \sum_{k=0}^{n} p_{n,k,j}\, h_k(\Delta)$$

for every $(n-1)$-dimensional simplicial complex Δ.

(b) If $h_k(\Delta) \geq 0$ for $0 \leq k \leq n$, then $h(sd(\Delta), x)$ has (nonnegative coefficients and) only real roots.

Theorem (Kubitzke–Nevo, 2009)

If Δ is Cohen–Macaulay (over some field) of dimension $n-1$, then $h(sd(\Delta), x)$ is unimodal with a peak in one of the middle positions $n/2$ or $(n\pm1)/2$, i.e.

- $$\begin{aligned} h_0(sd(\Delta)) &\leq h_1(sd(\Delta)) \leq \cdots \leq \\ &\leq h_j(sd(\Delta)) \geq \cdots \geq \\ &\geq h_n(sd(\Delta)) \end{aligned}$$

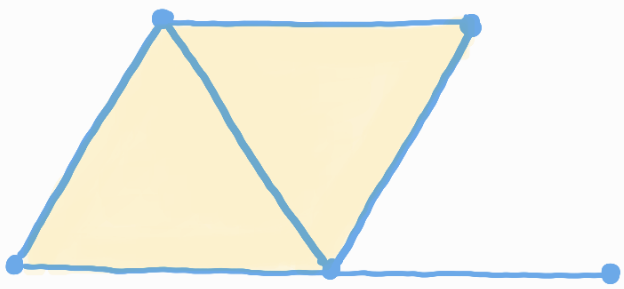
with $j \in \{n/2, (n\pm1)/2\}$.

Edgewise subdivision. Let

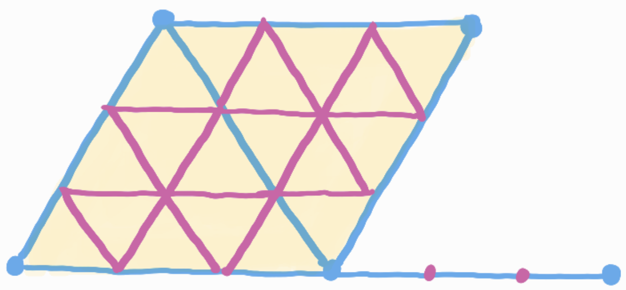
r = positive integer

Δ = simplicial complex of dimension $n-1$

$\mathrm{esd}_r(\Delta)$ = r-fold edgewise subdivision of Δ.



Δ



$\mathrm{esd}_3(\Delta)$

$r = 3$

Theorem. Fix an $r \in \mathbb{Z}_{>0}$.

(a) (Brenti–Welker, 2009) There exist $p_{n,k,j} \in \mathbb{N}$ such that

$$h_j(\mathrm{esd}_r(\Delta), x) = \sum_{k=0}^{n} p_{n,k,j}\, h_k(\Delta)$$

for every $(n-1)$-dimensional simplicial complex Δ.

(b) (Jochemko, 2018) If $r \geq n$ and $h_k(\Delta) \geq 0$ for $0 \leq k \leq n$, then $h(\mathrm{esd}_r(\Delta), x)$ has (nonnegative coefficients and) only real roots.

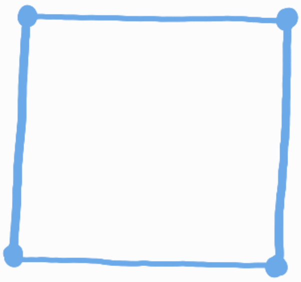
Remark. The $p_{n,k,j} \in \mathbb{N}$:

- can be interpreted in terms of permutation enumeration, in the case of $sd(\Delta)$,
- are essentially the entries of Holte's amazing matrices studied by Diaconis-Fulman (2009), in the case of $esd_r(\Delta)$.

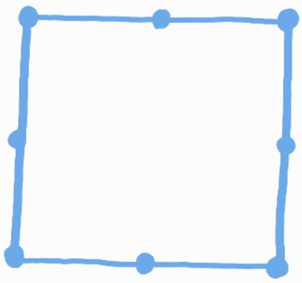
Remark. There are similar results for antiprism triangulations (A-Brunink-Kubitzke, 2022)

Definition. A triangulation Δ' of a simplicial complex Δ is called *uniform* if $f(\Delta'_F, x)$ depends only on $\dim(F)$, $F \in \Delta$.

