

Chain polynomials of face
lattices of polytopes and
geometric lattices:

A tale of two conjectures

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Let

P = finite poset of rank $n-1$

$\Delta(P)$ = order complex of P

= {chains in P }

$c_k(P)$ = # k -element chains in P

= $f_{k-1}(\Delta(P))$.

Example. If

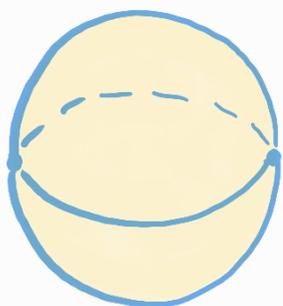
Γ = regular cell complex

P = face poset of Γ

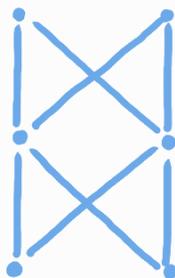
then

$\Delta(P)$ = barycentric subdivision
of Γ .

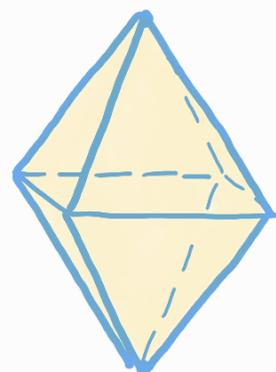
(a)



Γ



P

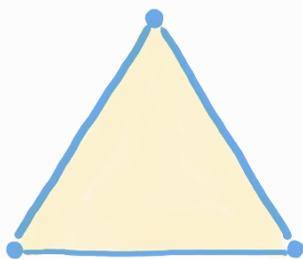


$\Delta(P)$

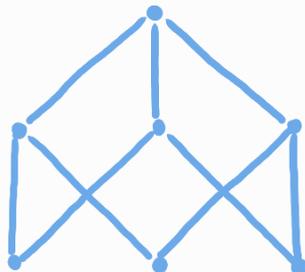
$$c_0(P) = 1, \quad c_1(P) = 6$$

$$c_2(P) = 12, \quad c_3(P) = 8$$

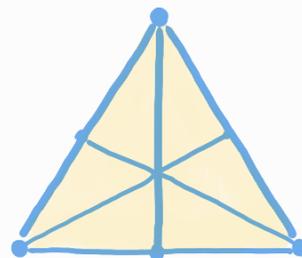
(b)



Γ



P

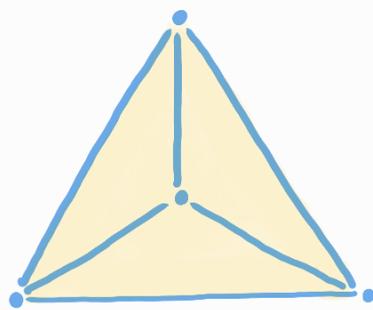


$\Delta(P)$

$$c_0(P) = 1, \quad c_1(P) = 7$$

$$c_2(P) = 12, \quad c_3(P) = 6$$

Remark. The order complex $\Delta(P)$ is **flag**, meaning that every clique in its one-skeleton is a face of $\Delta(P)$.



not flag

Definition. The f, h -polynomials of $\Delta(P)$ are defined as

$$f(\Delta(P), x) = \sum_{k=0}^n f_{k-1}(\Delta) x^k$$

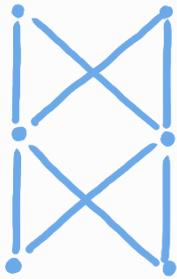
$$= \sum_{k=0}^n c_k(P) x^k$$

= chain polynomial of P

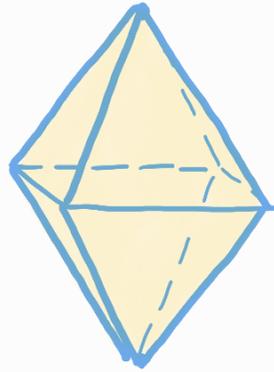
$$h(\Delta(P), x) = \sum_{k=0}^n c_k(P) x^k (1-x)^{n-k}$$

$$= (1-x)^n f(\Delta, \frac{x}{1-x}).$$

Example. (a)