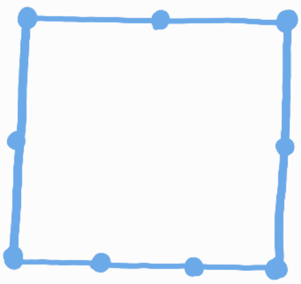
Δ



Δ'

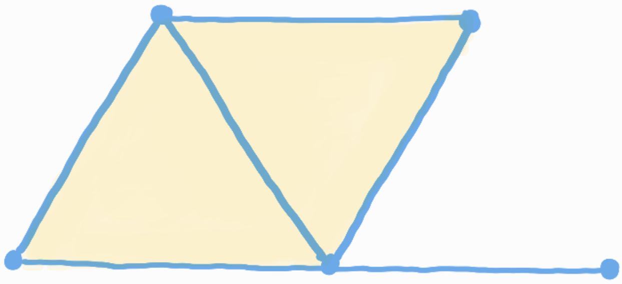


uniform

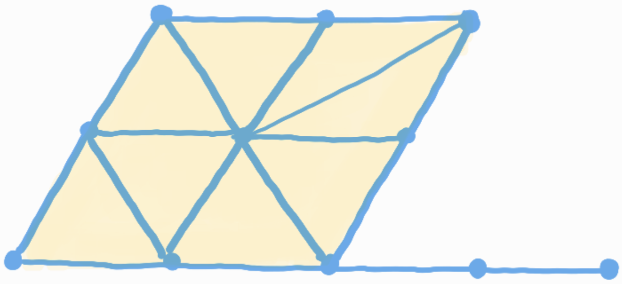


not uniform

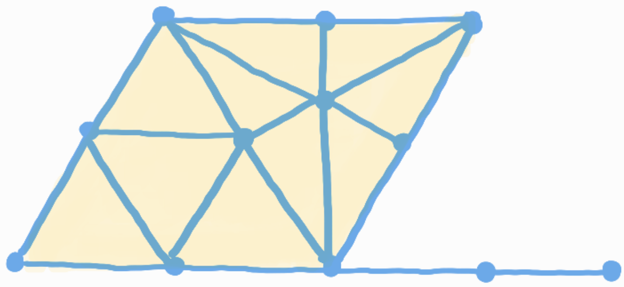
Δ



Δ'