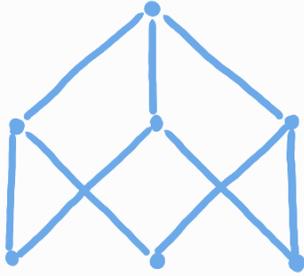
P



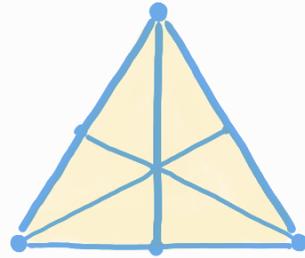
$\Delta(P)$

- $f(\Delta(P), x) = 1 + 6x + 12x^2 + 8x^3$
- $h(\Delta(P), x) = (1-x)^3 + 6x(1-x)^2 + 12x^2(1-x) + 8x^3$
 $= (1+x)^3$

(b)



P

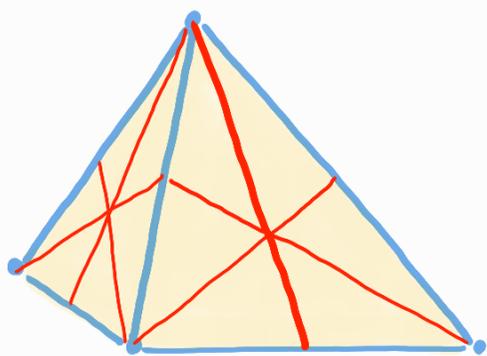


$\Delta(P)$

- $f(\Delta(P), x) = 1 + 7x + 12x^2 + 6x^3$
- $h(\Delta(P), x) = (1-x)^3 + 7x(1-x)^2 + 12x^2(1-x) + 6x^3$
 $= 1 + 4x + x^2.$

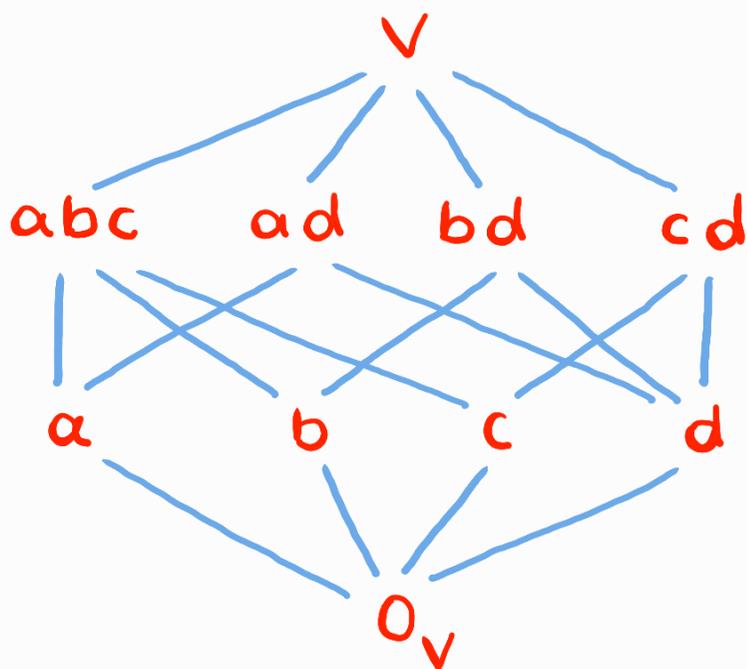
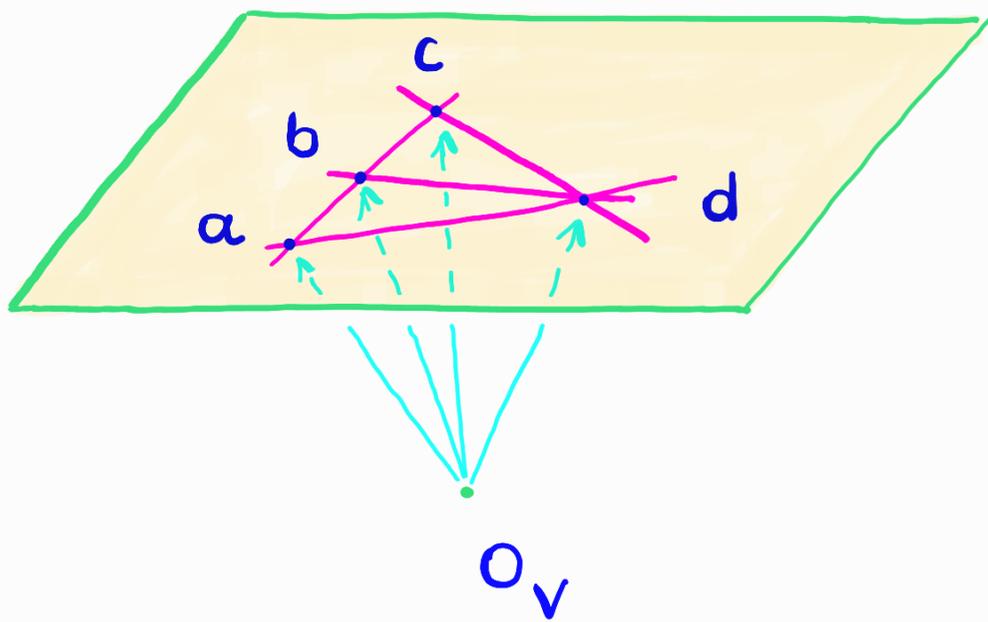
Conjecture (Brenti-Welker, 2008)

$f(\Delta(P), x)$ is real-rooted if P is the face lattice of a polytope.



Conjecture (A-Kalambogia Evangelinou)

$f(\Delta(P), x)$ is real-rooted if P is a geometric lattice (the lattice of flats of a matroid).



Remark

- (a) $f(\Delta(P), x)$ is real-rooted \Leftrightarrow
 $h(\Delta(P), x)$ is real-rooted.
- (b) For geometric lattices, even
the unimodality of $h(\Delta(P), x)$
is open.

Even stronger conjectures make sense.

Conjecture (unpublished)

$f(\Delta(P), x)$ is real-rooted if P is any rank-selected subposet of

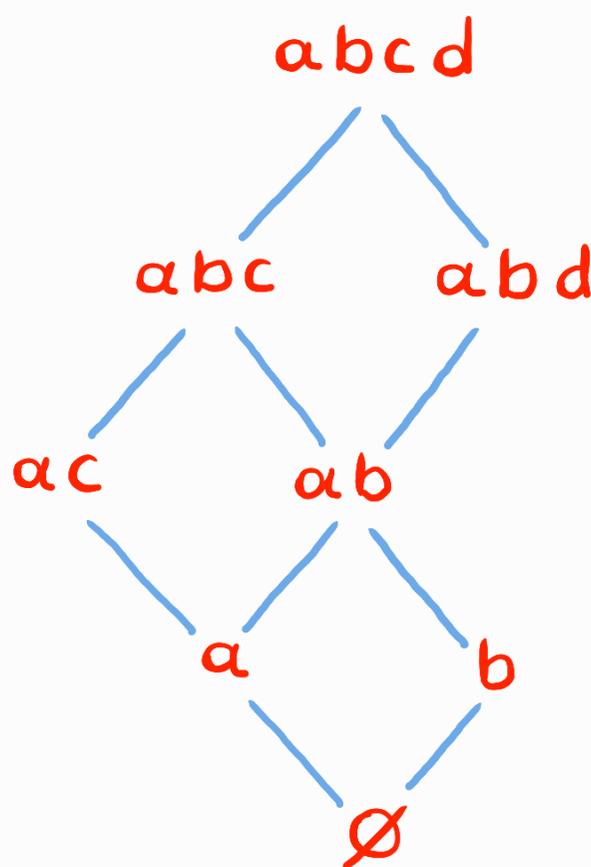
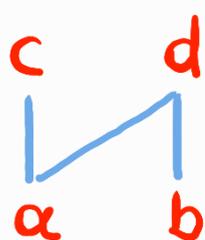
- the face lattice of a polytope, or
- a geometric lattice.

Question For which finite posets P is $f(\Delta(P), x)$ real-rooted?

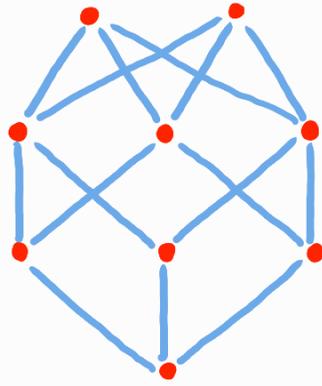
Question Is $f(\Delta(P), x)$ real-rooted for every doubly Cohen-Macaulay poset P ?