uniform



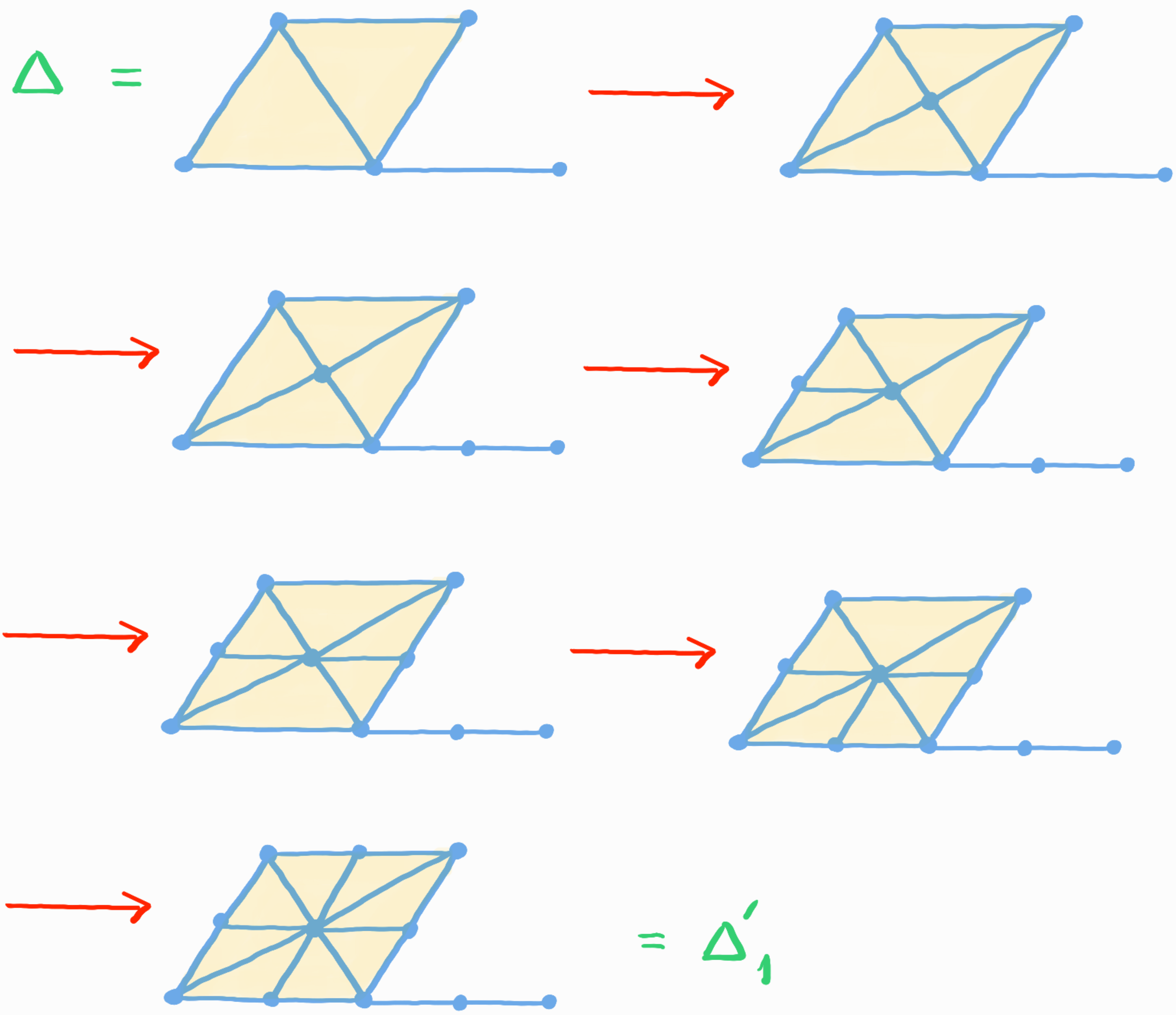
not uniform

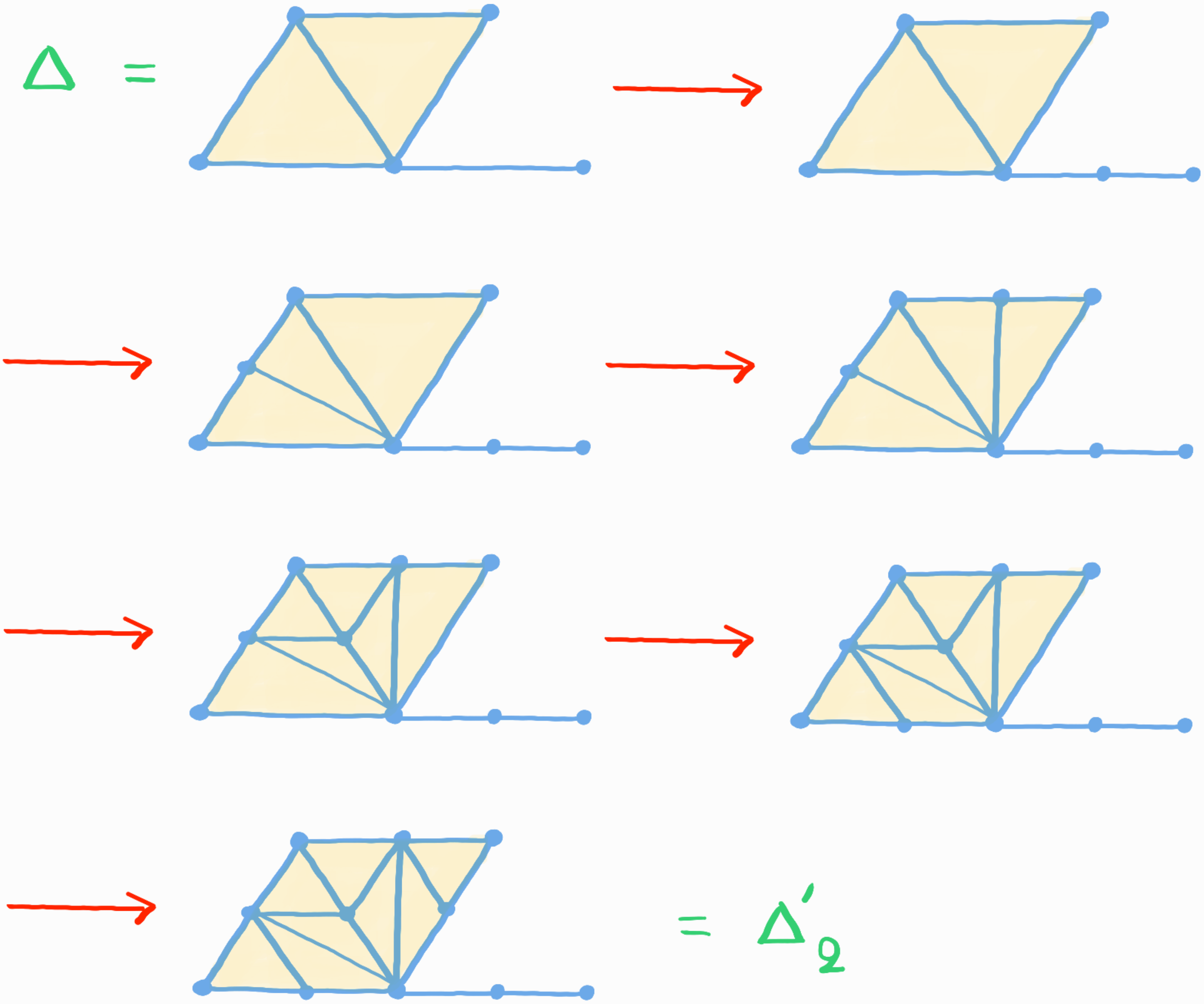
Prototypical examples of uniform triangulations of Δ are

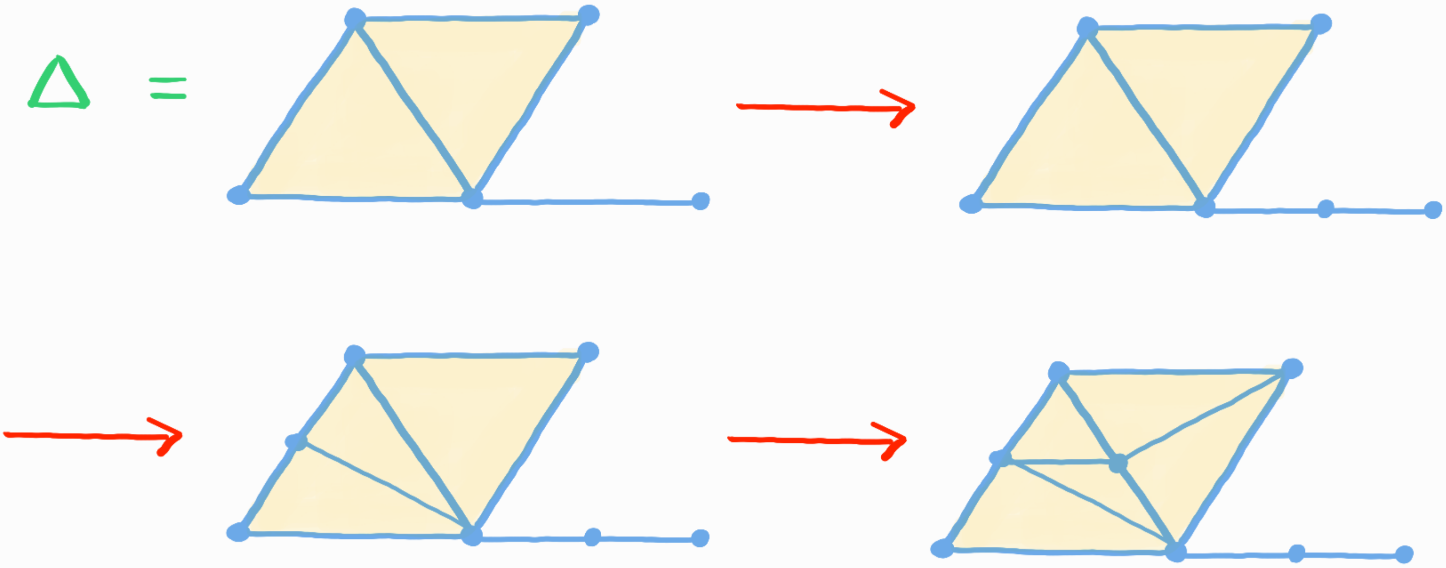
- the trivial subdivision
- $sd(\Delta)$ and $esd_r(\Delta)$