Some answers:

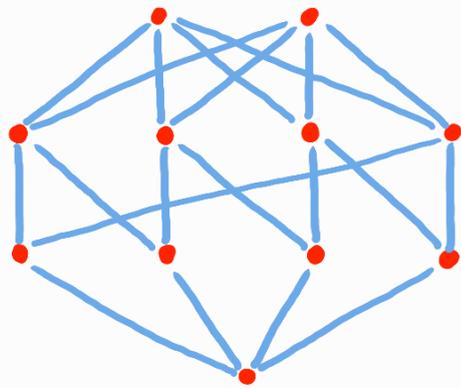
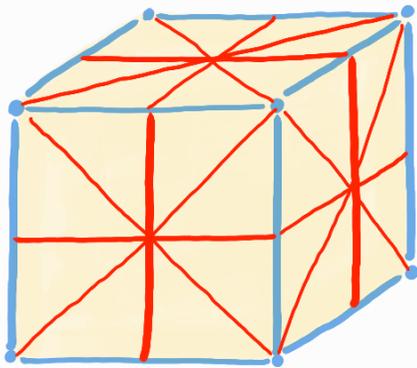
- No for distributive lattices (Stembridge, 2007)



- Yes for Cohen-Macaulay simplicial posets (Brenti-Welker, 2008)



- Yes for face lattices of cubical polytopes (more generally, for shellable cubical posets, [A 2021](#)).



- Yes for $P \times Q$, if so for P and Q and $h(\Delta(P), x)$, $h(\Delta(Q), x)$ have nonnegative coefficients ([essentially, Wagner 1992](#)).

- Yes for subspace lattices and partition lattices of types A and B (A-KE, 2023).
- Yes for the lattices of flats of generalized paving matroids and Dowling lattices (Brändén - Saut Maia Leite)
- Yes for rank-selected subposets of Cohen-Macaulay simplicial posets (A-KE, 2023).

- Yes for noncrossing partition lattices associated to finite Coxeter groups (A-KE, 2023).
- Yes for $(3+1)$ -free posets (Stanley, 1998).

Basic method: Express $h(\Delta(P), x)$ as a nonnegative linear combination of real-rooted polynomials with nonnegative coefficients which have a common interleaver.

Recall the for real-rooted polynomials $p(x), q(x) \in \mathbb{R}[x]$ with roots

- $\dots \leq \alpha_2 \leq \alpha_1 \leq 0$

- $\dots \leq \beta_2 \leq \beta_1 \leq 0$

we say that $p(x)$ **interlaces** $q(x)$

if $\dots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1 \leq 0$ and

write **$p(x) < q(x)$** .

Let

P = simplicial poset of
rank n

$f_{k-1}(P)$ = # elements of rank k

$$h(P, x) = \sum_{k=0}^n f_{k-1}(P) x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^n h_k(P) x^k.$$

Note. If P is Cohen-Macaulay,
then $h_k(P) \geq 0$ for every k .

Proposition (Brenti - Welker, 2008)

For every simplicial poset P of rank n

$$h(\Delta(P), x) = \sum_{k=0}^n h_k(P) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \sum_{w \in \mathfrak{S}_{n+1} : w(1) = k+1} x^{\text{des}(w)}$$

for $n \in \mathbb{N}$. Equivalently,

$$\frac{P_{n,k}(x)}{(1-x)^{n+1}} = \sum_{m \geq 0} m^k (1+m)^{n-k} x^m.$$

Note

$$\begin{pmatrix} P_{n,0}(x) \\ P_{n,1}(x) \\ \vdots \\ P_{n,n}(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x & 1 & \dots & 1 \\ x & x & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x & x & & x \end{pmatrix} \begin{pmatrix} P_{n-1,0}(x) \\ P_{n-1,1}(x) \\ \vdots \\ P_{n-1,n-1}(x) \end{pmatrix}$$

and hence $P_{n,i}(x) < P_{n,j}(x)$ for $0 \leq i < j \leq n$.

Corollary For every Cohen-Macaulay simplicial poset P of rank n , $h(\Delta(P), x)$ is real-rooted and is interlaced by the Eulerian polynomial $A_n(x) := p_{n,0}(x)$.