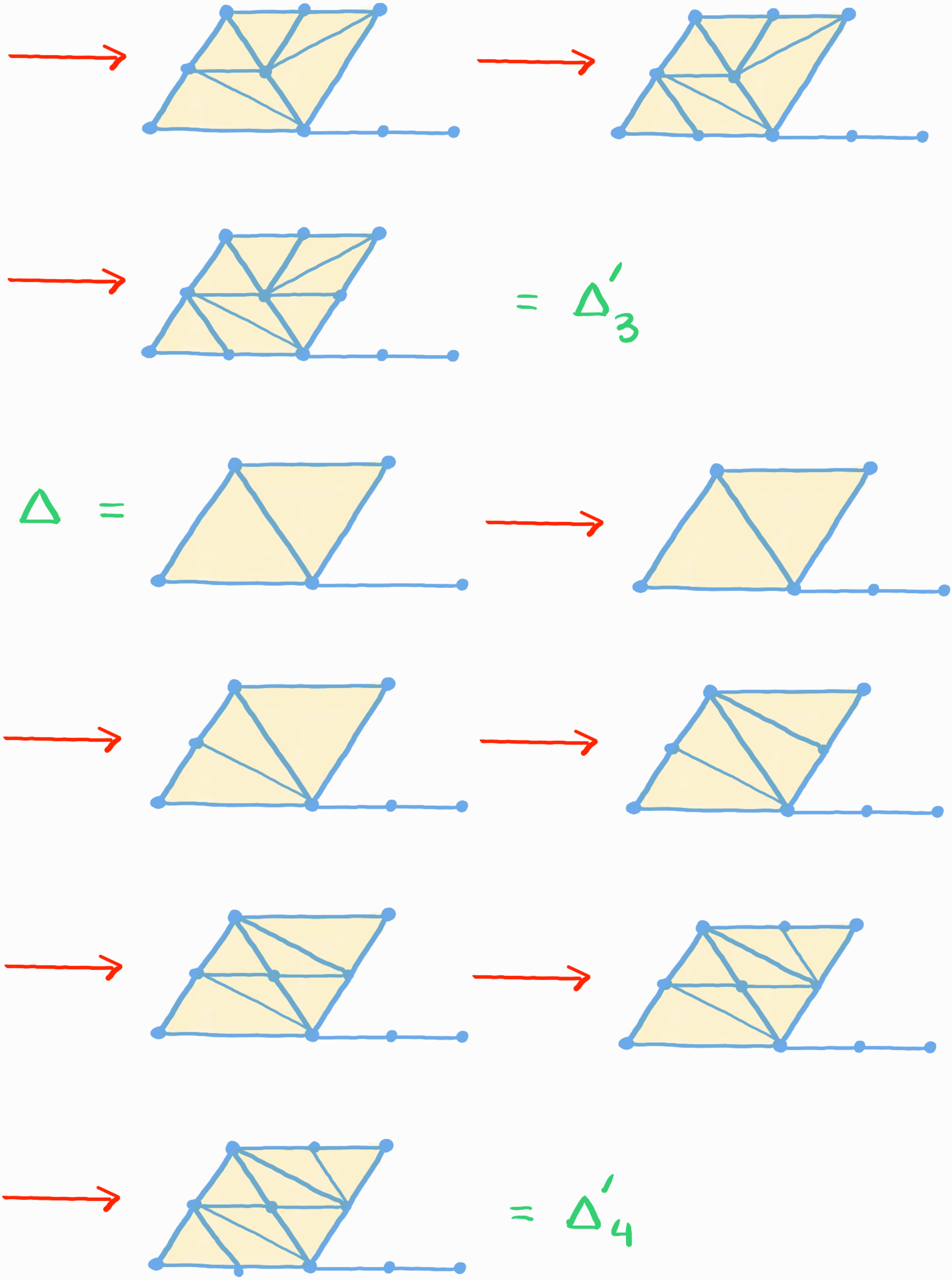
Example (Hetyei–Nevo, 2016)

Tchebyshev triangulations of Δ are obtained by edge subdividing Δ along each edge in some order.




$\Delta =$
$= \Delta'_2$


$\Delta =$



$= \Delta'_3$
$\Delta =$
$= \Delta'_4$

All Tchebyshev triangulations of Δ are uniform triangulations with the same f-vector.

Other examples of uniform triangulations include

- antiprism triangulations
- interval triangulations
- r-colored barycentric subdivisions.

$r=2$

III. Main results.

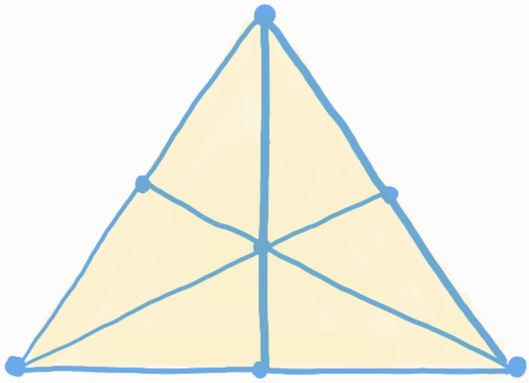
Let

Δ = simplicial complex of dimension $n-1$

Δ' = uniform triangulation of Δ

f_{ij} = number of $(i-1)$-dimensional faces of Δ'_F for any $(j-1)$-dimensional $F \in \Delta$.

Terminology. We call Δ' $\mathcal{F}$-uniform, where $\mathcal{F} = (f_{ij})_{0 \le i \le j \le n}$.



$$F = \begin{array}{c|c|c|c|c} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline 2 & 1 & 3 & 2 & 0 \\ \hline 3 & 1 & 7 & 12 & 6 \end{array}$$

Theorem (A, 2022). Given $\mathcal{F}$, there exist $p_{\mathcal{F},n,k,j} \in \mathbb{N}$ such that

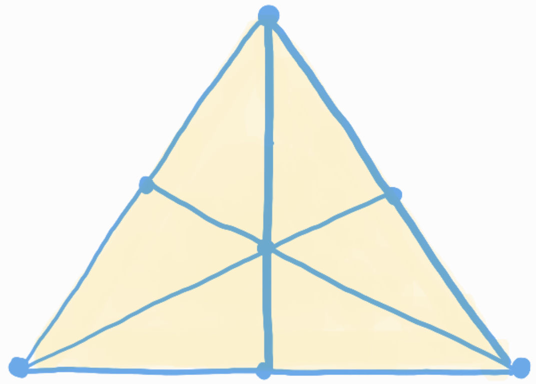
$$h_j(\Delta', x) = \sum_{k=0}^{n} p_{\mathcal{F},n,k,j}\, h_k(\Delta)$$

for every $(n-1)$-dimensional simplicial complex Δ and every $\mathcal{F}$-uniform triangulation Δ' of Δ. Equivalenly,

$$h(\Delta', x) = \sum_{k=0}^{n} h_k(\Delta)\, p_{\mathcal{F},n,k}(x)$$

for some $p_{\mathcal{F},n,k}(x) \in \mathbb{N}[x]$, $0 \leq k \leq n$.

For barycentric subdivision and $n=3$



$$P_{F,n,k}(x) = \begin{cases} 1+4x+x^2, & k=0 \\ 4x+2x^2, & k=1 \\ 2x+4x^2, & k=2 \\ x+4x^2+x^3, & k=3. \end{cases}$$

Notation.

σ_n = $(n-1)$-dimensional simplex

$h_F(\Delta, x)$ = $h(\Delta', x)$ for any F-uniform triangulation Δ' of Δ.

Theorem (A, 2022).

(a) $P_{\mathcal{F},n,0}(x) = h_{\mathcal{F}}(\sigma_n, x)$ and

$$P_{\mathcal{F},n,k}(x) =$$

$$P_{\mathcal{F},n,k-1}(x) + (x-1)\, P_{\mathcal{F},n-1,k-1}(x)$$

for $1 \leq k \leq n$.

(b) $x^n P_{\mathcal{F},n,k}(1/x) = P_{\mathcal{F},n,n-k}(x)$

for $0 \leq k \leq n$.

(c)

$$P_{\mathcal{F},n,k}(x) = \sum_{r=0}^{n} \ell_{\mathcal{F}}(\sigma_r, x) \cdot$$

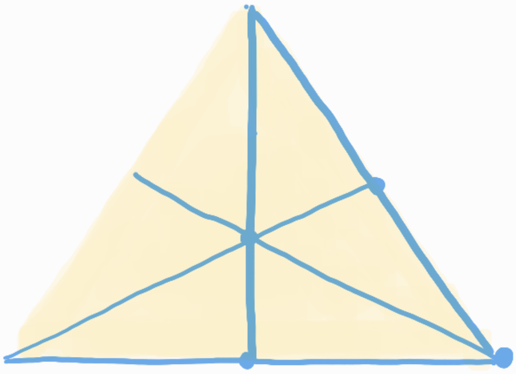
$$\sum_{i=0}^{r} \binom{n-k}{i} \binom{k}{r-i} x^{k-r+i}$$

where

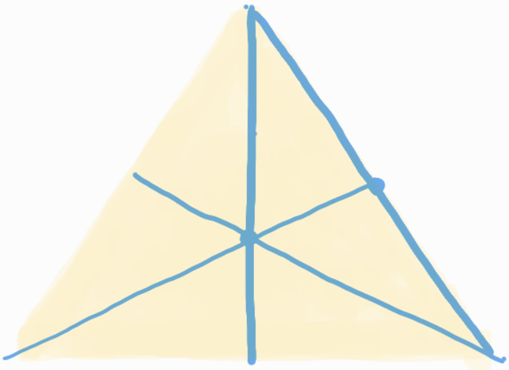
$$\ell_F(\sigma_n, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} h_F(\sigma_k, x)$$

is the local h-polynomial of any F-uniform triangulation of σ_n.

(d) $p_{F,n,k}(x)$ is equal to the h-polynomial of the relative simplicial complex obtained from any F-uniform triangulation of σ_n by removing all faces lying on k facets of $\partial\sigma_n$.



$$P_{\mathcal{F},3,1}(x) = 4x + 2x^2$$



$$P_{\mathcal{F},3,2}(x) = 2x + 4x^2$$

Example. For the trivial subdivision we have

$$P_{\mathcal{F},n,k}(x) = x^k$$

for $0 \leq k \leq n$.

Example. For barycentric subdivision we have

$$P_{F,n,k}(x) = \sum_{w \in \mathfrak{S}_{n+1} \,:\, w(1) = k+1} x^{\operatorname{des}(w)}$$

where

$\mathfrak{S}_n$ = group of permutations of $\{1, 2, \dots, n\}$

$\operatorname{des}(w) = \#\{ i \in [n-1] : w(i) > w(i+1)\}$ for $w \in \mathfrak{S}_n$.

Question. For which uniform triangulations $h(\Delta) \geq 0$ implies that $h_{\mathcal{F}}(\Delta, x)$ is real-rooted?