Let

\mathcal{P} = cubical poset of rank $n+1$

$f_k(\mathcal{P})$ = # elements of rank $k+1$

$$\tilde{\chi}(\mathcal{P}) = -1 + \sum_{k=0}^n (-1)^k f_k(\mathcal{P})$$

$$h(\mathcal{P}, x) = \sum_{k=0}^{n+1} h_k(\mathcal{P}) x^k$$

$$= \frac{1}{1+x} \left\{ 1 + \sum_{k=0}^n f_k(\mathcal{P}) x^{k+1} \left(\frac{1-x}{2} \right)^{n-k} + (-1)^n \tilde{\chi}(\mathcal{P}) x^{n+2} \right\}.$$

Proposition (A, 2023) For every cubical poset P of rank $n+1$,

$$h(\Delta(P), x) = \sum_{k=0}^{n+1} h_k(P) P_{n,k}^B(x),$$

where

$$\frac{P_{n,k}^B(x)}{(1-x)^{n+1}} = \begin{cases} \sum_{m \geq 0} (2m+1)^{\binom{n}{m}} x^m, & k=0 \\ \sum_{m \geq 0} 4m (2m-1)^{\binom{k-1}{m}} (2m+1)^{\binom{n-k}{m}} x^m, & 1 \leq k \leq n \\ \sum_{m \geq 1} (2m-1)^{\binom{n}{m}} x^m, & k=n+1. \end{cases}$$

Note The $p_{n,k}^{\mathcal{B}}(x)$ satisfy a recursion similar to the one satisfied by the $p_{n,k}(x)$.

Corollary For every shellable cubical poset P of rank $n+1$, $h(\Delta(P), x)$ is real-rooted and is interlaced by the Eulerian polynomial

$$B_n(x) := p_{n,0}^{\mathcal{B}}(x).$$

Let

P = simplicial poset of rank n
with minimum element $\hat{0}$

$\rho = \rho: P \rightarrow \{0, 1, \dots, n\}$
= rank function of P

and for $S \subseteq \{1, 2, \dots, n\}$ let

$P_S = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}\}$
= S -rank selected subposet
of P .

Proposition (A-KE 2023+) For every simplicial poset P of rank n and every $S \subseteq \{1, 2, \dots, n\}$

$$h(\Delta(P_S), x) = \sum_{k=0}^n h_k(P) P_{n,k}^{n+1-S}(x),$$

where

$$P_{n,k}^S(x) = \sum_{\substack{w \in \mathfrak{S}_{n+1} \\ \text{Des}(w) \subseteq S}} x^{\text{des}(w)}$$

$$w \in \mathfrak{S}_{n+1} : w(1) = k+1$$

$$\text{Des}(w) \subseteq S$$

Lemma For $S \subseteq \{1, 2, \dots, n\}$

$$P_{n,k}^S(x) = \begin{cases} \sum_{i=k}^{n-1} P_{n-1,i}^{S-1}(x), & 1 \notin S \\ x \sum_{i=0}^{k-1} P_{n-1,i}^{S-1}(x) + \sum_{i=k}^{n-1} P_{n-1,i}^{S-1}(x), & 1 \in S. \end{cases}$$

As a result, $P_{n,i}^S(x) < P_{n,j}^S(x)$ for $0 \leq i < j \leq n$.

Corollary For every Cohen-Macaulay simplicial poset P of rank n and every $S \subseteq \{1, 2, \dots, n\}$, $h(\Delta(P_S), x)$ is real-rooted and interlaced by

$$A_n^S(x) = \sum_{w \in \mathfrak{S}_n : \text{Des}(w) \subseteq S} x^{\text{des}(w)}$$

Let

- $\Pi_n =$ lattice of partitions of $\{1, 2, \dots, n\}$
- $A_n = \{1\} \times \{1, 1, 2\} \times \{1, 1, 1, 2, 2, 3\} \times \dots$
 $\times \{1, 1, \dots, 1, \dots, n-2, n-2, n-1\}$

and for $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}) \in A_n$

- $\text{des}(\sigma) = \# i \in \{1, 2, \dots, n-2\} :$
 $\sigma_i \geq \sigma_{i+1}.$

Proposition (A-KE 2023+)

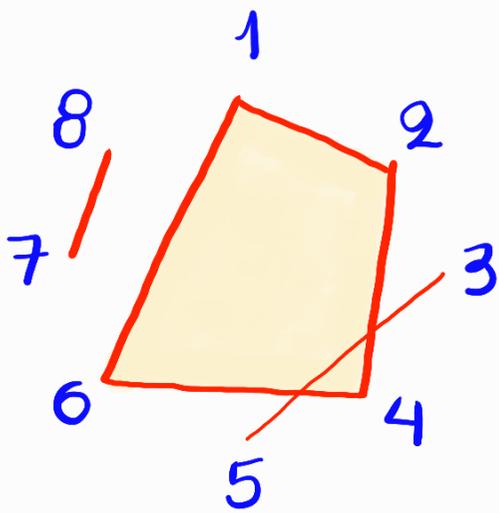
$$h(\Delta(\Pi_n), x) = \sum_{\sigma \in A_n} x^{\text{des}(\sigma)}$$

is real-rooted for every $n \in \mathbb{N}$.

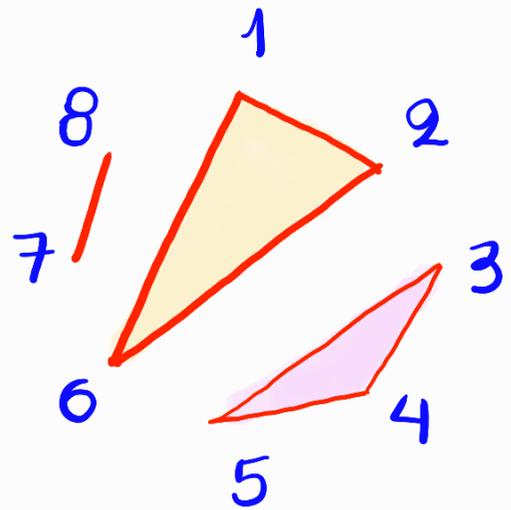
Let

- W = finite real reflection group of rank n
- NC_W = lattice of noncrossing partitions associated to W
- χ = Coxeter type of W

Remark. For $\mathcal{X} = A_{n-1}$, NC_W is the set of partitions of $\{1, 2, \dots, n\}$ whose blocks do not 'cross each other', partially ordered by refinement.



crossing



noncrossing

Theorem (A - Douvropoulos - KE)

(a) $h(\Delta(NC_W), x)$ is a nonnegative linear combination of the polynomials $p_{n,k}(x)$ for $0 \leq k \leq n$ and hence real-rooted.

(b) Let D_n be the set of words $w = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ such that

$$1 \leq |a_1|, a_2, \dots, a_n \leq n-1$$

and call $i \in \{1, 2, \dots, n-1\}$ a descent if

- $|a_i| > a_{i+1}$ or
- $a_i = a_{i+1} > 0$.

Then, $h(\Delta(\text{NC}_w), x)$ is equal to

$$\frac{1}{n+1} \sum_{w \in \{1, 2, \dots, n+1\}^n} x^{\text{des}(w)}, \quad X = A_n$$

$$\sum_{w \in \{1, 2, \dots, n\}^n} x^{\text{des}(w)}, \quad X = B_n$$

$$\sum_{w \in D_n} x^{\text{des}(w)}, \quad X = D_n$$

(c) Let

$$S_r \left(\sum_{n \geq 0} a_n x^n \right) = \sum_{n \geq 0} a_{rn} x^n$$

be the zeroth r -Veronese operator.

Then, $x^n h(\Delta(\text{NC}_W), 1/x)$ equals

$$\left\{ \begin{array}{l} \frac{1}{n+1} S_{n+1} \left(x(1+x+\dots+x^n)^{n+1} \right), \quad \mathcal{X} = A_n \\ S_n \left(x(1+x+\dots+x^{n-1})^{n+1} \right), \quad \mathcal{X} = B_n \\ S_{n-1} \left((x+x^2)(1+x+\dots+x^{n-2})^{n+1} \right), \quad \mathcal{X} = D_n \end{array} \right.$$

As a result (Jochemko, 2021),
 $h(\Delta(NC_w), x)$ has a real-rooted
symmetric decomposition with
respect to n for every irreducible
 w .

Thank you for your attention

Ευχαριστώ για την προσοχή σας