Recall that for real-rooted polynomials $p(x), q(x) \in \mathbb{R}[x]$ with roots

- $\cdots \leq \alpha_2 \leq \alpha_1$
- $\cdots \leq \beta_2 \leq \beta_1$

we say that $p(x)$ *interlaces* $q(x)$ if $\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1$ and write $p(x) \prec q(x)$.

A sequence

$$(p_0(x), p_1(x), \dots, p_n(x))$$

of real-rooted polynomials is called interlacing if $p_i(x) \prec p_j(x)$ for $0 \le i < j \le n$.

Fact. If $(p_0(x), p_1(x), \dots, p_n(x))$ is an interlacing sequence of real-rooted polynomials with nonnegative coefficients, then

$$\sum_{k=0}^{n} c_k p_k(x)$$

is real-rooted for all $c_k \ge 0$.

Definition. We say that $\mathcal{F}$

(a) has the interlacing property if

$$(P_{\mathcal{F},m,k}(x))_{0 \leq k \leq m}$$

is an interlacing sequence of real-rooted polynomials for every $m \leq n$

(b) has the strong interlacing property if

- $h_{\mathcal{F}}(\sigma_m, x)$ is real-rooted for $m < n$

- $\theta_{\mathcal{F}}(\sigma_m, x) :=$

$$h_{\mathcal{F}}(\sigma_m, x) - h_{\mathcal{F}}(\partial\sigma_m, x)$$

is either identically zero, or a real-rooted polynomial of degree $m-1$ which is interlaced by $h_{\mathcal{F}}(\sigma_{m-1}, x)$:

$$h_{\mathcal{F}}(\sigma_{m-1}, x) \prec \theta_{\mathcal{F}}(\sigma_m, x),$$

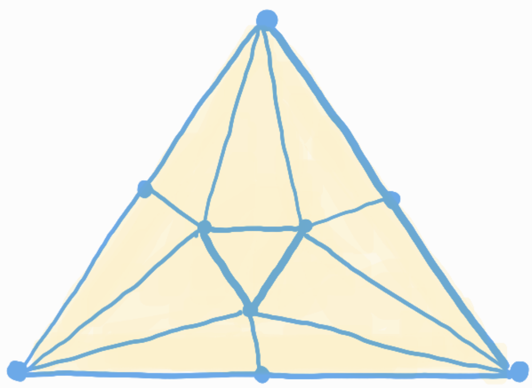
for $m \leq n$.

Remark. The strong interlacing property can be verified in several special cases of interest.

Theorem (A, 2022). Suppose that $\mathcal{F}$ has the strong interlacing property. Then, $\mathcal{F}$ has the interlacing property.

In particular, $h_{\mathcal{F}}(\Delta, x)$ has only real roots, provided that $h_{\kappa}(\Delta) \geq 0$ for every κ.

Example. Let Δ' be obtained by the antiprism construction from the barycentric subdivision of the $(n-2)$-skeleton of Δ.



$n=3$

For $n=4$ $\mathcal{F}$ has the interlacing property but not the strong one, since $\theta_{\mathcal{F}}(\sigma_4, x) = 3x + 11x^2 + 3x^3$ is not interlaced by $h_{\mathcal{F}}(\sigma_3, x) = 1+4x+x^2$.

Symmetric decompositions. We recall that every polynomial $f(x) \in \mathbb{R}[x]$ of degree $\leq n$ can be written uniquely as

$$f(x) = a(x) + x\, b(x)$$

where

- $\deg(a(x)) \leq n$
- $\deg(b(x)) \leq n-1$
- $x^n a(1/x) = a(x)$
- $x^{n-1} b(1/x) = b(x)$

This expression is the *symmetric decomposition* of $f(x)$ with respect to n.

This decomposition is called

- nonnegative, if both $a(x)$ and $b(x)$ have nonnegative coefficients
- real-rooted, if so are $a(x)$ and $b(x)$
- real-rooted and interlacing if $a(x)$ and $b(x)$ are real-rooted and $b(x) \prec a(x)$.

Note. If $f(x)$ has a nonnegative and real-rooted symmetric decomposition with respect to n, then $f(x)$ is unimodal with a peak at position $\lfloor (n+1)/2 \rfloor$.

Note. If Δ triangulates an $(n-1)$-dimensional ball and

$$\theta(\Delta, x) = h(\Delta, x) - h(\partial\Delta, x),$$

then

$$h(\Delta, x) = h(\partial\Delta, x) + x \cdot \theta(\Delta, x)/x$$

is the symmetric decomposition of $h(\Delta, x)$ with respect to $n-1$.

Theorem (A–Tzanaki, 2021).
Suppose that $\mathcal{F}$ has the strong interlacing property.

(a) $h_{\mathcal{F}}(\Delta, x)$ has a nonnegative, real-rooted symmetric decomposition with respect to n for every $(n-1)$-dimensional simplicial complex Δ such that $h_k(\Delta) \geq 0$ and

$$\sum_{i=0}^{k} h_i(\Delta) \leqslant \sum_{i=0}^{k} h_{n-i}(\Delta)$$

for $0 \leq k \leq n$ (special case of barycentric subdivision due to Brändén–Solus, 2021).

(b) This decomposition is interlacing if, additionally,

$$\frac{h_0(\Delta)}{h_n(\Delta)} \leqslant \frac{h_1(\Delta)}{h_{n-1}(\Delta)} \leqslant \cdots \leqslant \frac{h_n(\Delta)}{h_0(\Delta)}. \quad (*)$$

Note. The inequalities (*) imply the inequalities

$$h_i(\Delta) \leqslant h_{n-i}(\Delta)$$

for $0 \leqslant i \leqslant n$, studied by Swartz, 2006, Adiprasito-Papadakis-Petrotou, 2021.

IV. Some questions

Question. Which Cohen-Macaulay simplicial complexes Δ satisfy the inequalities $(*)$?

Do these hold for every doubly Cohen-Macaulay simplicial complex Δ of dimension $n-1$?

Note (Mu-Welker, 2024). They hold iff

$$\begin{pmatrix} h_n(\Delta) & h_{n-1}(\Delta) & \cdots & h_0(\Delta) \\ h_0(\Delta) & h_1(\Delta) & \cdots & h_n(\Delta) \end{pmatrix}$$

is TP (totally positive).

Moreover, if Δ satisfies (*) and

$$H_{\mathcal{F}} = (p_{\mathcal{F},n,k,j})_{0 \le k,j \le n}$$

is TP_2 (true for barycentric and edgewise subdivisions), then every $\mathcal{F}$-uniform triangulation of Δ satisfies (*) as well.

Note. The interlacing property implies that $H_{\mathcal{F}}$ is TP_2.

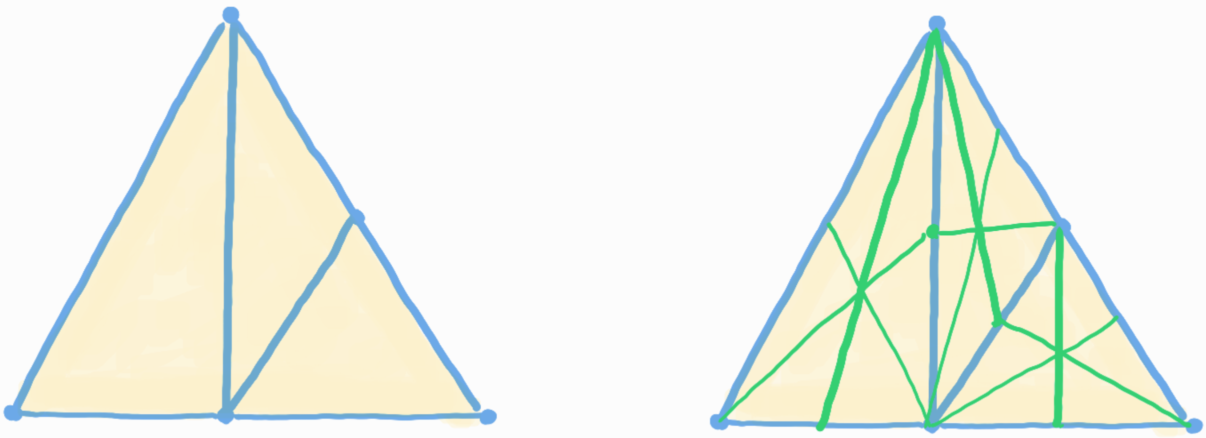
Question. Does the strong interlacing property imply that $H_{\mathcal{F}}$ is TP?

Question. Does the strong interlacing property imply that the local h-polynomial

$$\ell_{\mathcal{F}}(\sigma_n, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} h_{\mathcal{F}}(\sigma_k, x)$$

is real-rooted?

Question. Does the strong interlacing property imply that $\ell_V(\Delta,x)$ is real-rooted for every F-uniform triangulation Δ of any triangulation Γ of 2^V?



Γ $\qquad\qquad$ $\mathrm{sd}(\Gamma)$

Theorem (A, 2024). True for barycentric and edgewise subdivisions.

Question. Does the strong interlacing property imply that

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h_F(\sigma_k, x)$$

is real-rooted?

Question. Which uniform triangulations satisfy the strong interlacing property? E.g.

(a) Is the strong interlacing property preserved by barycentric subdivision?

(b) Is the strong interlacing property preserved by r-fold edgewise subdivision?

Note (A, 2024). (b) holds for $r=2$.

Thank you for your attention

Ευχαριστώ για την προσοχή σας

Спасибо за внимание!